# QUARKS WITH A PION CONDENSATE — A NEW PHASE OF MATTER*

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On the basis of the $\sigma$-model, it is argued that there may exist a new phase of matter at densities of the order of a few nuclear densities, and temperatures below $\sim$ 100 MeV. The phase consists of a gas of quarks with constituent-like mases, submerged in a periodic chiral field — a "pion condensate". The appearance of this phase is a general feature of the $\sigma$-model and other models based on the chiral dynamics (*e.g.* the NJL model), and it occurs for a broad range of model parameters. Phenomenological consequences on the physics of dense matter are discussed, in particular we describe interesting magnetic properties of the phase.

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## 1. Introduction

This talk will have two aspects: one is the physics of dense matter, with possible phenomenological consequences for neutron stars or RHIC. The other aspect is more formal, and concerns analysis of a class of periodic solutions to the $\sigma$-model [1] in $3+1$ dimensions. Our basic results [2,3] can be stated as follows:

1) At densities of the order of a few nuclear densities, and at temperatures below $\sim$ 100 MeV, chiral quark models predict a new phase of matter — quark gas in a "pion condensate" [2]. The result is obtained at the mean-field level, and for a wide range of the model parameters.
2) We have found solutions to the $\sigma$-model in 3+1 dimensions which are not translationally uniform, but nevertheless are analytic down to the one-loop level [3]. This may serve as a test ground for various approximate

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methods, as well as illustrate difficulties with chiral models at the one-loop level. We will discuss these issues at the end of this talk.

Let us begin with a reminder of the traditional view on the QCD phase diagram, which is shown in Fig. 1. The horizontal axis is the temperature $T$, and the vertical axis is the quark chemical potential $\mu$ (equal to 1/3 of the baryon chemical potential). The value of the chiral condensate $\langle\bar{\psi}\psi\rangle$ is used as the order parameter. At low values of $T$ and $\mu$ the value of $\langle\bar{\psi}\psi\rangle$ is nonzero (phase with spontaneously broken chiral symmetry, labeled "N"), and in the outer region $\langle\bar{\psi}\psi\rangle$ vanishes (phase with restored chiral symmetry, labeled "R"). The boundary between the phases corresponds to the chiral transition.

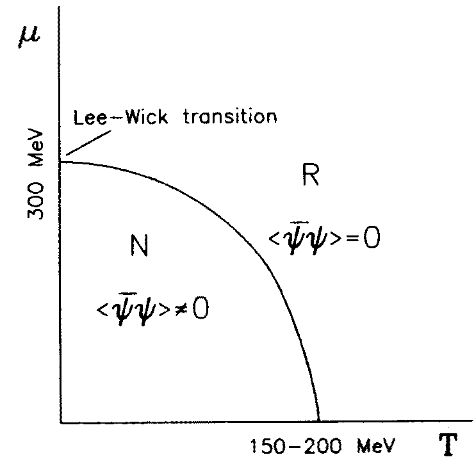



Fig. 1. The traditional phase diagram of QCD. At low values of the quark chemical potential $\mu$ and temperature $T$ the system is in the chirally broken phase, "N", with $\langle\bar{\psi}\psi\rangle \neq 0$. For higher $\mu$ and $T$ the chiral symmetry is restored (phase "R") and $\langle\bar{\psi}\psi\rangle = 0$.

The chiral transition is well established for $\mu = 0$, *i.e.* for zero net baryon density (same amount of quarks and antiquarks). For this case there are realistic lattice QCD calculations [4], which indicate that the transition occurs at temperatures of the order of 150-200 MeV (Fig. 1). However, for $\mu \neq 0$ (nonzero baryon density) at present there are no reasonable lattice calculations. Attempts have been made, but with realistic fermions the lattice size is very small ($4^4$). Thus departing from the $\mu = 0$ line in Fig. 1 presents, for the fundamental theory, a serious difficulty. On the other hand, the finite baryon density region can be analyzed in effective chiral models. The transition from the broken to the restored phase at $T = 0$, which occurs as one increases $\mu$, is usually referred to as the Lee-Wick transition [5] (see Fig. 1).

What we want to point out now is that previous analyses of quark matter carried in the framework of effective chiral models lacked an important degree of freedom — they all assumed that the value of $\langle\bar{\psi}\psi\rangle$ is spatially uniform. Departure from this assumption leads to the appearance of new phases in quark matter. The example presented in this talk is analogous to the "neutral pion condensate" of nuclear physics [6-8], and the basic result is depicted schematically in Fig. 2. In addition to the "N" and "R" phases present in Fig. 1, we find a new phase, labeled "C", which is the "pion condensed" phase. This phase has not only $\langle\bar{\psi}\psi\rangle \neq 0$, but also the expectation value $\langle\bar{\psi}i\gamma_5\tau^3\psi\rangle \neq 0$, and their values oscillate in the coordinate space:

$$\begin{aligned}\langle\bar{\psi}(\vec{r})\psi(\vec{r})\rangle &= \phi\cos\left[\vec{q}\cdot(\vec{r}-\vec{r}_0)\right],\\ \langle\bar{\psi}(\vec{r})i\gamma_5\tau^3\psi(\vec{r})\rangle &= \phi\sin\left[\vec{q}\cdot(\vec{r}-\vec{r}_0)\right],\\ \langle\bar{\psi}(\vec{r})i\gamma_5\vec{\tau}\psi(\vec{r})\rangle &= 0\,.\end{aligned} \tag{1}$$

We have introduced $\phi$ and $\vec{q}$ as parameters, which will later be determined dynamically. The value of the vector $\vec{r}_0$ is irrelevant, and we choose for simplicity $\vec{r}_0 = 0$. Of course, the "normal" phase "N" has $\langle\bar{\psi}(\vec{r})\psi(\vec{r})\rangle =$ const, $\langle\bar{\psi}(\vec{r})i\gamma_5\tau^a\psi(\vec{r})\rangle = 0$, and in the restored phase "R" all condensates vanish.

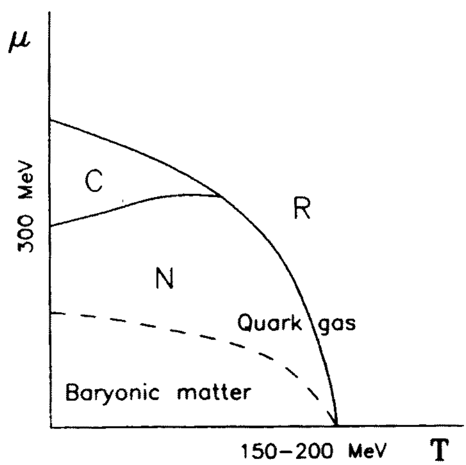



Fig. 2. Our speculation, based on the results of the $\sigma$-model to be presented in this talk: in addition to the phases of Fig. 1, a new phase, "C", appears. This phase has the chiral condensates of the form (1). The quark chemical potential $\mu$ at which the phase shows up corresponds to baryon densities of the order of a few nuclear saturation densities. The region inside the dashed line corresponds to hadronic matter (confinement), which is not described by our approach. The outer region corresponds to the quark gas (deconfinement).

Fig. 2 shows that at sufficiently low temperatures (in our model, below 100 MeV), as we increase the baryon density of the system we first pass from the "normal" phase to the "pion condensed" phase, which happens at a density of the order of a few nuclear densities (such densities correspond to the value of $\mu$ at which the phase shows up), and at still higher densities the transition to the restored phase occurs. In this talk we will demonstrate how the behavior of Fig. 2 is realized in the $\sigma$-model.

A digression is in place at this point. There is another prominent phase transition in QCD — the deconfinement phase transition. Again, the lattice gauge calculations (with $\mu = 0$) show that it occurs at temperatures around 150-200 MeV, which coincides with the temperature of the chiral transition. At low densities and temperatures the quarks are clustered in hadrons. Since in our calculation we assume that the quarks are not clustered, but constitute a free gas, our approach is justified only in the deconfined region (above the dashed line in Fig. 2). We can see that there is room for the new phase only if (at a given temperature) "declustering" occurs at a lower density than the chiral restoration. This situation may be viewed as follows: at low densities we have isolated hadrons. As the density is increased, the "bags" start to overlap, and the quarks can percolate. This is a geometrical effect, and it is hard to imagine why it should occur at the same density as the chiral restoration, which is a dynamical effect. Thus, it is possible that there exists a quark gas phase with broken chiral symmetry. In this talk we assume it as a conjecture, and show that if this happens, then the system develops a "pion condensed" phase.

It would certainly be desirable to describe the hadronic phase within the model, but this is beyond the scope of our current research.

## 2. Pion condensate in the $\sigma$-model

You have already heard today the talks by Ripka [9] and Christov [10] about models based on chiral dynamics. The relation of these models to QCD is discussed in Ref [11]. Ripka [9] and Christov [10] concentrated on the Nambu-Jona-Lasìnio model [12], and we shall work in the framework of the Gell-Mann-Lévy $\sigma$-model. The differences are irrelevant at this point and will be discussed in some detail later.

The Lagrangian of the linear SU(2)×SU(2) $\sigma$-model [1] has the form

$$\mathcal{L} = \bar{\psi}\left[i\partial\!\!\!/ - g\left[\sigma + i\gamma_5\tau^a\pi^a\right]\right]\psi + \tfrac{1}{2}\left(\partial^\mu\sigma\right)^2 + \tfrac{1}{2}\left(\partial^\mu\pi^a\right)^2 - V\left[\sigma,\pi^a\right], \quad (2)$$

where $\psi$ is the quark field, $g$ is the coupling constant, $\sigma$ and $\pi$ are the SU(2) chiral fields and $V$ is the "Mexican Hat" potential [1]

$$V\left[\sigma,\pi^a\right] = \tfrac{\lambda^2}{4}\left[\sigma^2 + \pi^a\pi^a - \nu^2\right]^2 - c\sigma - V_0, \quad (3)$$

which generates the spontaneous breaking of chiral symmetry. In the vacuum $\sigma_{vac} = -F_\pi$, where $F_\pi = 93$ MeV is the pion decay constant. The values of parameters in Eq. (3) are related to the pion and $\sigma$ masses in the following way:

$$\lambda^2 = \frac{m_\sigma^2 - m_\pi^2}{2F_\pi^2}, \quad \nu^2 = \frac{m_\sigma^2 - 3m_\pi^2}{m_\sigma^2 - m_\pi^2}, \quad c = m_\pi^2 F_\pi . \tag{4}$$

The constant $V_0$ is such that $V$ vanishes in the vacuum. In the following we shall work in the limit of exact chiral symmetry ($m_\pi = 0$). The value of the $\sigma$ mass will be left as the parameter of the model. The preferred value for $m_\sigma$ is around 1 GeV, which then could be identified with the scalar-isoscalar resonance $f_0$. Looking at Fig. 3 we can see that $m_\sigma$ controls the mesonic volume energy in the restored phase ($\sigma = \pi^a = 0$), which equals $B = m_\sigma^2 F_\pi^2/8$. This is the energy needed to break the chiral condensate. For $m_\sigma = 1200$ MeV, which is the preferred value of Ref. [13], $B$ equals to 140 MeV/fm$^3$. The second parameter of the model is the coupling constant $g$. Models of the nucleon use $g$ between 4 and 5 [13-15]. Instead of $g$ we shall frequently use $m_0 = gF_\pi$, the quark mass in the vacuum. In this talk we shall concentrate on the effect of the appearance of the new phase, and this will not depend on a particular selection of the parameters.

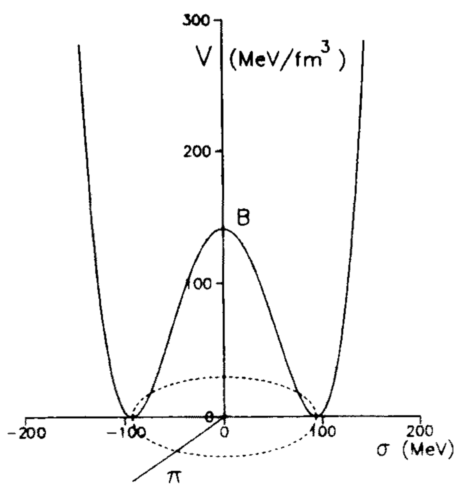



Fig. 3. The "Mexican Hat" potential (3) for $m_\sigma = 1200$ MeV and $m_\pi = 0$. The quantity B, the value at $\sigma = \vec{\pi} = 0$, is the volume energy necessary to break the chiral condensate.

In phenomenological models [13-15] the system (2) is treated in the so-called mean-field approximation: the $\sigma$ and $\pi$ fields are classical c-number fields, and only valence quarks are included. For a stationary case the Euler-Lagrange equations assume the form

$$\left[-i\vec{\alpha}\cdot\vec{\nabla}+g\beta\left(\sigma+i\gamma_5\tau^a\pi^a\right)\right]\varphi_j=E_j\varphi_j\,, \tag{5a}$$

$$\nabla^2\sigma=\frac{\delta V}{\delta\sigma}+gN_c\sum_{j\in\mathrm{val}}\bar{\varphi}_j\varphi_j\,, \tag{5b}$$

$$\nabla^2\pi^a=\frac{\delta V}{\delta\pi^a}+gN_c\sum_{j\in\mathrm{val}}\bar{\varphi}_j i\gamma_5\tau^a\varphi_j\,, \tag{5c}$$

where $\varphi_j$ are the Dirac spinors, $E_j$ are the corresponding singleparticle energies, "val" denotes the set of all valence orbitals, and $N_c = 3$ is the number of colors.

Various solutions of Eq. (5) are known. In models of baryons one uses the hedgehog solution [13-15]. To describe space-uniform matter one applies the simplest solution which has a constant $\sigma$, vanishing $\pi$, and plane-wave Dirac spinors [16]. A very interesting class of solutions of Eq. (5) has been found by Dautry and Nyman [8]. These authors have used them to describe the pion condensation in nucleonic matter. Here we shall use the solution desrcibing the "neutral pion condensate". It has the following form for the chiral fields:

$$\sigma=-m/g\cos\left(\vec{q}\cdot\vec{x}\right)\,,\qquad \pi^3=-m/g\sin\left(\vec{q}\cdot\vec{x}\right)\,,\qquad \pi^1=\pi^2=0\,. \tag{6}$$

It represents a chiral field oscillating in the $\vec{q}$ direction, with amplitude $m/g$. For the special case $\vec{q}=0$ $m$ plays the role of the quark mass. Note that the solution has the property that $\sigma^2+\pi^2=(m/g)^2$ is constant in space, hence it lives in the bottom of the "Mexican Hat" (Fig. 3), and "winds" around it as one goes along the $\vec{q}$ direction in space.

Solutions discussed in this talk will have saturated isospin (equal amount of up and down quarks). More general cases are discussed in [2].

The Dirac equation (5a) with the fields (6) acquires the form

$$\left[-i\vec{\alpha}\cdot\vec{\nabla}+\beta m\exp\left(i\gamma_5\tau^3\vec{q}\cdot\vec{x}\right)\right]\varphi_p\left(\vec{x}\right)=E\left(\vec{p}\right)\varphi_p\left(\vec{x}\right)\,. \tag{7}$$

Using a trick with similarity transformations [8], Eq. (7) can be easily solved. Introducing space-independent spinors $\chi(\vec{p})$,

$$\chi\left(\vec{p}\right)=\exp\left(\tfrac{1}{2}i\gamma_5\tau_3\vec{q}\cdot\vec{x}+i\vec{p}\vec{x}\cdot\right)\varphi_p\left(\vec{x}\right)\,, \tag{8}$$

we find that the Dirac equation (7) becomes

$$\left(\vec{\alpha} \cdot \vec{p} - \tfrac{1}{2}\vec{\alpha} \cdot \vec{q}\gamma_5\tau_3 + \beta m\right) \chi(\vec{p}) = E(\vec{p})\,\chi(\vec{p}) \,. \tag{9}$$

The remarkable simplification is that this equation is space-independent, and further diagonalization is trivial [8]. The spectrum is

$$\begin{aligned} E(\vec{p}) &= +\,\varepsilon^{\pm}(\vec{p}) \quad \text{for positive energy states,} \\ E(\vec{p}) &= -\,\varepsilon^{\pm}(\vec{p}) \quad \text{for negative energy states,} \end{aligned} \tag{10}$$

where

$$\varepsilon^{\pm} = \left[p_{\perp}^2 + p_{\parallel}^2 + m^2 + \tfrac{q^2}{4} \pm q\sqrt{p_{\parallel}^2 + m^2}\right]^{1/2} . \tag{11}$$

We have introduced $\vec{p}_{\perp}$ and $p_{\parallel}$ as the components of $\vec{p}$ perpendicular and parallel to $\vec{q}$. The $\pm$ sign in (11) distinguishes the types of quasiparticles [8]. The dependence of spectrum (10-11) on the wave vector $\vec{q}$ is drawn schematically in Fig. 4. The solid line is for $\vec{p} = 0$, and the dashed lines are for $p_{\perp} = 0$, $p_{\parallel} = m$ (long dash) and for $p_{\perp} = m$, $p_{\parallel} = m$ (short dash). The four branches: $E^+$, $E^-$, $-E^+$ and $-E^-$ are indicated in the figure. We also note that the spectrum is charge symmetric. We also note that levels belonging to the $E^-$ branch go down as $q$ increases. This immediately indicates the possibility of lowering the energy by increasing $q$. The energy of occupied valence quarks is decreased as we increase the value of $q$, and this is the key to the formation of the pion condensate. Of course, there are also other terms in the system's total energy (see Eq. (12)), but the basic understanding of the effect can be extracted from Fig. 4.

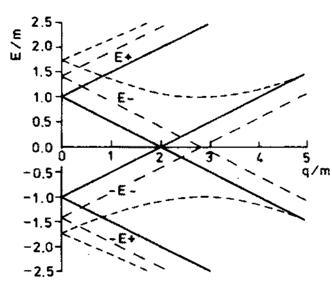



Fig. 4. The spectrum (10-11) of the Dirac equation (5a) with chiral fields (6). The continuous lines are for $p_{\perp} = p_{\parallel} = 0$, the long dash lines are for $p_{\perp} = 0$ and $p_{\parallel} = m$, and the short dash lines are for $p_{\perp} = p_{\parallel} = m$. The labels $E^+$, $E^-$, $-E^+$ and $-E^-$ denote the four branches in the spectrum. In the valence approximation, the positive energy levels are occupied, up to some Fermi energy.

It is easy to verify that the ansatz (6), together with the corresponding forms for the Dirac spinors $\varphi_p$, are in fact solutions of the Euler-Lagrange equations (5). What has to be determined now are the values of $m$ and $\vec{q}$ for which the system acquires minimum energy. The energy density $\mathcal{E}$ of our system consists of three contributions: from valence quarks, from kinetic terms of the $\sigma$ and $\pi$ fields, and from the "Mexican Hat" potential $V$:

$$\mathcal{E} = N_c N_f \int \frac{d^3p}{(2\pi)^3} \left[\varepsilon^+(\vec{p})\,\theta\left[\varepsilon_F - \varepsilon^+(\vec{p})\right] + \varepsilon^-(\vec{p})\,\theta\left[\varepsilon_F - \varepsilon^-(\vec{p})\right]\right]$$
$$+ \tfrac{1}{2} q^2 m^2/g^2 + m_\sigma^2/\left(8F_\pi^2\right)\left[m^2/g^2 - F_\pi^2\right]^2 , \qquad (12)$$

where $N_f = 2$ is the number flavors, $\varepsilon_F$ is the Fermi energy and $\theta$ is the step function. The expression for the baryon density $\mathcal{N}$ is [2]

$$\mathcal{N} = N_f \int \frac{d^3p}{(2\pi)^3} \left[\theta\left[\varepsilon_F - \varepsilon^+(\vec{p})\right] + \theta\left[\varepsilon_F - \varepsilon^-(\vec{p})\right]\right] . \qquad (13)$$

We now have to minimize $\mathcal{E}$ with respect to $\vec{q}$ and $m$ at a fixed density $\mathcal{N}$. This amounts to minimizing the grand thermodynamical potential $\Omega$ at zero temperature,

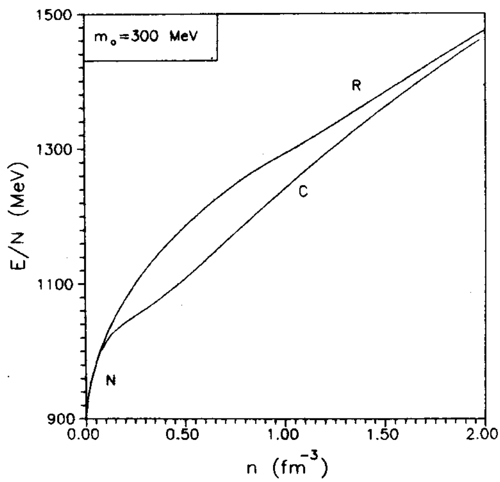
$$\Omega(m, \vec{q}) = \mathcal{E} - N_c \varepsilon_F \mathcal{N} . \qquad (14)$$

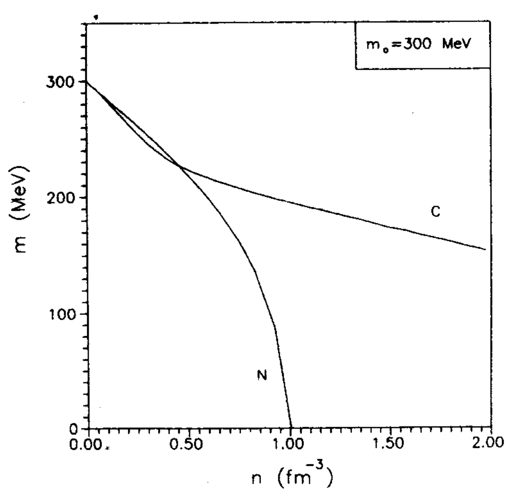
Of course, in this case $\varepsilon_F$ is just the quark chemical potential.

In their work Dautry and Nyman [8] made nonrelativistic approximation on expressions (13) and (14). This was justified in the application to nucleonic matter. In our case this would not be justified, since the quarks are lighter than the nucleons, and also we want to go to much higher densities. Fortunately, integrals appearing in (13) and (14) can be done analytically [2] and there is no reason to make any approximations to the formulas.

The results of our calculation of quark matter are shown in Fig. 5 (case $m_0 = gF_\pi = 300$ MeV) and Fig. 6 (case $m_0 = 500$ MeV). Fig. 5a shows the zero-temperature equation of state. The energy per baryon, $E/N$, is plotted *vs* baryon density, here denoted by $n$. The upper curve is the result of the calculation where we have imposed $\vec{q} = 0$, as has been always done in other works on the quark matter in chiral models. This curve consists of two parts: at low densities (label "N") it has nonzero expectation value of $\sigma$, and the chiral symmetry is broken. The value of $m = -g\sigma$ is plotted in Fig. 5b. We can see that the curve "N" drops to zero at the density 1 fm$^{-3}$, and at this point the chiral restoration occurs. Beyond this point the curve in Fig. 5a continues as the restored phase, "R". But this curve is not the ground state. At densities above 0.07 fm$^{-3}$ the ground state is described by the curve "C", which is the phase with pion condensation, and has nonzero

$q$ (see Fig. 5c). Looking again at Fig. 5b, we notice that for the "C" phase $m$ decreases more slowly than for the "N" phase. At very high densities, outside the frame of Fig. 5c, the value of $m$ drops to 0. This phase transition is first order. We have also checked that the transition from the normal to the condensed phase is second order.



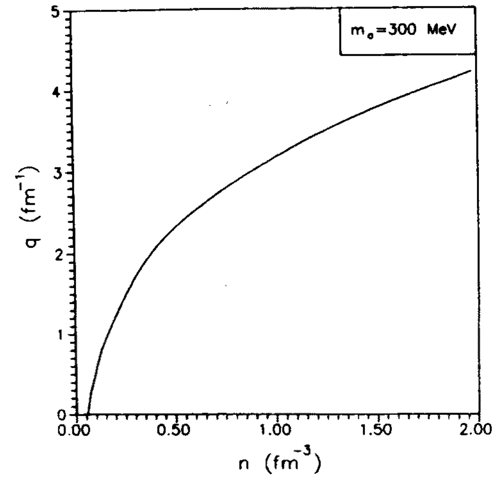



Fig. 5. The results for $T = 0$ and $m_0 = gF_\pi = 300$ Mev. Figure (a) shows the equation of state, *i.e.* the dependence of energy per baryon, $E/N$, *vs* baryon density $n = N/V$. The upper curve consists of two pieces: normal phase, "N", at low densities, and restored phase, "R", at higher densities. The transition between "N" and "R" occurs at the points where $m$ drops to zero on curve "N" if Fig. 5b. The lower curve in Fig. 5a is the phase with the pion condensate, "C". It is the ground state of the system, for densities higher than 0.07 $\mathrm{fm}^{-3}$, which is the point where $q$ raises from zero in Fig. 5c. Figure (b) shows the behavior of optimum $m$ on the baryon density. It drops from the value $m_0 = gF_\pi = 300$ MeV at $n = 0$, to zero at $n$ around 1 $\mathrm{fm}^{-3}$. The curve "C" decreases more slowly than "N", and it approaches 0 at very high densities, outside the scope of the plot. Figure (c) shows the dependence of q on baryon density.

Fig. 6 shows the same as Fig. 5 for a higher value of $g$ ($m_0 = gF_\pi = 500$ MeV). The qualitative behavior does not change. The only difference worth mentioning is that the curves in the equation of state (Fig. 6a) have a minimum and the system develops saturation [2]. Recalling our remark in the introduction on the baryonic matter, we should stress that our results are not realistic at low densities, lower than, say, twice the nuclear saturation density. Thus, at low densities there is baryonic matter, and only at higher densities our assumption of the quark gas can hold.

## 3. Equation of state and magnetic properties of the pion-condensed quark matter

Quark matter at low temperatures can exist in cores of neutron stars provided the neutron matter equation of state is sufficiently soft. We can read from Figs 5a and 6a that at densities of the order of a few nuclear densities, the condensed phase has the energy density lower than the "N" or "R" phase by a few tens MeV. Thus it leads to an equation of state

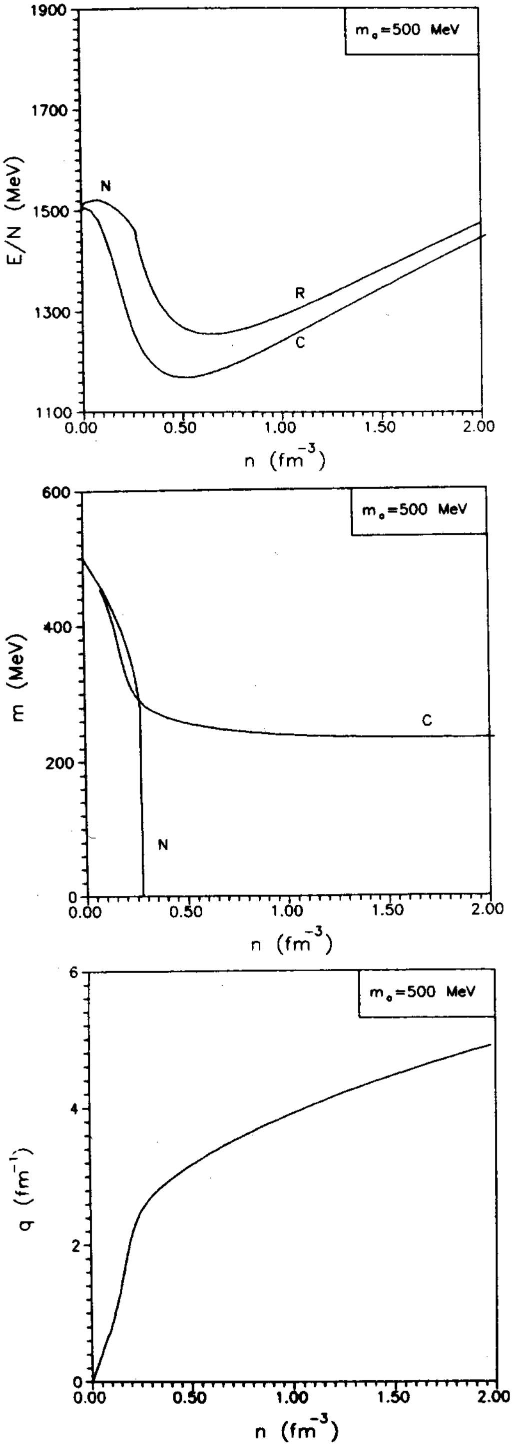



**Fig. 6.** Same as Fig. 5 for $gF_\pi = 500$ MeV.

which is even softer than for the massless quark gas. In Fig. 7 we compare this equation with several conventional equations of state which are used to describe dense matter. What we can see is that our equation of state, labeled "Q", is not so much different from some traditional equations. Therefore it will not lead to significantly different results in, for example, the radius-mass dependence of the neutron star. To have a chance of detecting the pion-condensed quark matter we have to look for other signatures.

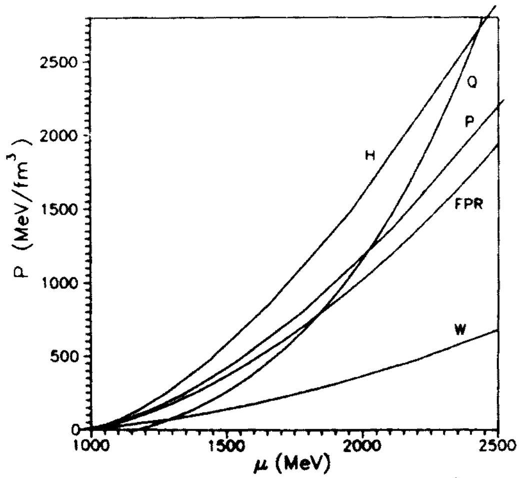



Fig. 7. Comparison of our equation of state with several conventional equations, drawn on the pressure-baryon chemical potential [27] plot. "Q" labels the pion-condensed quark gas for the case $m_0 = gF_\pi = 500$ MeV. The other curves correspond to Walecka's equation of state "W" [16,17], Pandharipande's neutron "P" [18] and hyperon "H" [19] equations of state, and for the Friedman-Pandharipande-Ravenhall "FPR" [20] equation of state.

An interesting possibility is given to us by the fact that the phase possesses **nonzero magnetization** [2]. Assuming Dirac magnetic moments for the quarks we have $\mu_u = -2\mu_d$, where $\mu_u$ and $\mu_d$ are the up and down quark magnetons, respectively. The magnetization $M$ is given by the formula

$$M = g\left(\mu_u s_u + \mu_d s_d\right) , \tag{15}$$

where $s_u$ and $s_d$ are the spin densities of up and down quarks, and $g = 2$ is the gyroscopic factor. The spin densities can be calculated [2] from the definitions

$$s_{u(d)} = \tfrac{1}{2}\left\langle \bar{\psi}\left(1 \pm \tau_3\right)\gamma_0 \Sigma_z \psi\right\rangle , \tag{16}$$

where $\Sigma_z = \frac{1}{2}\gamma_5\gamma_0\gamma^3$. The point now is that whereas for the normal phase both $s_u$ and $s_d$ vanish, for the pion condensed phase they do not, and, as a consequence we get nonzero magnetization (15). The appearance of this interesting feature can be traced back to the spectrum of Fig. 3. We populate valence (positive energy) levels up to some Fermi energy. There are two branches: $E^+$ and $E^-$. The easiest interpretation of these branches can be done in the nonrelativistic approximation [8]. Then the $E^-$ branch contains levels with the spin antiparallel to the isospin: $u_\downarrow$ and $d^\uparrow$, and the $E^+$ contains the states with the spin parallel to the isospin: $u^\uparrow$ and $d_\downarrow$. The $E^-$ branch is energetically favored, hence there are always more states in the system with antiparallel spin and isospin. As a result it gives the nonzero spin densities in Eq. (16).

Our results indicate that in centers of neutron stars there may exist cores (or shells) with pion-condensed quark matter, which possess a net magnetization. Such cores, presumably with some macroscopic domain structure, could contribute to the magnetic moment of the star, which in this case would be a pulsar.

## 4. Finite temperatures

At the mean-field level, the extension of our calculation to finite temperatures is straightforward. One has to thermalize the quark degrees of freedom only, because, as will be explained in the following, the $\sigma$ and $\pi$ should not be treated as independent dynamical degrees of freedom. One therefore should not include mesonic thermal excitations to the partition function. We obtain the following expression for the grand thermodynamical potential [16]:

$$\Omega\left(T, \mathcal{V}, \mu; m, \vec{q}\right)$$

$$= -T\mathcal{V} \int \frac{d^3p}{(2\pi)^3} \sum_{a=+,-} \left[\ln\left(1 + e^{(\mu-\varepsilon^a)/T}\right) + \ln\left(1 + e^{(\bar{\mu}-\bar{\varepsilon}^a)/T}\right)\right]$$

$$-\mathcal{V} p_{\sigma-\pi} , \qquad (17)$$

where $T$ is the temperature, $\mathcal{V}$ — the system's volume, $\mu$ — the quark chemical potential, $m$ and $\vec{q}$ — the collective parameters introduced earlier, and $\gamma = N_c N_f$. The quantities $\varepsilon^a$ ($a = +, -$) are the quark (positive energy) eigenvalues (11). The second term in the brackets comes from the antiquarks. We have introduced $\bar{\mu}$ as the antiquark chemical potential, and $\bar{\varepsilon}^a$ as the antiquark eigenvalues. Because the baryon number is conserved, $\bar{\mu} = -\mu$, and because the spectrum (10) is symmetric, $\bar{\varepsilon}^a = -\varepsilon^a$. The last term in expression (17) comes from the "Mexican Hat" potential in the Lagrangian (2). At the mean-field level, the mesonic contribution is [16]

$$p_{\sigma-\pi} = -V\left[\sigma, \vec{\pi}\right] . \tag{18}$$

The equilibrium conditions

$$\left.\frac{\partial \Omega}{\partial m}\right|_{T,\mathcal{V},\mu} = 0\,, \quad \left.\frac{\partial \Omega}{\partial \vec{q}}\right|_{T,\mathcal{V},\mu} = 0 \tag{19}$$

determine the optimum values of the collective parameters $m$ and $\vec{q}$. The thermodynamical quantities are obtained from the standard expressions,

$$p = -\Omega/\mathcal{V}\,,$$

$$\mathcal{E} = \gamma \int \frac{d^3p}{(2\pi)^3} \sum_{a=+,-} \left[\varepsilon^a n^a(\vec{p}) + \bar{\varepsilon}^a \bar{n}^a(\vec{p})\right],$$

$$\mathcal{N} = N_f \int \frac{d^3p}{(2\pi)^3} \sum_{a=+,-} \left[n^a(\vec{p}) - \bar{n}^a(\vec{p})\right]. \tag{20}$$

The quark and antiquark distribution functions are, of course

$$n^a(\vec{p}) = \left(1 + e^{(\varepsilon^a - \mu)/T}\right)^{-1}, \quad \bar{n}^a(\vec{p}) = \left(1 + e^{(\bar{\varepsilon}^a - \bar{\mu})/T}\right)^{-1}. \tag{21}$$

At present, we do not yet have the results of the full optimization (19). In order to speed up the numerical calculation, we have reduced the number of variational parameters from 2 to 1, by setting

$$m = m_0 = gF_\pi \tag{22}$$

in the massive quark phase. This situation would occur in the nonlinear $\sigma$-model, since the constraint (22) corresponds to the condition $\sigma^2 + \vec{\pi}^2 = F_\pi^2$. This simplification does not influence our qualitative conclusions, since the effect is determined by the dependence on $\vec{q}$.

For our presentation, most interesting are the phase diagrams. Fig. 8a shows the $\mu$-$T$ phase diagram for the calculation with $m_\sigma = 1200$ MeV and $m_0 = gF_\pi = 300$ MeV. We notice the existence of three distinct phases, as announced in the introduction in Fig. 2. The new phase, labeled "C", shows up at values of the quark chemical potential around 400 MeV, and at temperatures up to 100 MeV. Fig. 8b shows the baryon density-temperature phase diagram. Since the phase transition to the "R" phase is first-order, we also have regions with coexistence of phases, labeled "C/R" (condensed and restored phases) and "N/R" (normal and restored phases). For other values

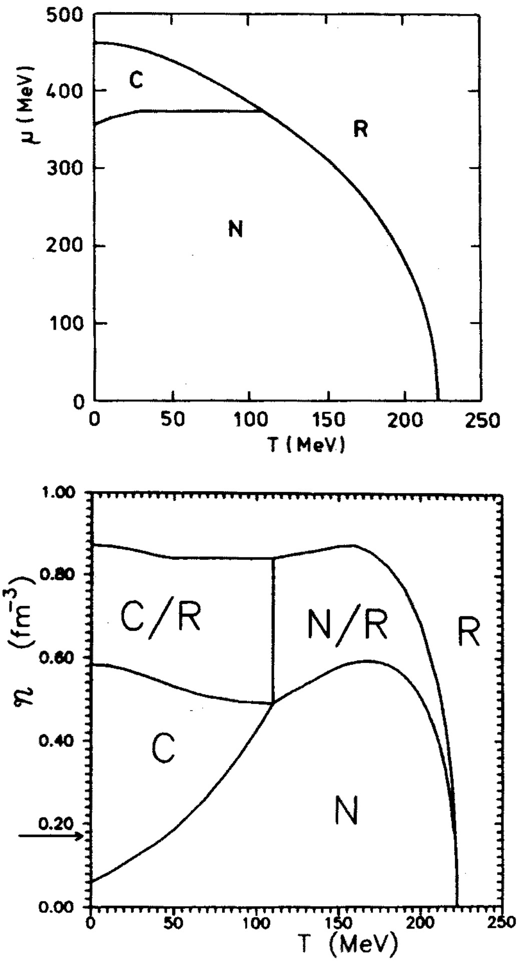



Fig. 8. Finite temperature results, obtained in a calculation with frozen $m$, $m = m_0 = 300$ MeV: (a) The $\mu$-$T$ diagram — the phase with the pion condensate, "C", appears up to temperatures around 100 MeV. (b) The corresponding baryon density-temperature diagram — because the transition to the restored phase is first-order, regions of coexistence of phases appear. The arrow shows the nuclear saturation density.

of $m_\sigma$ and $g$ the behaviour is qualitatively the same as for the parameters of Fig. 8.

The conclusion one has to draw at this point is that, at least in the framework of effective chiral models, one cannot make the assumption that the mean fields describing condensates are translationally invariant. Allowing for spatial dependence of the condensates lowers significantly the energy, as well as introduces interesting magnetic effects. The pion-condensed phase is present at densities of the order of several nuclear densities, at which one is tempted to make the usual assumption of transitional invariance.

## 5. Finite cutoffs in effective models

Now we want to make a few remarks to show that our results do not depend on the particular choice of the effective chiral model. The prototype chiral model is the Nambu–Jona-Lasinio (NJL) model [12]:

$$\mathcal{L}_{\mathrm{NJL}} = \bar{\psi} i \partial\!\!\!/ \psi + \tfrac{G}{2}\left[\left(\bar{\psi}\psi\right)^2 + \left(\bar{\psi} i\gamma_5 \tau^a \psi\right)^2\right] . \tag{23}$$

The model is traditionally treated at the one-loop level, and a cutoff is introduced to make calculations sensible. At this level, Lagrangian (23) is equivalent to the following Lagrangian:

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi + g\left[\sigma\bar{\psi}\psi + \pi^a \bar{\psi} i\gamma_5\tau^a\psi\right] - \tfrac{1}{2}\mu^2\left(\sigma^2 + \pi^a\pi^a\right) , \tag{24}$$

where $\sigma$ and $\pi$ do not have explicit kinetic pieces, and serve as nondynamical constraint fields. From (24) we get

$$\sigma = g/\mu^2 \langle\bar{\psi}\psi\rangle , \qquad \pi^a = g/\mu^2 \left\langle\bar{\psi} i\gamma_5\tau^a\psi\right\rangle , \tag{25}$$

where the brackets denote expectation values in a given physical state. Because of (25), the Dirac equation following from (23),

$$i\partial\!\!\!/\psi + G\left[\langle\bar{\psi}\psi\rangle\psi + \left\langle\bar{\psi} i\gamma_5\tau^a\psi\right\rangle i\gamma_5\tau^a\psi\right] = 0 \tag{26}$$

is identical to the Dirac equation resulting from (24), provided

$$G = g^2/\mu^2 . \tag{27}$$

The above is referred to as the partial bosonization of the NJL model [21]. Now, working up to the one-loop level in the quark fields, and substracting the vacuum contribution (*i.e.* for $\sigma = -F_\pi$ and $\pi^a = 0$) we get the effective action

$$S_{\mathrm{oneloop}} = \mathrm{Tr}_\Lambda \log\left[i\partial\!\!\!/ + g\left(\sigma + i\gamma_5\tau^a\pi^a\right)\right] - \mathrm{Tr}_\Lambda \log\left[i\partial\!\!\!/ - gF_\pi\right] - \tfrac{1}{2}\int d^4x\, \mu^2\left(\sigma^2 + \pi^a\pi^a\right) . \tag{28}$$

The subscript $\Lambda$ reminds of the presence of the cutoff. When expanded in the mesonic fields, the effective Lagrangian (28) contains the kinetic piece

$$\tfrac{1}{2} Z(\Lambda)\left[\left(\partial^\mu\sigma\right)^2 + \left(\partial^\mu\pi^a\right)^2\right] . \tag{29}$$

To have the canonical normalization of the $\sigma$ and $\pi$ fields, *i.e.* to reproduce the experimental value of $F_\pi = 93$ MeV, one has to choose $\Lambda$ in such a way that [15]

$$Z(\Lambda) = 1\,. \tag{30}$$

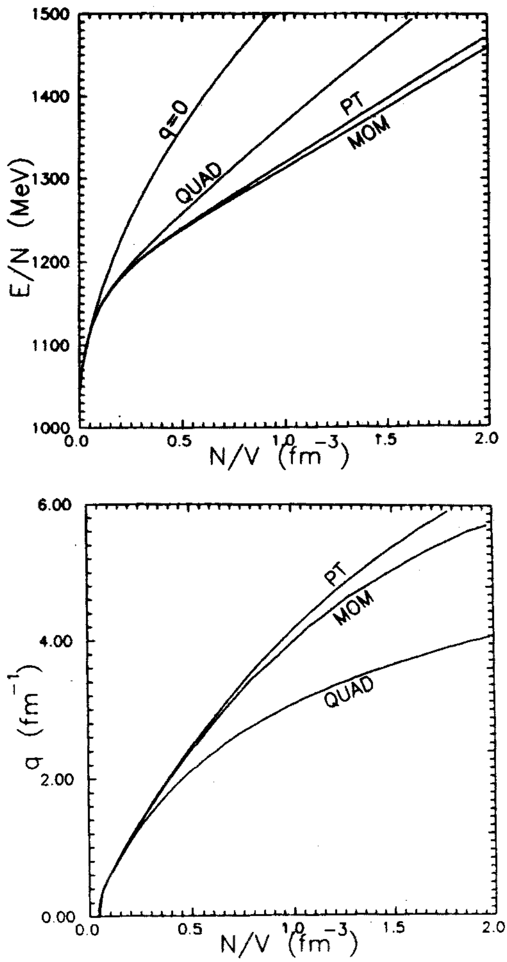



Fig. 9. Comparison of results of various chiral models. (a) The energy per baryon *vs* baryon density for the $\sigma$-model in the valence approximation (QUAD), and for the Nambu–Jona-Lasinio model with the proper time (PT) and four-momentum (MOM) regulators. (b) The dependence of $q$ on baryon density for the models of Fig. 9a.

We have repeated our calculation for the pion-condensed phase in the described above NJL model. The results (at zero temperature) are shown in Fig. 9 for the quark mass $m = gF_\pi = 345$ MeV, which is the preferred value of Dyakonov and Petrov [17]. In the NJL model various regularization methods can be applied, in principle leading to different results [22]. Fig. 9a shows the energy per baryon *vs* baryon density for the pion-condensed phase in the NJL model with the proper time regularization (curve labeled PT), and with the four-momentum cutoff (MOM). The curve labeled QUAD corresponds to the pion-condensed phase in the so-called quadratic approximation to the effective action (28), which is the assumption we made throughout the remaining part of this talk, and it just leads to the Lagrangian (2). The curve labeled "$q = 0$" is the normal phase, common to all above-mentioned models. What we can see from the figure is that in the NJL model with the proper time and four-momentum regulators the effect of the appearance of the new phase is even stronger — the equation of state is softer. The values of $q$ (see Fig. 9b) are larger (curves PT, MOM) than in the case of the $\sigma$-model (curve labeled QUAD).

Thus we have demonstrated th existence of the pion-condensed phase of quark matter does not depend on dynamical details of the model.

Concerning the effective chiral models with cutoffs, using the ansatz (6) to of us (WB and MK) have shown [22] that, as soon as one departs from the quadratic approximation to the fermion determinant in (28), the model predictions become sensitive to the particular choice of the regulator. Thus, there are ambiguities in effective chiral models with cutoffs. These ambiguities can only be resolved by invoking QCD and working harder to get a better effective theory.

## 6. One-loop physics

Now we shall briefly summarize some formal results described in detail by two of us (WB and MK) in Ref. [3]. As we have already mentioned, the great simplification in analyzing the system comes from the fact that the integrals in (12) and (13) are analytic. It allows us to get an analytic expression for the one-fermion-loop effective action in background fields (6). For the case of $m = gF_\pi$ we have found [3] the following result for the renormalized one-fermion loop contribution

$$\mathcal{E} = \tfrac{1}{2}F_\pi^2 q^2 - \frac{\gamma}{16\pi^2}\left\{\frac{q^4}{24} + \theta(q-2m)\left[m^2\left(m^2+q^2\right)\log\left(\frac{q+\sqrt{q^2-4m^2}}{2m}\right) - \left(\frac{13m^2}{12}+\frac{q^2}{24}\right)q\sqrt{q^2-4m^2}\right]\right\}. \quad (31)$$

The one-boson-loop contribution can also be evaluated in a closed form [3]. Results such as (31) are useful for two reasons. First, they demonstrate general features and difficulties of renormalized effective models, such as nonanalitycities [23] or vacuum instability [24]. Secondly, they allow us to test various approximate methods [25] used in the formalism of the effective action, such as derivative expansions. For our system subsequent terms in such approximations can be separately evaluated and convergence properties of various series can be studied in detail [3].

To our knowledge, the form of the chiral field in Eq. (6) is the only known spatially nonuniform case for which the one-loop contributions to the effective action can be evaluated in a closed form in a 3 + 1 dimensional model. For spatially uniform fields the task is, as is well known [26], extremely simple. On the other hand, for hedgehog solitons the one-loop contribution is achieved by tedious numerical calculations [15]. Our case shows all the features of the spatially nonuniform solution, but maintains simplicity.

## 7. Conclusion

What we have shown is that in chiral models at densities of the order of a few nuclear densities, and at temperatures up to about 100 MeV, there is a new phase of matter: gas of quarks with constituent-like masses, submerged in the background of the chiral field whose configuration has the form of the "pion condensate". The result seems to be very general and does not depend on dynamical detail, or model parameters. Of course, a valid question is whether effects not included can spoil our conclusion. In nuclear physics, the interest in the pion condensation was ended after it was realized that the short-range correlations weakened the pion-exchange force such that the effect was pushed to very high densities. In quark matter we are interested in high densities, and certainly correlation effects, such as *e.g.* the RPA correlation energy, should be studied. The most important question is, of course, whether the "C" phase exists in nature, and whether it can be shown not within effective models, but using QCD. We feel, however, that as long as chiral dynamics determines the property of the system, it reproduces all physical effects, and our result should be taken with this in mind.

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