SPIN FACTORS, BERRY PHASES AND WILSON LOOPS*

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We review the path integral formulation for relativistic spinning particles and show how the spin factor appears as a geometrical phase. We also explain how the three dimensional spin factor can be expressed as a Wilson loop in a Chern-Simons gauge theory.

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1. Introduction

In these lectures we review some recent work on the path integral formulation of quantum mechanics for spinning particles. We do not intend to give a complete picture of the status of this much studied problem[1] but will concentrate on the formulation in terms of so called spin factors. Very roughly speaking, the spin factor $\Phi[C]$ is a phase which depends on the path

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[1] Some relevant references will be given in Section 2.2.

$C = x(\tau)$ and when included in an ordinary path integral (*i.e.* sum over space-time histories C) for spinless particles turns it into a path integral for particles with spin:

$$A_{\text{no spin}} = \int\limits_C e^{-S[C]} a[C] \to A_{\text{spin}} = \int\limits_C e^{-S[C]} \Phi[C] a[C]. \tag{1}$$

Here A is some quantity of interest like the partition function, or the propagator, a is some function of the path C, and S the corresponding action.

We will approach the spin factor from various different directions. We start by a short recollection of the first quantized version of relativistic quantum theory in the path integral formulation. Although this is standard material, it is not included in most textbooks with the result that many students believe that relativistic quantum mechanics can only be formulated as a field theory. We give a simplified derivation of the relativistic path integral formula for the scalar propagator starting from the more familiar second quantized result. We then discuss how to incorporate spin and discuss two different ways to arrive at the spin factor.

The next main objective is to understand the spin factor as a geometrical phase. In line with the pedagogical aim of this review, we introduce the Berry phase, essentially following Berry's original paper, and then go on to discuss the spin factor. When the relation to the Berry phase is established, we can use known results to calculate an explicit expression in the three dimensional case. We then show that in four or higher dimensions, the spin factor is still a geometrical phase, but this time of the non–Abelian type.

The last section starts in a seemingly completely unrelated way, namely with a discussion of anyons and their relation to Chern–Simons gauge theory. The connection to the previous discussion is made when we establish that in three dimensions, the spin factor is the expectation value of a Wilson loop in a Chern–Simons theory. In connection with this we also briefly discuss the relation to Witten's topological field theory and the closely related issue of how to regularize the Wilson loops.

2. The spin factor

2.1 Path integrals for relativistic spinless particles

Consider spinless particles with mass m. From the standard theory of the Klein–Gordon field, we have the Euclidean space propagator,

$$\Delta(x, y) = \langle x| \frac{1}{p^2 + m^2} |y\rangle, \tag{2}$$

where we used a bra-ket notation for the Fourier transform. Since we can think of the integrand in (2) as the imaginary time propagator for a non relativistic particle with mass 1/2, we can immediately write

$$\Delta(x,y) = \int_0^\infty dt \int_{x(0)=y}^{x(1)=x} \mathcal{D}(x(s))\, e^{-S[x]},$$

$$S[x] = \int_0^t ds \, \left[\tfrac{1}{4}\dot{x}^2 + m^2\right]. \tag{3}$$

As written, S depends on the parametrization of the path, but it can be turned into a reparametrization invariant form by introducing an "einbein" field e as follows,

$$ds = \tfrac{1}{2} e(u)\, du, \qquad \tfrac{1}{2}\int_0^1 e(u)\, du = 1. \tag{4}$$

It can be shown [1] that integrating over t is the same as integrating over gauge inequivalent $e(u)$. Thus, modulo the (infinite) gauge volume of the reparametrization group,

$$\Delta(x,y) = \int \mathcal{D}(e) \int_{x(0)=y}^{x(1)=x} \mathcal{D}(x(s))\, e^{-S_{\rm cl}[x]},$$

$$S_{\rm cl}[x] = \tfrac{1}{2}\int_0^1 du \, \left(\dot{x}^2 e^{-1} + e m^2\right). \tag{5}$$

We might have guessed this, since $S_{\rm cl}[x]$ is simply a reparametrization invariant form of the usual classical (euclidian) action for a relativistic particle,

$$\widetilde{S}_{\rm cl}[x] = m \int_0^1 du \sqrt{\dot{x}^2}, \tag{6}$$

which is nothing but the mass times the length of the space time path. Classically $S_{\rm cl}$ and $\widetilde{S}_{\rm cl}$ are equivalent, as can easily be seen by eliminating the auxiliary field e by the equation of motion. It is shown in [1] how to use (6) in the path integral. Both forms will appear in subsequent Sections.

So far we discussed the propagator, but a similar expression can be derived for the partition function $Z = \det^{-1}(p^2 + m^2)$. The partition function gets contributions from an arbitrary number of particles but for a non-interacting theory this exponentiates so the free energy $F = -\ln Z$ is simply the contribution from a single particle. The corresponding path integral expression is,

$$F = \mathrm{Tr}\ln(p^2 + m^2) = \oint \tilde{\mathcal{D}}(x(s)) e^{-m\int_0^\infty ds\, \sqrt{\dot{x}^2}}, \tag{7}$$

where the integral is over closed paths, and where we used the form (6) for the classical action.

2.2 Path integrals for spinning particles

Ever since Feynman formulated quantum mechanics in path integral language [2] it has been a challenge to incorporate spin in this description, preferably in an intuitively appealing way. Several methods have been used to achieve this goal, amongst them coherent-states quantization [3–8], the use of anticommuting variables and local supersymmetry [9–12], geometric quantization [13], phase space path integrals [14] and the introduction of terms depending on the extrinsic geometry of the world-line [15]. In addition, recent work by Polyakov [16] on spin in connection with Bose–Fermi transmutation in 2+1 dimensions has inspired comments and generalizations [17–21]. We shall not attempt even to briefly review this large body of work, but concentrate on the spin factor formulation referred to above. The spin factor itself can be derived in several different ways. One method is to rewrite the partition function for a Dirac field as a path integral in the same way as we did above for the Klein–Gordon case [22,23]. The expression corresponding to (3) for the free energy reads,

$$F = \int_0^\infty \frac{dt}{t} \oint \mathcal{D}(x)\mathcal{D}(p) e^{i\int_0^1 ds\, (\dot{x}p - ip_\mu\gamma^\mu - im)}, \tag{8}$$

where again the integral is over closed loops. This expression is more difficult to handle than (3). First, it is a matrix expression, and second the integral is not Gaussian[2]. Strominger carefully regularized the path integral by discretizing the proper time variable in N steps. Following his treatment one obtains,

$$F = \oint \tilde{\mathcal{D}}(x(s))\, \Phi[C] e^{-m\int_0^\infty ds\, \sqrt{\dot{x}^2}}, \tag{9}$$

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[2] There is no p^2 term, and a formal integration gives the nonsensical result $\dot{x}_\mu = \gamma_\mu$.

where the **spin factor** is given by

$$\Phi[C] = \lim_{\Delta s \to 0} \mathrm{Tr} \prod_{k=1}^{N} \tfrac{1}{2}(1 + \gamma \cdot \dot{x}). \tag{10}$$

Here $\Delta s = T/N$, where T is the proper time of the path. In this connection we should also refer the reader to the interesting recent work by Korchemsky [24,25] where this derivation is worked out in detail together with a discussion of many of the mathematical properties of the spin factor in different dimensions.

We now discuss a second way to derive the spin factor which is due to Polyakov [1,26]. The idea is to start from a **classical** action for a spinning particle. Such actions differ from the usual bosonic ones, in that they include anticommuting numbers (*i.e.* Grassman variables). Without any further comments we now write down the path integral expression for the partition function for a spin half particle,

$$W = \int_P \mathcal{D}x^\mu(s)\mathcal{D}e(s) \int_{AP} \mathcal{D}\psi^\mu(s)\mathcal{D}\phi(s)\mathcal{D}\chi(s)\, \mathrm{e}^{-S}, \tag{11}$$

where

$$S = \tfrac{1}{2} \int_0^T ds\, (\dot{x}^2 e^{-1} - em^2 - \psi\dot{\psi} - \phi\dot{\phi} + 2\chi\dot{x}\psi + 2me\chi\phi). \tag{12}$$

The integration in (11) is over periodic functions $x^\mu(s)$, $e(s)$ and anticommuting antiperiodic functions $\phi(s)$, $\chi(s)$ and $\psi^\mu(s)$. Here $\mu = 0, ..., D-1$ where D is the dimension of the (Euclidean) spacetime and we have suppressed contracted indices.

The action (12) and its (super)symmetries have been extensively discussed in the literature [9–12] and a short discussion relevant for the present calculation can be found in [27].

Imposing the ghost-free gauge condition $\phi = 0$ the fermionic part of the path integral can be calculated:

$$\begin{aligned} \Phi &= \int_{AP} \mathcal{D}\psi^\mu \mathcal{D}\chi \exp\left(\tfrac{1}{2} \int_0^T ds\, (\psi\dot{\psi} - 2\chi\dot{x}\psi) \right) \\ &= \widetilde{\mathrm{Tr}} \int_{AP} \mathcal{D}\chi \,\mathrm{P}\exp\left(-i \int_0^T ds\, \chi \frac{v}{\sqrt{2}} \mathrm{e}^\mu \gamma_\mu \right), \end{aligned} \tag{13}$$

where P denotes path ordering and we have rewritten $\dot{x}^\mu$ as

$$\dot{x}^\mu = ve^\mu, \quad e^\mu e_\mu = 1. \tag{14}$$

In the last equality in (13) we have translated the path integral into operator language using the rules of canonical quantization of fermionic systems [12]. For convenience we have introduced a normalized trace $\widetilde{\mathrm{Tr}}$ such that $\widetilde{\mathrm{Tr}}\,1 = 1$. The details of this calculation can be found in [27].

Next we use time-slicing to perform the χ-integral. Apart from a factor that modifies the bosonic part of the measure in F, the result is

$$\Phi = \lim_{\Delta s \to 0} \widetilde{\mathrm{Tr}} \prod_{k=1}^{N} e_k^\mu \gamma_\mu, \tag{15}$$

where again $\Delta s = T/N$ and $e_k^\mu = e^\mu(T - k\Delta s)$.

Using the identity

$$e_k^\mu \gamma_\mu e_{k+1}^\nu \gamma_\nu = 1 - \frac{2\Delta s}{4} e_k^\mu \dot{e}_k^\nu (\gamma_\mu, \gamma_\nu) + O((\Delta s)^2) \tag{16}$$

we can rewrite the spin factor on the form given by Polyakov, for a closed curve [26,1]:

$$\Phi = \widetilde{\mathrm{Tr}}\mathrm{P}\, \exp\left(-\tfrac{1}{2} \int_0^T ds\, e^\mu \dot{e}^\nu \sigma_{\mu\nu} \right), \tag{17}$$

where $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$. The spin factor for an open curve is the matrix obtained by removing $\widetilde{\mathrm{Tr}}$.

Note that the expression (10) looks quite different from (15), but it can be shown that it gives the same final expression (17).

The spin factor in (17) is very natural from a geometric point of view. The matrix of which the trace is taken in (17) gives the spinor version of the Fermi-Walker (FW) transport. The FW transport of a vector along a curve keeps the tangential component of the vector constant and makes no rotation of the vector in planes orthogonal to the tangent. This means that the angular velocity tensor of the transport is $e^\mu \dot{e}^\nu - e^\nu \dot{e}^\mu$ (where e^μ is the unit tangent vector to the curve) and our statement about the role played by the matrix in the spin factor follows. From the interpretation of (17) as the trace of an $SO(D)$ rotation operator in a spinor representation it is clear that Φ is real. This can also be proved by using properties of charge conjugation matrices.

Equation (17) is not quite correct as it stands. It is well known from quantum field theory that a minus sign goes with each fermion loop, and a clear discussion of why unitarity makes this necessary has been given by Feynman [28]. Basically it is the sign associated with a rotation by 2π of a spin $\frac{1}{2}$ particle that has to be compensated for. The 2π-rotation sign is present in the spin factor, the compensating statistics sign we have to insert by hand. This is of course irrelevant in single particle problems, but when comparison is made with spin factors deduced from results derived by many particle methods it must be included.

The evaluation of the covariant spin factor (17) is difficult for a general curve since it involves a path-ordered exponential of non-commuting operators. In two dimensions, however, there are no ordering problems, and (17) is easily evaluated, and one finds

$$\Phi_2[C] = -\cos\left(\tfrac{1}{2}\int_0^T ds\,\omega\right) = (-1)^n, \tag{18}$$

where n is the number of self intersections of the curve C, and where we included the statistics sign referred to above. It is possible to relate this result to the 2–dimensional Ising model on a square lattice [1,26]. It is well known (see, *e.g.* [29] that its partition function can be written

$$Z = \exp\left(\sum_C k^{L(C)}(-1)^{n(C)}\right), \tag{19}$$

where k is a constant, C denotes closed loops on the lattice with length $L(C)$ and $n(C)$ self intersections. In the continuum limit this looks like the partition function for a free field theory with first-quantized transition amplitudes $\sim \sum_C \exp[-mL(C)](-1)^{n(C)}$ for the excitations (*c.f.* the discussion at the end of Sect. 2.1 above). This corresponds to a relativistic theory with spin factor $(-1)^n$; *i.e.* a theory of relativistic 2–dimensional spin $\frac{1}{2}$ particles. That the Ising model is closely related to spin $\frac{1}{2}$ particles in two dimensions is known on other grounds as well.

3. $\Phi[C]$ as a Berry phase, and the Thomas connection

3.1 The Berry phase

We start with a brief reminder of what a Berry phase is [30]. Assume that we have a Hamiltonian $\mathcal{H}(R_i(t))$ which depends on some external parameters $R_i(t)$. We can now define a complete set of *instantaneous eigenstates* $|n(R_i(t))\rangle$ by

$$\mathcal{H}|n(R_i(t))\rangle = E_n(t)|n(R_i(t))\rangle, \tag{20}$$

and we shall for the time being assume these states to be nondegenerate. We also assume that they are continuously defined around any closed curve in the parameter space. If the time dependence is slow, the adiabatic theorem tells us that if a system is in the state $|\Psi_n(0)\rangle = |n(R_i(0))\rangle$ at $t = 0$, then it will evolve into the state

$$|\Psi_n(t)\rangle = e^{i\alpha}|n(R_i(t))\rangle, \tag{21}$$

under the action of the **time dependent** H. Note that (20) define the instantaneous eigenstates only up to phases that can be picked arbitrarily at any given time, so the phase α in (21) is purely conventional. Berry's fundamental observation was that although this is true in general, if we consider adiabatic evolution where the parameters R_i return to their original values, the phase α is uniquely determined and has a deep geometrical meaning. To demonstrate this we make the *ansatz*

$$|\Psi_n(t)\rangle = e^{i\gamma_n(t)} e^{-i\int_0^t dt' E_n(t')} |n(R_i(t))\rangle, \tag{22}$$

where we have separated out the **dynamical phase** $e^{-i\int_0^t dt' E_n(t')}$. Direct substitution in the time dependent Schrödinger equation gives

$$\dot{\gamma}_n = i\dot{\vec{R}} \cdot \vec{A}_n, \tag{23}$$

where

$$\vec{A}_n(\vec{R}) = i\langle n(\vec{R})|\vec{\nabla}_{\vec{R}}|n(\vec{R})\rangle\,. \tag{24}$$

For $\vec{A}_n$ to be well defined it is important that the states were assumed to be continuous in the parameters $\vec{R}$. Integrating (23) around a closed curve in parameter space yields

$$\gamma_n^B = \oint d\vec{R} \cdot \vec{A}_n. \tag{25}$$

It is now clear that the **Berry phase factor** $e^{i\gamma_n^B}$ does not depend on how we choose the phases in (20), since a change in phase

$$|n(R_i(t))\rangle \to e^{i\alpha(t)}|n(R_i(t))\rangle \tag{26}$$

corresponds to the gauge transformation

$$\vec{A}_n \to \vec{A}_n - \vec{\nabla}_{\vec{R}}\, \alpha(\vec{R}) \tag{27}$$

and the Berry phase factor, being a Wilson loop of the gauge potential (24), is gauge invariant.

As a concrete example, which will be important in the following, we consider a spin in a time dependent magnetic field of constant magnitude B, *i.e.* we study the Hamiltonian

$$H = -B\,\hat{e}(t)\cdot\vec{\sigma}, \tag{28}$$

where $\hat{e}(t)$ is a time dependent unit vector, and $\vec{\sigma}$ the Pauli matrices. If we define $\hat{e}\cdot\vec{\sigma}|\hat{e};\pm\rangle = \pm|\hat{e};\pm\rangle$, then for $\vec{e} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $\gamma^1 = \sigma^1$, $\gamma^2 = \sigma^2$ the + states are

$$|\hat{e};+\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}, \tag{29}$$

and the vector potential for the + states is given by $A_\theta = 0$ and $A_\phi = -\sin^2\theta/2$, *i.e.* the vector potential of a magnetic monopole. The Berry phase is now easily calculated

$$\gamma_+(C) = \int_0^{2\pi} d\phi A_\phi = -\int_0^{2\pi} d\phi \sin^2\theta/2 = -\tfrac{1}{2}\Omega, \tag{30}$$

where Ω is the solid angle swept out by the unit vector $\hat{e}$ during a complete cycle.

3.2 The spin factor in three dimensions

We are now ready to analyze the spin factor in three (Euclidian) dimensions. Define the Hamiltonian,

$$i\mathcal{H}(s) \equiv \tfrac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu} = \frac{i}{2}(\hat{e}\times\dot{\hat{e}})\cdot\vec{\sigma} = \frac{i}{2}\vec{\omega}\cdot\vec{\sigma}, \tag{31}$$

where $\hat{e} = \dot{\vec{x}}$ (and $\dot{\vec{x}}^2 = 1$), and note that $\Phi[C]$ in (17) is just the "time" evolution operator for a spin half in the magnetic field $\vec{\omega}$. (Our "time" is the length along the curve.) We now introduce a continuous set of normalized eigenstates to the spin along the tangent of the curve: $\hat{e}(s)\cdot\vec{\sigma}|\hat{e}(s),\pm\rangle = \pm|\hat{e}(s),\pm\rangle$. By writing

$$\exp(-i\Delta s\mathcal{H}(s)) = \tfrac{1}{2}\left[1 + (\hat{e}(s+\Delta s)\cdot\vec{\sigma})(\hat{e}(s)\cdot\vec{\sigma})\right] + O(\Delta s^2) \tag{32}$$

it is easy to see that the operator $\exp[-i\Delta s\mathcal{H}(s)]$ does nothing but evolve $|\hat{e}(s),\pm\rangle$, the eigenstate of $\hat{e}(s)\cdot\vec{\sigma}$, into the state $|\hat{e}(s+\Delta s),\pm\rangle$, which is an eigenstate of $\hat{e}(s+\Delta s)\cdot\vec{\sigma}$. Geometrically this follows from (31) by noting that $\vec{\omega}$ is the angular velocity of $\hat{e}$. As already mentioned, this is the quantum mechanical counterpart of Fermi–Walker transport. Thus we have the time evolution

$$|\Psi(s),\pm\rangle = e^{i\gamma_\pm(s)}|\hat{e}(s),\pm\rangle. \tag{33}$$

Note that (33) is exact; no adiabatic assumption is involved. This calculation differs from conventional calculations of Berry phases in that the states $|\hat{e}(s),\pm\rangle$ that evolve into each other are not eigenstates of the Hamiltonian. In fact, $\langle\hat{e}(s),\pm|\mathcal{H}(s)|\hat{e}(s),\pm\rangle = 0$, which follows from (31) and $\vec{\omega}\cdot\hat{e}=0$, so the dynamical phase is identically zero and we will be left with a geometrical phase, characteristic of the curve only. (A general discussion of systems exhibiting such "pure" Berry phases has been given by Aharonov and Anandan [31].) To see this we follow the derivation of the Berry phase, *i.e.* we substitute in the Schrödinger equation and integrate around the curve to get

$$\gamma_+ = i\oint_C ds\langle\hat{e}(s),+|\frac{d}{ds}|\hat{e}(s),+\rangle = \oint_C d\vec{x}\cdot\vec{A}, \tag{34}$$

where $\vec{A}(s) = i\langle\hat{e}(s),+|\vec{\nabla}|\hat{e}(s),+\rangle$. As shown in the previous subsection, $\gamma_\pm = \mp\Omega/2$ where Ω is the solid angle subtended by $\hat{e}$ [30], so the final expression for the spin factor becomes,

$$\Phi_3[C] = -\mathrm{Re}\exp\left(i\oint_C d\vec{x}\cdot\vec{A}\right) = -\mathrm{Re}\, e^{-\frac{1}{2}i\Omega}, \tag{35}$$

where Re denotes real part, and where we again included the statistics sign.

3.3 $\Phi[C]$ as non-abelian Berry phases, and Thomas precession

As we already saw, the FW transport of spinors, defined by the matrix in (32), has the property that eigenvectors of $e(0)\cdot\gamma$ are transported into eigenvectors of $e(s)\cdot\gamma$ with the same eigenvalue. This is true even when the eigenvalue is degenerate as is the case in dimensions higher than three, (as can be easily seen from the explicit expressions for the gamma matrices).

Thus if we consider the degenerate eigenstates $|e(0);\alpha\rangle$ of $e(0)\cdot\gamma$ with eigenvalue 1, we have

$$\mathrm{P}\exp\{-\int_0^s ds\, e^\mu \dot{e}^\nu \tfrac{1}{4}[\gamma_\mu,\gamma_\nu]\}|e(0);\alpha\rangle = |e(s);\beta\rangle U_{\beta\alpha}(s), \tag{36}$$

where $\{|e(s);\alpha\rangle\}$ is a complete set of orthonormal eigenstates of $e(s)\cdot\gamma$ with eigenvalue 1 that are continuously defined around the curve and the index β which labels the degenerate eigenstates is summed over. In differential form

$$-\tfrac{1}{4}[e\cdot\gamma,\dot{e}\cdot\gamma]|e(s);\beta\rangle U_{\beta\alpha} = \frac{d}{ds}|e(s);\beta\rangle U_{\beta\alpha} + |e(s);\beta\rangle\frac{d}{ds}U_{\beta\alpha}. \qquad (37)$$

Since $\langle e(s);\alpha|[e\cdot\gamma,\dot{e}\cdot\gamma]|e(s);\beta\rangle = 0$ we conclude

$$\frac{d}{ds}U_{\alpha\beta} = -\langle e(s);\alpha|\frac{d}{ds}|e(s);\gamma\rangle U_{\gamma\beta} = -\dot{e}^{\mu}\langle e(s);\alpha|\frac{\partial}{\partial e^{\mu}}|e(s);\gamma\rangle U_{\gamma\beta}. \qquad (38)$$

Solving this equation we get the contribution of eigenstates of $e\cdot\gamma$ with eigenvalue 1 to the spin factor for a closed curve

$$\Phi_{+} \propto \sum_{\alpha} U_{\alpha\alpha}(T) = \mathrm{Tr}\,\mathrm{P}\exp[i\int_{0}^{T} ds\,\dot{e}^{\mu}\mathbf{A}_{\mu}], \qquad (39)$$

with the **non–Abelian** vector potential $\mathbf{A}_{\mu}$ given by

$$A_{\mu}^{\alpha\beta} = i\langle e(s);\alpha|\frac{\partial}{\partial e^{\mu}}|e(s);\beta\rangle. \qquad (40)$$

That the Berry phase factor turns into a non–Abelian Wilson loop in the case of adiabatic evolution of a degenerate subspace of the Hilbert space, was first discussed by Wilczek and Zee [32]. The arbitrariness involved in defining $\{|e(s);\alpha\rangle\}$ again corresponds to gauge transformations of this vector potential. States with eigenvalue -1 give a similar contribution, Φ_{-}, and $\Phi = \Phi_{+} + \Phi_{-}$.

That "boosts" around a curve give rise to a spatial rotation is familiar from the phenomenon of Thomas precession (see, *e.g.* [33]), and under slightly restricted circumstances we can use the vector potential formulation above to make the relation between Thomas precession and the spin factor explicit. A natural set of eigenstates $\{|e(s);\alpha\rangle\}$ is obtained by picking a reference "time" axis and "boosting" eigenstates $\{|\alpha\rangle\}$ of γ^0 with eigenvalue 1 in this direction to eigenstates of $e(s)\cdot\gamma$:

$$|e(s);\alpha\rangle = U(e)|\alpha\rangle. \qquad (41)$$

This is a continuous set provided the curve is nowhere tangent to the "time" axis, *i.e.* , $e\cdot\gamma \neq \pm\gamma^0$ always. With this choice of basis, the vector potential must belong to the Lie algebra of $SO(D)$ and thus takes the form

$$\begin{aligned}\dot{e}^{\mu}A_{\mu}^{\alpha\beta} &= i\dot{e}^{\mu}\langle\alpha|U(e)^{\dagger}\frac{\partial}{\partial e^{\mu}}U(e)|\beta\rangle \\ &= i\dot{e}^{\mu}\langle\alpha|\left(\tfrac{1}{4}\omega_{\mu}^{0k}[\gamma_0,\gamma_k] + \tfrac{1}{8}\omega_{\mu}^{kl}[\gamma_k,\gamma_l]\right)|\beta\rangle \\ &= \tfrac{1}{8}i\dot{e}^{\mu}\omega_{\mu}^{kl}\langle\alpha|[\gamma_k,\gamma_l]|\beta\rangle. \qquad (42)\end{aligned}$$

$\omega_T^{kl} \equiv \dot{e}^\mu \omega_\mu^{kl}$, $k, l = 1, \ldots, D-1$ is the angular velocity tensor for "spatial" rotation (rotations in planes orthogonal to our chosen "time" axis) which arises because the "boosts" in different directions do not commute, *i.e.* , the angular velocity tensor of the Thomas precession [33]. With $\{|\alpha\rangle\}$ replaced by eigenstates of γ^0 with eigenvalue -1, the same calculation gives the vector potential for the contribution of eigenstates of $e(s) \cdot \gamma$ with eigenvalue -1 to Φ. Since $[\gamma^0, [\gamma^k, \gamma^l]] = 0$ we get as the final expression for Φ

$$\Phi = \widetilde{\mathrm{Tr}}\, \mathrm{P} \exp\left(-\int_0^T ds\, \tfrac{1}{8}\omega_T^{kl}[\gamma_k, \gamma_l]\right). \tag{43}$$

4. Spin factors from Chern-Simons gauge theory

4.1 From Anyons to C-S gauge theory

In this section we shall concentrate on the three dimensional case where there is a remarkable field theoretic expression for the spin factor as a Wilson loop in a Chern-Simons gauge theory. In order to understand the deep connections between spin, statistics and gauge interactions in three dimensions, we should recall some facts about anyons.

The simplest way to understand what anyons are, is to consider the following Hamiltonian for two identical particles of mass m,

$$\mathcal{H} = \frac{\vec{P}^2}{4m} + \frac{(\vec{p} - \vec{a})^2}{m}, \tag{44}$$

where $\vec{P}$ and $\vec{p}$ are the total and relative momenta respectively, $\vec{r}$ the relative distance, and

$$a^i = \frac{\theta}{\pi}\varepsilon^{ij}\frac{r_j}{r^2}. \tag{45}$$

To understand the significance of this vector potential, we calculate the corresponding magnetic field,

$$b = -\varepsilon^{ij}\partial_i a_j = 2\theta\delta^2(\vec{r}). \tag{46}$$

This means that $\vec{a}$ is a pure gauge everywhere except at $\vec{r} = 0$, where there is a pointlike flux. Classically, the particles are free, but in a quantum mechanical treatment the (long range) vector potential gives rise to a phase exactly like in the Aharonov–Bohm (AB) effect [34]. Also in analogy with the AB effect, we can trade the vector potential for a phase in the wave function, by performing the gauge transformation,

$$\vec{a} \rightarrow \vec{a} + \vec{\nabla}\frac{\theta}{\pi}\phi,$$
$$\psi \rightarrow \exp\left(i\frac{\theta}{\pi}\phi\right)\psi, \tag{47}$$

where ϕ is the polar angle of the relative coordinate $\vec{r}$. Note that this gauge transformation is singular and the resulting wave function is multivalued. Exchange of the two particles, corresponds to a π rotation of $\vec{r}$, and according to (47), the wavefunction picks up a phase θ. In the case of fermions the wave function changes sign under exchange of two identical particles. Thus, if we take $\theta = \pi$, and let the Hamiltonian (44) act on bosonic (*i.e.* symmetric) wave functions, it can either be thought of as describing interacting bosons, or free fermions, the necessary sign change under exchange arising as an AB phase[3]. Clearly the system is periodic in θ with period 2π so all even multiples of π correspond to bosons and all odd ones to fermions. We now have the possibility of taking θ not being a multiple of π — such "fractional statistics" particles are called anyons.

It is easy to generalize the above Hamiltonian to an N particles system. The resulting picture is that of a collection of thin "solenoids" interacting *via* a long range gauge potential. A couple of points are worth mentioning. There is no dynamics in the gauge field (since it is simply a function of the positions and momenta of the particles) and consequently no photons. Also the interaction (44) is not that of charge — flux tube composites — there is a long range static vector potential, but no electrostatic potential [36][4]. Another interesting property of anyons is that they carry fractional spin, $\theta/2\pi$, and hence there is a generalized spin statistics connection. For a general discussion of anyons [38] we refer to the review paper [39].

Now we give another description of the many anyon system based on a coherent state path integral formulation[40]. We start from the second quantized form of the many body version of the Hamiltonian (44)

$$\mathcal{H} = \int d^2r\,\phi^*(\vec{r})\left(-\frac{1}{2m}\left(-i\vec{\nabla} - \vec{a}(\vec{r})\right)^2\right)\phi(\vec{r}), \tag{48}$$

where

$$a^i(\vec{r}) = \frac{\theta}{\pi e}\varepsilon^{ij}\int d^2r'\,\frac{r_j - r'_j}{|\vec{r} - \vec{r}'|^2}\phi^*(\vec{r}')\phi(\vec{r}') \tag{49}$$

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[3] For a discussion of the relation between the phase associated with a real rotation and the one coming from a permutation in the wave function we refer to [35].

[4] For a discussion of charge — flux tube composites, or cyons, see [37].

and where we have set $\hbar = c = 1$. This Hamiltonian describes a system of identical particles with mass m which are created by the complex field operator ϕ. The particles interact *via* a two-dimensional "statistical" gauge potential $\vec{a}$. If we take the ϕ field to be bosonic, the above Hamiltonian describes "anyons" obeying θ statistics.

From (49) we immediately get the following expression for the "statistical" gauge field b

$$b(\vec{r}) = -\varepsilon^{ij}\partial_i a_j(\vec{r}) = \frac{2\theta}{e}\,|\phi(\vec{r})|^2 \equiv s|\phi(\vec{r})|^2 \left(\frac{2\pi}{e}\right), \tag{50}$$

which corresponds to associating $\theta/\pi = s$ units of flux to each particle.

We incorporate the constraint (50) by means of a Lagrange multiplier field a_0, to get the following coherent state path integral representation for the partition function

$$Z[A^\mu] = \int \mathcal{D}(\phi)\mathcal{D}(a_i^{\mathrm{T}})\mathcal{D}(a_0)\, \exp\left(iS[\phi, a_i^{\mathrm{T}}, a_0]\right), \tag{51}$$

where a_i^{T} is a transverse gauge field (*i.e.* satisfying $\partial^i a_i^{\mathrm{T}} = 0$), and $S = \int dt d\vec{r}\, \mathcal{L}$ with

$$\mathcal{L} = i\phi^*\partial_0\phi - \mathcal{H}(\phi) + \mu\phi^*\phi - a_0\Big(\frac{e^2}{2\theta}\varepsilon^{ij}\partial_i a_j^T + e\phi^*\phi\Big). \tag{52}$$

The term $\sim \varepsilon^{ij}$ in this expression is nothing but the Chern–Simons action in radiation gauge [41–43]. The covariant form of the Chern–Simons term (see Sect. 4.2 below) can be obtained by reintroducing the (infinite) gauge volume (*i.e.* by reversing the usual Faddeev–Popov gauge fixing procedure [44]).

4.2 The spin factor as a Wilson loop: Writhe, Twist & Linking.

We shall now make the promised connection between the spin factor $\Phi_3[C]$ in (17) and the Wilson loop in a Chern–Simons gauge theory [16]. For this we consider the non-relativistic version of the model (52)

$$S = \int d^3x\, \left(|(\partial_\mu + iA_\mu)\varphi|^2 - m^2|\varphi|^2\right) + S_A\,, \tag{53}$$

where $S_A = \frac{1}{4\theta}\int d^3x\, \epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho$ is the covariant Chern–Simons action. As discussed above, this model describes anyons. This has also been established by other methods [45, 46]. In particular, for $\theta = \pi$ the excitations are spin $\frac{1}{2}$ fermions. Using the identity [47]

$$Z[A] = \int \mathcal{D}\varphi\mathcal{D}\varphi^* \, e^{iS}$$
$$= \sum_{N=0}^{\infty} \frac{1}{N!} \int \prod_{i=1}^{N} \mathcal{D}x_i \exp\left(i \sum_{i=1}^{N} \int ds_i \left(m\sqrt{\dot{x}_i^2(s_i)}\right.\right.$$
$$\left.\left. + \dot{x}_i^{\mu} A_{\mu}(x_i)\right) + iS_A \right), \tag{54}$$

(with S given by (53)), and integrating out the gauge field we get

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \int \prod_{i=1}^{N} \mathcal{D}x_i \exp\left(i \sum_{i=1}^{N} \int ds_i \, m\sqrt{\dot{x}_i^2} \right)$$
$$\times \exp\left(i\theta \sum_{i=1}^{N} W[C_i] + 2i\theta \sum_{i<j} L[C_i, C_j] \right), \tag{55}$$

where

$$W[C] = \frac{1}{4\pi} \int_C dx^{\mu} \int_C dy^{\nu} \frac{x^{\sigma} - y^{\sigma}}{|x-y|^3} \epsilon_{\mu\nu\sigma} \tag{56}$$

$$L[C_i, C_j] = \frac{1}{4\pi} \int_{C_i} dx^{\mu} \int_{C_j} dy^{\nu} \frac{x^{\sigma} - y^{\sigma}}{|x-y|^3} \epsilon_{\mu\nu\sigma}, \quad i \neq j \,. \tag{57}$$

W is the so-called writhe of the closed curve C and $L[C_i, C_j]$ is an integer, the Gauss linking number for the curves C_i and C_j [48]. L is a topological invariant whereas W is not. The integral in (56) is well-defined; however, it is not a smooth limit of the integral in (57). The Wilson loop in (54) is ambiguous; by using the gauge $\partial_{\mu} A^{\mu} = 0$ one obtains (55,56). This implies a physically sensible definition of (54), as will be discussed in Sect. 4.3 below [49]. For $\theta = \pi$, corresponding to Bose-Fermi transmutation, L disappears from (28) and we can identify $\exp(i\pi W[C])$ as the spin factor for a closed loop. We showed in [17] that W is related to Ω by

$$W = \frac{\Omega}{2\pi} + k \,, \tag{58}$$

where k is an odd integer (see below). Thus the spin factor deduced from this D=3 model does coincide with the one obtained for D=3 spin $\frac{1}{2}$ particles, a result that lends support to the idea that the excitations in this model are spin $\frac{1}{2}$ fermions. Here we should also note that the analysis from (53) and onwards can also be made for $\theta \neq \pi$. In this case (53) is known to describe anyons, at least in the non-relativistic limit.

The concept of writhe of a closed curve is perhaps not a familiar one. We give here a brief discussion of some of its properties.

The writhe of a closed curve is closely related to the Gauss linking number. If a framing of the closed curve is introduced one has

$$W = L - T\,, \tag{59}$$

where L is the linking number of the original curve and the curve at the tip of the frame vector. T is the twist, *i.e.* the integrated angular rotation of the frame vector about the tangent vector divided by 2π [48]. The twist is related to the solid angle

$$\Omega = -2\pi T \bmod (2\pi)\,. \tag{60}$$

For a squashed knot Ω is a multiple of 2π and W is an integer. Ω and W vary smoothly when the curve is deformed continuously. When the curve intersects itself W changes by 2. For a circle $W = -1$. (58) then follows from (59) and (60). (58) and (59) can be used to evaluate W for a curve. In particular, (58) shows that the non-integer part of W is equal to the non-integer part of $\Omega/2\pi$. This provides an intuitive understanding of the writhe.

For an explicit example of how to calculate the writhe of a curve in three dimensions, we refer to [27].

4.3 The need for regularization

In the previous Section, we stressed that the integral in (56) is finite and equal to the writhe, W, of the curve C; no regularization needed. On the other hand (56) is the expectation value of a Wilson loop, and Witten has argued that it is a topological invariant since, in a pure C-S theory, neither the Wilson loop, nor the action depend on the metric [50]. Writhe, however, does depend on the shape of the curve (and hence the metric), but if one instead defines:

$$\widetilde{\Gamma}[C] \equiv \lim_{\epsilon\to 0} \oint_C dx^\mu \oint_C dy^\nu \varepsilon_{\mu\nu\sigma} \frac{(x - y - \epsilon\hat{n}(\vec{y}))^\sigma}{|\vec{x} - \vec{y} - \epsilon\hat{n}(\vec{y})|^3} = \theta\, L[C,\hat{n}]\,, \tag{61}$$

where the unit normal $\hat{n}(\vec{y})$ defines a "framing" of the curve C, one obtains a topological invariant. In fact L is nothing but the linking number between the two edges of the ribbon defined by the curve C and the frame $\hat{n}$. The apparent contradiction between (56) and (61) is resolved by noting that the limit $\epsilon \to 0$ is not smooth, since for a general twisted ribbon one has $W = T - L$, where T is the twist of the ribbon. It is natural to ask which one of the two above definitions is of physical relevance. We show that in a theory with an F^2 term in the action, introducing a framing has no effect, and the result is unambiguously the Polyakov one *i.e.* $\theta W[C]$.

We take the following Lagrangian for the gauge field,

$$\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4\theta} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \tag{62}$$

and obtain the (Landau gauge) propagator

$$\Delta_{\mu\nu}(\vec{x} - \vec{y}) = \frac{1}{4\pi} \left[-\varepsilon_{\mu\nu\rho} \frac{\hat{r}^\rho}{\mu r^2} \left(1 - (1 + \mu r) e^{-\mu r}\right) + g_{\mu\nu} \frac{e^{-\mu r}}{r} \right], \tag{63}$$

where $r = |\vec{x} - \vec{y}|$. The pure C-S theory is obtained in the limit $e^2 \to \infty$ (keeping θ constant), which gives the result (56). To see how the presence of the F^2 term modifies the fields close to the loops consider,

$$A_\mu(\vec{x}) = \oint_C dy^\nu \Delta_{\mu\nu}(\vec{x} - \vec{y}) = \oint_C dy^\nu \left(f(r) \varepsilon_{\mu\nu\rho} \hat{r}^\rho + g_{\mu\nu} \frac{e^{-\mu r}}{4\pi r} \right), \tag{64}$$

where $f(r) \sim 1/(\mu r^2)$ for $r \gg 1/\mu$ and $f(r) \to \mu$ for $r \to 0$, so the singularity connected to the C-S term is smoothened and instead we have a usual Coulomb type singularity (the last term in (64)). Since the long range part $\sim 1/\mu r^2$ is pure gauge, the field strength is smooth and concentrated to a region $\sim 1/\mu$ around C. For a purely timelike stretch of the curve C this is easily understood. The point-like magnetic flux present in the pure C-S theory is spread out over a region with size $\sim 1/\mu$, but we also see the screened electric field due to the charge [46]. Since there is no short distance singularity left in the term $\sim \varepsilon^{\mu\nu\rho}$ it is clear that a framing cannot change the result (56). It is also clear that for **two** curves C_1 and C_2 separated by a distance $\gg 1/\mu$ one obtains θL where L is the linking number of the two curves. This leads to the statistical phase discussed in [17]. For curves that come within the distance $1/\mu$ from each other the magnetic fluxes overlap and no simple interpretation in terms of statistics is possible. This proves the assertion that by adding an F^2 term for the gauge field one will unambiguously get the result (56) for the Wilson loops. For more details connected to the regularization procedure, we refer to Ref. [49].

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