SELFSIMILAR CASCADE STRUCTURE IN MULTIPARTICLE PRODUCTION PROCESSES*

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We discuss recent developments in the study of multiplicity fluctuations where scaling properties are searched for and partly established in full phase space ("KNO scaling") and limited domains of phase space ("Intermittency"). Such scaling properties are expected in parton cascade models based on QCD which are well established in e^+e^--annihilation with timelike parton evolution. In the other collision processes, in particular soft processes, similar phenomena are observed but no common theoretical description is available yet.

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1. Introduction

From the study of multiparticle production processes we can learn about the strong interaction dynamics. The fundamental theory, QCD, describes the interaction between quarks and gluons, the partons, from which the hadrons are built up. Scattering processes involving partons at large momentum transfers Q^2 can be treated in QCD perturbation theory. This theory, supplemented with an appropriate assumption or model on the transition from partons to hadrons, has enjoyed many successes in the interpretation of the multiparticle production phenomena, in particular, the hadron jet production.

Out of the problems which are studied in multiparticle production we emphasize the following: (a) **Properties of the QCD cascade.** The evolution of the parton final state can be treated in QCD perturbation

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theory. For small parton multiplicities there are exact calculations up to $O(\alpha_s^2)$ which include quantum effects. For large multiplicities one can derive the multiparton cross sections in leading logarithmic approximation (LLA) with various refinements. The same branching processes of partons are repeated at different scales during the evolution and one obtains a parton cascade with selfsimilar structure. The question arises which limiting scale Q_0 should be chosen to allow the application of perturbation theory with sufficiently small coupling. Another question is whether and how the selfsimilar structure of the partonic branching process could be verified experimentally. **(b) Strong interactions at small Q^2.** At the end of the cascade the partons recombine to hadrons at relative distances of $\sim$ 1 fm. It is not yet clear how to derive this process from the fundamental theory and so one has to construct "hadronisation models" which take into account consistently the experimental results. In processes with initial hadrons, one has to deal with a similar problem, the parton exchange at low momentum transfer. **(c) Universal properties of various collision processes.** Only in the e^+e^- annihilation process there is a good understanding of how the parton cascade is initiated and evolves. Other hard scattering processes (example μp scattering) require additional information about the parton structure of initial hadrons. The role of QCD in the description of "soft" processes, *i.e.* the untriggered collision of hadron or nuclei with their large cross sections of O(10-100 mb) is even less clear. On the other hand there is a large body of similarities in the phenomenology of all these different processes, though with differences in detail. An example are the scaling properties of multipicity distributions (KNO-scaling, intermittency) which we will discuss below. One can formulate the working hypothesis that the parton cascade structure is at the basis of this universality, and the differences of the different collision processes are due to the different initial conditions for such cascades [1,2] though there is not yet a generally accepted theory. **(d) Quark gluon plasma.** Besides the parton cascade process as considered in e^+e^- annihilation, there is the possibility of collective phenomena in heavy ion collisions, in particular of quark gluon plasma formation and a subsequent transition into the hadronic phase. It is therefore of great interest to compare the various collision processes and to look for trends which would indicate a phase transition.

In this lecture we focus on the consequences of selfsimilar cascade models of different degrees of sophistication to global multiplicity fluctuation and those in small phase space domains of different dimensions ("intermittency"). We also emphasise how the predictions could change for a system with a second order phase transition which also shows selfsimilarity properties.

2. Models for selfsimilar branching processes

Realistic models, such as the QCD parton cascade models can only be analysed by Monte Carlo methods. It is therefore desirable to consider first some simplified models and to study their properties, at least partly, by analytical methods.

2.1. Simplified "toy" models

(a) α model. This model represents a selfsimilar random cascade structure and was used by Białas and Peschanski [3] to describe multiplicity fluctuations in rapidity. One considers a sequence of cascade steps $n = 1, 2 \dots N$. In the first step the original rapidity interval Δy is divided into λ subintervals of size $\Delta y/\lambda$, in the next step these subintervals are divided further in the same way and so on until after N steps $\delta y_N = \Delta y/\lambda^N$. Any of the final $M = \lambda^N$ intervals can then be labeled by a set of indices $\{\alpha_j\}$ ($\alpha_j = 1 \dots \lambda$, $j = 1 \dots N$) refering to the particular path in the process of subdivision. The density X_N in one interval after N steps is calculated in the α-model from the product

$$X_N = w(\alpha_N) \dots w(\alpha_2) w(\alpha_1) X_0 , \tag{1}$$

where X_0 is the initial density in the full interval $\delta y_0 = \Delta y$ and $w(\alpha_i)$ are independent random variables with distribution $p(w)$ which is assumed not to depend on the step number and therefore define a selfsimilar branching process. In the simplest case w takes only two values w_a and w_b with probabilities p_a and $p_b = 1 - p_a$. Without loss of generality for normalized quantities one can choose also $\langle w \rangle = p_a w_a + p_b w_b = 1$. The model has then two arbitrary parameters p_a and w_a for a given branching structure defined by λ and N.

The 1-dimensional α-model can be generalised in a natural way to higher dimensions [4]. This is important if one wants to study more realistic cascades evolving in a 3d momentum space. In d dimensions one starts from an initial cube of volume $(\Delta y)^d$; this is subdivided into λ^d cubes of edge length $(\Delta y/\lambda)$ and so on for N steps. The density in one of the final volume elements is again constructed by multiplication of random numbers as in (1) . The fluctuations of multiplicity in the 1d projection of the 3d model are quite different from those in the above 1d model, as will be shown below.

(b) Branching models in 1+1d field theory. Another group of simple models is discussed by Chiu and Hwa [5,6]. A parton of virtuality $Q^2 > 0$ initiates a branching process with successive degradation of virtuality until a final cutoff Q_0^2 is reached in which case the parton is identified with the final particle. The longitudinal momentum fraction z in an infinite momentum frame is distributed according to a distribution $P(z)$. Contrary to the previous model the number of cascade steps fluctuates and so does the multiplicity of particles.

(c) Scale invariant cascade model. This model [7,8] again describes the evolution of a cascade from an initial scale Q^2 down to final hadrons of mass Q_0^2, but now in 3d momentum space. At each vertex an intermediate state of mass Q_p^2 decays isotropically into two states with masses Q_1^2, Q_2^2 with distribution

$$\frac{dn}{d\mu_1 d\mu_2} = F(\mu_1, \mu_2), \quad \mu_i = Q_i^2/Q_p^2 . \tag{2}$$

This distribution is finite and obeys exact scale invariance and is derived from phase space arguments. The model is realistic insofar as it reproduces the main features of the jet structure of the hadronic final states in e^+e^- collisions. However, the multiplicity fluctuations are predicted larger [8] than experimentally found.

2.2. QCD parton cascade models

The description of multiparticle production in a hard scattering process involving a large momentum transfer Q^2 can be based on QCD. In a first step there is an initial hard scattering process, such as $e^+e^- \to q\bar{q}$ or $eq \to eq$, for which the matrix element can be calculated in QCD or in the electroweak theory. In a second step the scattered partons radiate gluons and initiate in this way the parton cascade process. The theoretical treatment of this process in LLA and beyond has been improved over the last years [9]. An important step was the observation that in a particular gauge interferences between different Feynman diagrams can be neglected in the given approximation so that a probabilistic interpretation of the results and the application of Monte Carlo methods became feasible. Remarkably, this property could be maintained after including next to leading corrections from soft gluons by an "angular ordering" of outgoing partons. For reviews of these works see [10].

In the LLA the "decay" of a parton $a \to bc$ which transforms a final state of N partons into one with $N+1$ partons yields the following recursion formula for the respective cross sections in the limit of small angles

$$d\bar{\sigma}_{N+1} = d\bar{\sigma}_N \frac{dt}{t} dz \frac{\alpha_s}{2\pi} P_{a\to bc}(z) , \tag{3}$$

where an appropriate average or sum is taken over the polarisation states and the azimuthal decay angle. Here t is the virtual mass of parton a, z the energy fraction of a final parton, and $P_{a\to bc}(z)$ the respective Altarelli–Parisi splitting functions for the three possible subprocesses $q \to qg$, $g \to gg$

and g $\to$ q$\bar{\text{q}}$ [11]. For final soft gluons there is an infrared singular factor $P(z) \sim 1/z$. One can see that the density in a phase space interval is again constructed recursively by the multiplication of random functions. The approximate selfsimilarity of the branching process is due to the scale free evolution of virtualities dt/t and energies $dzP(z)$ but some scale breaking effects are introduced by the logarithmic dependence of the strong coupling constant α_s on the QCD scale Λ and also by a necessary cutoff in the final parton mass Q_0^2 and consequently in z.

2.3. Parton cascade and hadron final states

Several models have been proposed [9] which describe the transition from partons to hadrons. The most popular ones are the string model and the cluster model whereas a simple recipe is based on "local parton hadron duality (LPHD)" [12]. The preferred models today are based on the parton cascade with low cutoff $Q_0 \lesssim 1$ GeV. Models based on $O(\alpha_s^2)$ matrix elements do not give a consistent description of the data (see, for example, the intermittency analysis by DELPHI [13].)

(a) String fragmentation. This is the basis of the LUND-hadronisation model [14] (for some earlier results, see [15,16]). It is assumed that the parton final state corresponding to a particular cutoff Q_0 generates a color flux tube of limited transverse size of ~ 1 fm between the partons. If the partons are sufficiently energetic the string may break by the production of a new q$\bar{\text{q}}$ pair. Assuming a tunneling mechanism one obtains for the probability of the production of a pair of quarks, each with mass m_{q} and transverse momentum p_{T} in a static approximation the factor $\exp(-\frac{\pi}{\kappa}(m_{\text{q}}^2 + p_{\text{T}}^2))$ where $\kappa \approx 1$ GeV/fm is the string tension. In principle, such formulae would predict the rate for different flavors and transverse momenta, however there is considerable uncertainty on what quark (diquark) masses or string tension should be used for a dynamical system and so one has to introduce appropriate parameters into the formalism. Though the predictive power of the model is limited by the possibility to choose various parameters its overall ability to fit the data in detail, recently, for example, intermittency effects at LEP, is quite remarkable.

(b) Cluster fragmentation. In another approach it is assumed that at the end of the cascade, when the virtual mass becomes small (≤ 1 GeV) and the coupling constant large of $O(1)$, the gluons split into u$\bar{\text{u}}$ and d$\bar{\text{d}}$ pairs nonperturbatively [17] (for other work on clusters see [18]). Quark and antiquark pairs form color neutral objects of variable mass with a distribution approximately independent of the primary energy. These clusters are assumed to decay isotropically into the hadron states of the lowest $\text{SU}(3)_{\text{f}}$ meson and baryon multiplets according to a phase space probability. For a small fraction of the clusters in the high mass tail, above a certain fission threshold ($\sim$ 4 GeV), a decay with preferred direction of the string type is assumed. This model is impressive by the rather small number of adjustable parameters. Results can be derived from the computer program HERWIG.

(c) Local parton hadron duality. There is the interesting question to what extent the structure of the hadronic final state can be described by the partonic final state disregarding hadronisation alltogether. It is clear that this can only work in some approximate sense or for a suitable average (for example resonance phenomena are not present at the parton level).

A remarkable similarity of this type has been found by the Leningrad group [12]. The shape of the inclusive momentum spectra of hadrons is found to be rather similar to the inclusive momentum spectra of partons as calculated from the parton cascade in LLA including the soft gluon interferences with the cutoff $Q_0 = m_h$, the final hadron mass. This works reasonably well in dependence of hadron masses (π,K,p) [12], energy dependence [19] and particle rates [20]. Recent results from LEP on charged particles [19] which became available after this meeting, are shown together with TASSO data from lower energies [21] in Fig. 1. A very good description of these spectra for $\ln(p/1\,\text{GeV}) \gtrsim -1$, or $p \gtrsim 0.35\,\text{GeV}$, up to the phase space boundary can be seen for all energies. The fitted parameters are $\Lambda = 0.253\,\text{GeV}$ and the normalisation. Hadronisation models can fit these spectra too and the above similarity can be obtained in the LUND model, for example [22]. However, we would expect that in this case the spectrum shape, in particular for large momenta, is strongly dependent on the model parameters (such as fragmentation function, ρ/π ratio) which are only weakly constrained. The understanding of the physical origin of LPHD therefore remains as a puzzle and a challenge.

There had been suggestions to extend this duality beyond single particle inclusive spectra. It was noted [23] that the exclusive jet multiplicity distribution at low resolution as a function of a mass resolution parameter could be fitted by the parton cascade without hadronisation. It has been proposed that exclusive multi cluster states above the resonance region with mass resolution $\delta m \gtrsim 1$ GeV could be represented by the corresponding parton states which leads to an almost "exclusive LPHD". A possible theoretical scenario has been discussed assuming a duality in momentum space between a partonic and a hadronic cascade – not in obvious contradiction to QCD with confinement.

According to another suggestion [24] the normalised multiparticle corelation functions are proportional to the corresponding functions for partons. This property is found for the LUND model at very high energies (above 200 GeV) from an analysis of multiplicity distributions.

As the suggestion of the various forms of LPHD are not very well founded theoretically it appears to be an interesting research project to

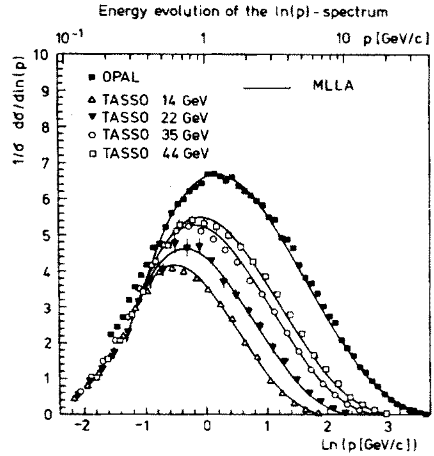

Fig. 1. Inclusive momentum spectra $d\sigma/d\ln p$ of charged particles at PETRA energies from TASSO [21] and at LEP from OPAL [19]. For momenta $p \gtrsim 300$ MeV these spectra are well described by the inclusive spectra of partons (mainly gluons) from the modified LLA in accordance with the hypothesis of "Local Parton-Hadron Duality" (from [19]).

establish phenomenologically the regime of validity and the limits of this remarkable similarity.

3. Global Multiplicity fluctuations and KNO scaling

A well known scaling property in multiparticle physics is the multiplicity scaling proposed in 1972 by Koba, Nielsen and Olesen (KNO) [25]. The probability $P_n(s)$ to find n particles in one event depends on the total energy $\sqrt{s}$ only through the scaling variable $z = n/\bar{n}(s)$

$$\bar{n}P_n(s) = \psi(n/\bar{n}), \qquad \bar{n} = \bar{n}(s). \tag{4}$$

In the derivation non asymptotic modifications $O(1/\bar{n})$ are admitted. The original derivation was based on Feynman scaling of the inclusive particle spectra, but this hypothesis is not supported by today's experiments anymore. Another derivation of the scaling law (4) was given by Polyakov [26] already before within a scale invariant field theory. His view of multiparticle production as a branching process with virtual mass degradation is already close to our present understanding of hard processes in terms of QCD parton cascades. In the following we discuss some recent experimental results, theoretical explanations and a possible extension to jet multiplicities.

3.1. Recent experimental results

KNO scaling has first been observed in soft hadronic collisions. Some finite energy corrections can be taken into account if (4) is reformulated in terms of central moments $D_q = < (n - \bar{n})^q >^{1/q}$ as

$$D_q(\bar{n}) = a_q(\bar{n} - c), \tag{5}$$

with nonvanishing constant c [27]; Eqs (4) and (5) agree for high energies. These relations (5) are found to be well satisfied up to the highest ISR energies ($\sqrt{s} \leq 60\,\text{GeV}$) [28], but a violation has been observed at the collider for $\sqrt{s} \geq 200\,\text{GeV}$ [29]. Remarkably, (5) is valid down to very small energies near threshold ("early KNO scaling") which is not expected from the theoretical arguments leading to (4).

In the last two years KNO scaling has also been established for hard processes, which was in dispute for a long time. In e^+e^--annihilation, new data from TASSO [30] and more recently from DELPHI at LEP [31] are consistent with KNO scaling and, for example, in disagreement with a Poisson distribution (for a review, see Ref. [32]). Other results became available from deep inelastic νN scattering [33,34]. Also in this case "early KNO scaling", almost down to threshold has been observed (see Fig. 2).

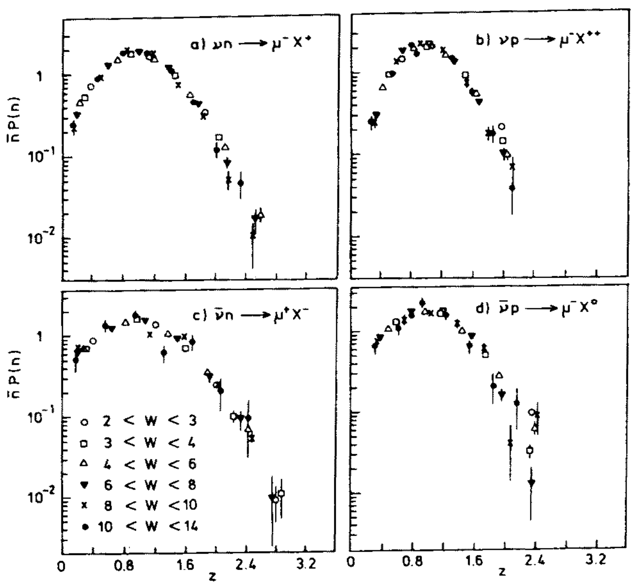

Fig. 2. Global multiplicity scaling ("KNO-scaling") also works in hard processes, shown are results for neutrino scattering processes [33]. The scaling works "early", *i.e.* almost down to threshold.

3.2. KNO scaling and cascade processes

KNO scaling is predicted in e^+e^--annihilation from scale invariant branching processes [26] and also for the QCD parton cascades [35,36]. As an example, we present an outline of the proof for the scale invariant cascade model discussed in (2.1) which can be derived rather easily [7,37].

If the initial state of mass $\sqrt{Q^2} = \sqrt{s}$ decays into two states with masses $\sqrt{Q_1^2}$ and $\sqrt{Q_2^2}$ one can write for the respective multiplicities

$$P_n(Q^2) = \sum_{\nu} P_\nu(Q_1^2) P_{n-\nu}(Q_2^2), \tag{6}$$

or for the generating function $\Phi(Q^2, z) = \Sigma\, P_n(Q^2) e^{nz}$

$$\Phi(Q^2, z) = \int d\mu_1 d\mu_2 F(\mu_1, \mu_2)\, \Phi(\mu_1 Q^2, z)\, \Phi(\mu_2 Q^2, z), \tag{7}$$

with the scale invariant mass distribution F as in (2) and with $Q_i^2 = \mu_i Q^2$. A solution of (7) can be found by differentiation in terms of moments $\overline{n^k}(Q^2) = \Phi^{(k)}(Q^2, 0)$ as

$$\overline{n^k}(Q^2) = c_k (Q^2/Q_0^2)^{\alpha k}, \tag{8}$$

where the constants c_k are solutions of

$$c_k = \sum \binom{k}{\nu} c_\nu c_{k-\nu} \int d\mu_1 d\mu_2 \mu_1^{\alpha\nu} \mu_2^{\alpha(k-\nu)} F(\mu_1, \mu_2). \tag{9}$$

From (8) one finds

$$\frac{\overline{n^k}}{\bar{n}^k} = \frac{c_k}{c_1^k} = \text{const} \tag{10}$$

which is equivalent to KNO scaling (4) at high energies. In particular, also, the multiplicity grows with energy like a power $\bar{n}(Q^2) = c_1 (Q^2/Q_0^2)^\alpha$.

From the Monte Carlo analysis of this model one also obtains "early scaling" as in (5). It is a result of the validity in this model of the same scaling law for F at all scales, in some average sense even down to the external mass. If one had two different dynamical regions, below and above a certain energy, one would get in general a violation of early scaling [38].

For the QCD parton cascade a general proof of KNO scaling has been obtained. This problem is more involved because of the singular structure of the evolution equation (infrared and mass singuarities) and the running of the strong coupling constant [35].

The numerical simulation of the QCD parton cascade [17] shows good KNO scaling, over the present energy range, but the scaling function is still different from the asymptotic limit, i.e. the approach to "asymptopia" is rather slow.

In hadronisation models one has a complicated interplay between fluctuations at the parton and the hadron level. It has been reported for the LUND hadronisation model that only both components together yield early scaling, but the parton component alone does not [22].

In soft hadronic collisions KNO scaling is not usually related to scale invariant branching processes but to multicomponent models of various kinds (for a review, see [39]). Early KNO scaling appears so far to be a property of all types of collision processes though each one with different scaling function $\psi(z)$ in (4). It is a challenge to understand the meaning of this universality, which we will meet again below in the intermittency studies.

3.3. Scaling of cluster multiplicities

In a strictly scale invariant model, such as the one discussed above in (2.1), the only scales are the total energy $\sqrt{Q^2}$ and the external mass $\sqrt{Q_0^2}$. Therefore the moments depend only on $\overline{n^k}(Q^2, Q_0^2) = \overline{n^k}(Q^2/Q_0^2)$. If we consider Q_0^2 now to be a variable mass of a cluster or of a jet of final state particles the multiplicity scaling also holds for fixed total energy and variable mass [23]

$$\bar{n}P_n(s, Q_0^2) = \psi(n/\bar{n}), \qquad \bar{n} = \bar{n}(s, Q_0^2). \tag{11}$$

This property has been tested by a Monte Carlo simulation of the simple scale invariant model applying the JADE cluster finding algorithm [40]. It would be interesting to see whether this also holds for the QCD cascade.

If such an extended multiplicity scaling could be established experimentally, it would be a further hint to an underlying scale invariant cascade mechanism. This would be particularly interesting for the case of soft hadronic collisions and a distinction from the multicomponent models may become possible.

4. Multiplicity fluctuations in limited domains of phase space and intermittency

4.1. Moment analysis and the intermittency hypothesis

Many discussions have recently been generated by the proposal [3] that the multiplicity fluctuations may reveal a selfsimilar structure if analysed in phase space domains of decreasing scale size. If selfsimilarity holds over a large range of scales one speaks of fractal structure which is known for various physical and geometrical systems or objects [41]. More specifically, for the selfsimilar multiplicity fluctuations in the multiparticle production the name "intermittency" was chosen in some analogy to fluctuations in turbulent fluids where also similar simple models had been applied ("α model") [42].

In a first analysis of this type Białas and Peschanski [3] studied the dependence of the multiplicity distribution in rapidity intervals of size δy (from a subdivision of the full interval Δy into $M = \Delta y/\delta y$ bins) on the scale size δy. The multiplicity distribution was analyzed in terms of scaled factorial moments

$$F^{(q)}(\delta y) = \frac{< n(n-1)\ldots(n-q+1) >}{< n >^q}, \tag{12}$$

where the average is taken over the M bins in one event and also over the event sample. If these moments follow a power law

$$F^{(q)}(\delta y) \sim (\delta y)^{-\alpha_q}, \tag{13}$$

over a range of δy scales one speaks of intermittency, and such behaviour is expected in simple models with selfsimilar dynamics as will be discussed in more detail below. A singular behaviour as in (13) implies ever increasing multiplicity fluctuations for decreasing δy scales and the occurrence of irregular "spikes", which have actually been observed [43] and provided one motivation for these studies.

The moments $F^{(q)}$ in (13) yield an estimate of the usual moments $C^{(q)} =< n^q > / < n >^q$ of the underlying probability distribution in the case of finite statistics if a Poisonnian noise is assumed [3]. For a Poisson distribution one obtains $F^{(q)} = 1$ for all q. Also, if $F^{(q)} \neq 0$ is observed for large q at small δy there is at least one "spike" of multiplicity q, so these moments can serve as filter for such spikes. For these properties the $F^{(q)}$ moments are often used but other methods have been proposed also in closer analogy to the theory of fractals [44].

4.2. Overview of experimental results and the theoretical discussion

A rise of factorial moments with decreasing δy has been observed in all types of collision processes: e^+e^- [45–47,13], μp [48], hh [49–51] and nuclear collisions [52,53]. As example we show in Fig. 3 the results on π^+p, K^+p collisions by the NA22 collaboration. The moments show a stronger rise in the region $\delta y > 1$ and a smaller rise in the region $\delta y < 1$ consistent with the power law (13). It is somewhat controversial whether and for which δy scale there is a saturation of the moments. The EMU01 collaboration working with heavy nuclei has found no further rise of the moments in a region with $\delta y \leq 0.1$ which was accessible within good experimental resolution [53]. The

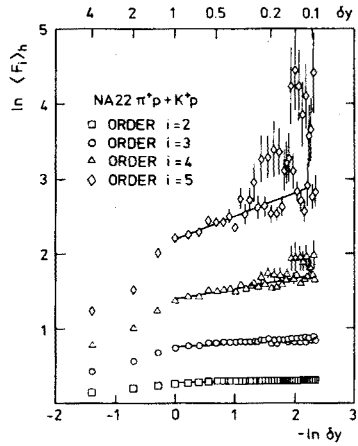

Fig. 3. Scaled factorial moments for a 1d analysis in rapidity. The straight lines for $\delta y < 1$ correspond to the power law of intermittency (from Ref. [50]).

intermittency slopes α_q are largest for the elementary processes (e^+e^-, μp) decrease for hadron–hadron collisions and become even smaller for heavier nuclei.

Results became also available for 2-dimensional analyses in the $y - \phi$ plane (ϕ is the azimuthal angle around the jet axis). In this case the effect is much larger and the slopes are bigger by factors ranging from 2 to 6 as was actually predicted for cascade models [8] (for a survey, see [4]).

Another interesting finding is due to Sarcevic and Satz [54]. They found a power law for moments as in (13) in the complementary range of large δy intervals near the maximal value $\delta y \lesssim y_{\max}$ to be valid in pp collisions of different energies. They also related this behaviour to scale invariant cascades.

A theoretical discussion goes on about the physical origin of the rise of the moments and, in particular, whether the effects can be explained by conventional ideas on particle production.

Recent results (which became available after this school) from LEP on e^+e^- annihilation [13,55] have been found in good agreement with the LUND parton shower model. The same was also reported from CELLO at PETRA energies [46] contrary to previous findings by TASSO [45]. In all other processes (μp, hh, AA) the conventionally used models did not give a satisfactory description of the data: for $\delta y \lesssim 1$ they give either no rise of moments at all or too weak a rise. The rise of moments for $\delta y \gtrsim 1$ can be explained by conventional short and long range correlations due to resonance or hadronic cluster decay.

The proposal that also for small $\delta y < 1$ conventional correlations of various types can be responsable for the moderate rise has been suggested by several authors [56–58]. They use exponential or Gaussian functions to parametrize the rapidity correlations and therefore predict a saturation of moments for $\delta y \to 0$ and so they do not require a selfsimilar structure. These models obtain acceptable fits of the data. Apparently the assumed correlations are stronger than those in explicite fragmentation models which have the known resonance effects included and nevertheless are unable to explain the data.

Another source of very short range fluctuations are **Bose-Einstein correlations** of identical particles [59]. Experimental studies comparing particle pairs with same and opposite charge come to the conclusion that the rise of moments cannot be entirely due to this effect [50,45,48], though it cannot be neglected [46].

The possibility that the power law observed in a limited δy range should be related to an underlying selfsimilar structure has been considered by various authors and will be discussed in more detail below. Selfsimilarity can naturally be realised by **random cascade processes** (see section 2); results have been obtained for the α-model [3], scale invariant cascade model [8] and QCD parton cascade models with or without hadronisation [60–64] Another realisation of selfsimilarity is found in statistical systems with a **second order phase transition.** This has been studied in the Ising model which describes a spin system on a lattice with nearest neighbour interactions [65–67]. Recently it has been suggested that these results could be relevant for the phase transition from the quark gluon plasma to a hadron gas which could occur in heavy ion collisions [68].

In pursuing the idea of selfsimilarity in the multiparticle data one has to consider the following problems.

(a) For **finite statistics** a singular behaviour as in (13) cannot be realised and below a certain scale δy the moments have to vanish when no more than one particle is left in any bin. With increasing statistics the moments at fixed small δy may rather suddenly jump from zero towards the asymptotic value (this happens in the model discussed in Ref. [8]) or only approach it slowly from below (see examples by Redner [69] and Kittel[70]). This requires some caution in the interpretation of data for small δy where data show large fluctuations.

(b) The **finite multiplicity** prevents a singular behaviour, even for infinite statistics. Here one may study the trends of the correlations with increasing energy (multiplicity) [71].

(c) Selfsimilarity can only be expected to lead to a power law of moments if the **dimensionality of the moment analysis** match the dimensionality of the space in which the scale invariant dynamics is acting [4]. Generally this requires a moment analysis in the three momentum space dimensions although under special circumstances already in lower dimensional space a power law could appear. This has been found in an analysis of the multidimensional α-model [4], for a model employing a correlation function with a powerlike singularity [72] and also for the Ising model [73].

In fact, whereas in one dimension of rapidity the validity of the power law is restricted to a limited region with $\delta y < 1$, in the two-dimensional $y - \phi$ analysis a power law is observed in the full range of scales presented in the μp [74] and in the πp/Kp [50] experiments (see Fig. 4). These are the best examples of intermittency up to now. A power law in a large range is also observed in e^+e^-annihilation at LEP [55] and in a restricted range analysed in heavy ion collisions [75]. A final discussion about the range of validity of a power law and selfsimilar dynamics has to await the results from the 3-dimensional analyses.

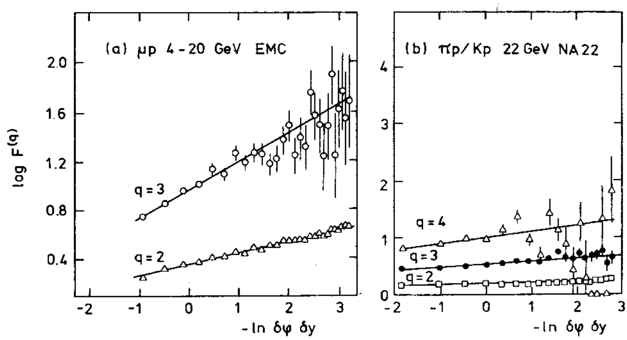

Fig. 4. Scaled factorial moments from a 2d analysis in rapidity and azimuthal angle around the beam for (a) μp data [74] (b) πp/kp data [50]. The data are consistent with a power law (straight lines) in the full parameter range presented suggesting intermittent behaviour in the higher dimensions.

In the following we discuss in more detail the expectations from random cascade models in multidimensional intermittency analyses.

4.3. Role of phase space dimensionality

Intermittency in one-dimensional models. In the 1d α-model the density is given by a multiplicative random process. Selfsimilarity occurs if the same random function enters at all scales (see Sect. 2.1). The moments of the probability distribution of X_N in one interval (Eq. (1)) can be calculated (using $F^{(q)} \approx C^{(q)}$, disregarding the statistical noise) as

$$C_N^{(q)} = (< w^q > / < w >^q)^N, \tag{14}$$

and with $N = \log(\Delta y/\delta y_N)/\log\lambda$ one obtains [3] the power law of intermittency (13) with exponent

$$\alpha_q = \log(< w^q > / < w >^q)/\log\lambda . \tag{15}$$

This power law occurs in the variable for which the branching process is defined. If the fluctuations are measured in another variable $\tilde{y}$ $(y = g(\tilde{y}))$, one finds in the limit $\delta\tilde{y} \to 0$ by inserting $\delta y \approx g'(\tilde{y})\delta\tilde{y}$ into Eq. (13), again a power law with the same exponent α_q as before, but with different normalisation of the moments.

The numerical analysis of the 1d field theoretical models discussed in Sect. 2.1 again yields a power behaviour in a large range of scales which is defined by the total energy $\sqrt{Q^2}$ and the final mass Q_0. Moreover, the slopes α_q are found to be independent of energy but strongly dependent on the type of parton splitting functions $P(z)$. The slopes for the infrared singular "gluon model" with $P(z) \sim z^{-1}$ for $z \to 0$ are more than an order of magnitude larger than those for the twin model with $P(z) = \delta(z - \frac{1}{2})$ or the Φ^3 model in six dimensions with $P(z) = 6z(1-z)$. The slope ratios α_q/α_2 on the other hand are the same within about 20% and the rise with q is between linear and quadratic.

An exact power law is also obtained in the 1d model by Sarcevic and Satz [54] which describes a 1d branching process of massive initial states into hadrons.

1d projections in multidimensional models. First we consider the 2d α-model for the $y - \phi$ plane (see Sect. 2.1) [4]. As the density in a 2d phase space element is again given by the product of random variables as in Eq. (1) a power law as in the 1d case is obtained.

Next we calculate the moments for the densities in the 1d projection onto the rapidity-axis, where we restrict ourselves to the case of two subintervals ($\lambda = 2$). The density $\hat{X}_N$ in one δy interval after N steps is calculated as a sum over the densities in the corresponding $M = 2^N$ $\delta\phi$-intervals. Inserting an extra branch at the beginning of the cascade one finds a recursion relation for the random densities

$$\tilde{X}_{N+1} = w(1)\,\tilde{X}_N(1) + w(2)\,\tilde{X}_N(2) . \tag{16}$$

which implies for the moments

$$\begin{aligned} \langle \widehat{X}^q_{N+1} \rangle &= \sum_{\nu=0}^{q} \binom{q}{\nu} \langle w^\nu \rangle \langle w^{q-\nu} \rangle \langle \widehat{X}^\nu_N \rangle \langle \widehat{X}^{q-\nu}_N \rangle \quad \text{for} \quad N \geq 1 \\ \langle \widehat{X}^q_0 \rangle &= 1 . \end{aligned} \tag{17}$$

For the normalized moments one obtains after division of (17) by $\langle X_N\rangle^q = 2^{Nq}$

$$\widehat{C}^{(q)}_{N+1} = \frac{1}{2^q}\sum_{\nu=0}^{q}\binom{q}{\nu}\langle w^\nu\rangle\langle w^{q-\nu}\rangle\, \widehat{C}^{(\nu)}_N \quad \text{for} \quad N \geq 1$$
$$\widehat{C}^{(q)}_0 = 1\,. \tag{18}$$

Starting from the initial condition one finds by iteration the form

$$\widehat{C}^{(q)}_N = a_q\left(1 + \sum_{i=2}^{q} b_i^{(q)}\left(\frac{\langle w^i\rangle}{2^{i-1}}\right)^N\right), \tag{19}$$

for $q = 2$ explicitly

$$\widehat{C}^{(2)}_N = \frac{1}{2-\langle w^2\rangle}\left((1-(\langle w^2\rangle - 1)\left(\frac{\langle w^2\rangle}{2}\right)^N\right), \tag{20}$$

We obtain the interesting result that these moments tend towards a finite value in the limit $N \to \infty$ or $\delta y \to 0$, provided that

$$\langle w^q\rangle < 2^{q-1}, \tag{21}$$

or, in case of the simple 2-valued distribution of w, if Max $(w_a, w_b) < 2$. Otherwise the moments would exponentially diverge, which does not correspond to the actual physical situation. A completely analogous result has already been obtained for the total multiplicity distribution in the 1d α-model; in this case the saturation of normalised moments corresponds to KNO scaling [76]. Therefore we conclude from these results: **if a system is intermittent in two dimensions it is not necessarily intermittent in the one dimensional projection**, and, in fact, it is not so in the 2-dimensional α-model.

Similar results have been obtained for other models. Białas and Seixas [72] studied a model with singular correlation functions. The two-particle distribution for example, was taken in the form

$$\rho_2(p_1, p_2) = \rho_1(p_1)\rho_1(p_2)Q_{12}^{-\alpha}. \tag{22}$$

Here $Q_{12}^2 = |\,(p_1^\mu - p_2^\mu)^2\,|$ measures the distance of particle 4-momenta p^μ and

$$\rho_1(p) = \exp\left(-6.25\sqrt{m_\pi^2 + p_\perp^2}\right) \tag{23}$$

represents the single particle distribution. The factorial moments are obtained by integration of the multiparticle distributions over the domain Δ,

$$F^{(2)}(\Delta) = \int\limits_{\Delta} \frac{d^3\vec{p}_1}{E_1} \int\limits_{\Delta} \frac{d^3\vec{p}_2}{E_2} \rho_2(p_1,p_2) / \left(\int\limits_{\Delta} \frac{d^3\vec{p}}{E} \rho_1(p) \right)^2 . \qquad (24)$$

Whereas in 3d space the moments are power behaved the moments are reduced in size and tend to saturate in the 2d or 1d projection.

More generally, it has been shown by Bożek and Płoszajczak [77] that for a 2d correlation function with factorising singularities in both dimensions intermittency is obtained in the 1d projection, whereas for a non-factorising singularity it is not.

Wosiek [73] has shown that whereas there is intermittency in the 2d Ising model in the fluctuation of the block spin (see below) saturation occurs in the 1d projection.

4.4. Multidimensional intermittency analysis

From the above it becomes clear that the analysis has to proceed to higher dimensions, *i.e.* up to three, in order to fully explore the fractal structure of multiparticle production. In the following we describe a method and apply it to some realistic models in 3d momentum space.

A natural choice of variables for a full 3d analysis would be the set y, ϕ and $p_\perp$, the transverse momentum of a particle with respect to the jet axis. However, one has to remove the influence of the trivial fluctuations which arise from a variation of the average multiplicities over the various bins, *i.e.* the variation of the inclusive particle distribution, if the "horizontal" average over the bins is taken. These fluctuations in rapidity can be avoided by restricting to the central rapidity region. Alternatively one can multiply the bin averaged moments with a factor which takes this variation into account [60]. Here we use an alternative method which is easy to apply also to the 3-dimensional case where this problem becomes essential. We choose new longitudinal and transverse variables for which the projected densities are constant.[1]

Consider first the rapidity variable y in which the particle density has a bell shaped distribution $dn/dy = \rho(y)$. We define the new variable by the transformation

$$\tilde{y}(y) = \int\limits_{y_{\min}}^{y} \rho(y')dy' / \int\limits_{y_{\min}}^{y_{\max}} \rho(y')dy' \qquad (25)$$

[1] For some more details and applications, see also [78]. A similar approach has also been suggested by Białas and Gazdzicki [79].

which can be constructed from the observed density $\rho(y)$. The new variable satisfies

$$\frac{dn}{d\tilde{y}} = \text{const}, \quad 0 \le \tilde{y} \le 1 . \tag{26}$$

Intervals of constant length $\delta\tilde{y}$ correspond to intervals of variable length δy_i in the original variable y, such that the average particle numbers in δy_i are the same, i.e. intervals near the peak of $\rho(y)$ are smaller than near the minimum. For a 2d analysis suitable variables are $\tilde{y}$ and ϕ (or y and ϕ with a y cut [8]).

In case of the 3d analysis we choose p_T as the third variable. This choice is suggested for there is no strong correlation between y and p_T (except in the fragmentation regions). We therefore restrict ourselves here to independent transformations of y and p_T, *i.e.* we only require constant densities in certain projections. Then we can obtain the transformed variable $\tilde{p}_T$ by the same formula as Eq. (25) in terms of the density $\rho(p_T)$. Because of the rapid variation of $\rho(p_T)$, at small p_T it is advantageous for numerical applications but completely equivalent to start from the distribution in the variable $l = \log(2p_T/\sqrt{s})$ instead of p_T and to calculate $\tilde{p}_T(l)$ from an integral over $\rho(l)$. With this choice the variable space is a cylinder with unit length and radius. To study intermittency one can subdivide the total space of each variable $\tilde{y}, \tilde{p}_T$ and ϕ in a sequence of steps $N = 1, 2, 3 \cdots$ into $2, 4, 8, \cdots$ subintervals of equal length, so that the total number of intervals M_d in a d-dimensional analysis is

$$M_d = 2^{dN} . \tag{27}$$

In the 1d analysis the interval size is then $\delta\tilde{y} = 1/2^N$.[2]

In Fig. 5 we show the behaviour of the moments $F^{(2)}$ as a function of N, the number of steps of subdivisions into halves, for different dimensions d of the variable space $\tilde{y}, \phi, \tilde{p}_T$ and for different realistic particle production models, as discussed in Sect. 2. The 1d moments always show an early saturation after some initial rise as expected from the above analysis of toy-models. The 2d moments rise more strongly but still show curvature or saturation. In case of the parton model, there is even a drop for large N, which comes from the angular cutoff which terminates the cascade. The 3d analysis finally shows the strongest rise. In case of the cascade model in (a) one obtains a rise consistent with the power law of intermittency which is a consequence of the underlying exact scale invariance of the model. In the QCD parton model (b) some deviation of the power law is visible which can be related to the scale violations for the running coupling constant α_s and

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[2] in [78] we also considered the division into equal intervals in the variable $\tilde{\ell} = \sqrt{p_T}$ and a $\tilde{l}$-dependent number of ϕ-intervals to avoid very elongated bins. Only small changes of results were found.

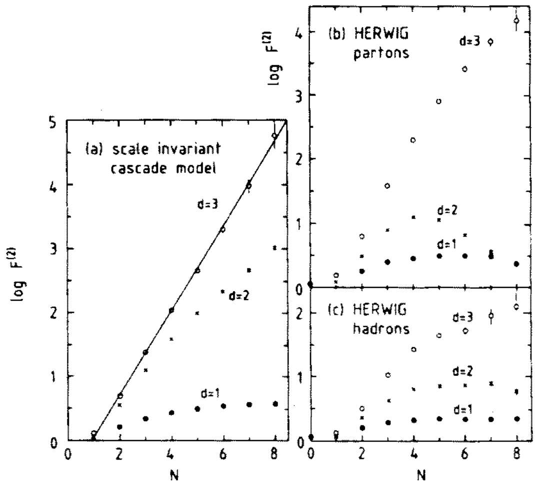

Fig. 5. Predictions from cascade models on factorial moments in different dimensions d using the variables $\tilde{y}, \phi, \tilde{p}_T$. The strongest rise of moments appears for $d = 3$, with power behaviour for the scale invariant model in (a); deviation from the power law occur in the parton models because of various scale breaking effects in (b, c). The $d = 1$ moments saturate; energies (a) $\sqrt{s} = 35$ GeV, (b, c) $\sqrt{s} = 90$ GeV. The MC calculations have been carried out in (a,c) with about 10,20,30k and in (c) with 20,40,60k events for $d = 1, 2, 3$ respectively.

to the cutoff. In the full hadronisation model there are in addition effects from resonances and heavy quark decays. One can see from this figure that the scale invariance of the dynamics and the scale breaking effects can be studied most directly in the 3d analysis where exact scale invariance leads to a power law of moments in the full range of scales.

4.5. Scaling properties of higher moments and multiparticle correlation structure

Power laws for higher moments. It was found for simple 3d scale invariant cascade models that the higher moments $F^{(q)}(\delta y)$ in the 1d analysis follow a modified power law [8]

$$F^{(q)}(\delta y) \sim (g(\delta y))^{\alpha_q}, \tag{28}$$

with arbitrary function $g(\delta y)$ or, equivalently, by expressing $g(\delta y)$ in terms of $F^{(2)}(\delta y)$

$$\log F^{(q)}(\delta y) = \frac{\alpha_k}{\alpha_2} \log F^{(2)}(\delta y) + d_k . \tag{29}$$

Subsequently these relations were found to be valid in a larger class of models (including the α-model, and parton cascade models) so that they seem to be another general property of scale invariant cascade models. Also, these relations are found to be fulfilled by experimental data [4]. Recently, these relations are found also to hold in higher dimensions in the same models with the same slope ratios α_q/α_2 in all dimensions and again agreement with available data for $d = 1$ and $d = 2$ is found [78]. As an example we show in Fig. 6 results from 1d and 2d analyses of μp and e^+e^--collisons.

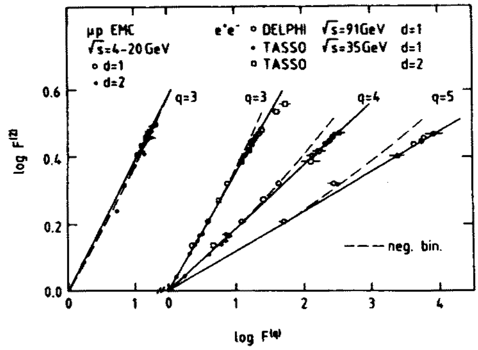

Fig. 6. Test of the linear relationship between the log of moments $F^{(q)}$ for $d = 1$ and $d = 2$ for μp collisions [48,47] and e^+e^- annihilation [45,13] (TASSO used a different normalisation than the other groups). Also shown are the predictions from the negative binominal distribution.

In the 2d α-model one can see in which way Eq. (29) arises. For the parameter range relevant to the experimental situation and large N the first term of the sum in Eq. (19) for the moments $\widehat{C}_N^q$ with coefficient $b_2^{(q)}$ yields the dominant contribution. Then all moments have the same dependence on $N(\delta y) \sim \log \delta y_N$ and one finds

$$\log C_N^{(q)} \approx \log a_q + b_2^{(q)} \big(\langle w^2\rangle/2\big)^{N(\delta y)} \tag{30}$$

which is of the form (29). A more general derivation which could be taken over to other models is still missing.

It is interesting to compare these results with other suggestions on the relation of higher moments or, equivalently, of multiparticle correlation functions to the second moment on the two-particle correlation function.

Negative binomial distribution (NB). Multiplicity distributions in full and restricted rapidity ranges can be well fitted by the NB distribution. This was first observed for p$\bar{\text{p}}$ collisions by the UA5 collaboration [80] and subsequently for various other processes [81]. The higher moments of the NB distribution are determined by a recursion relation without free parameters from $F^{(2)}$ (see *e.g.* [71])

$$F^{(q)} = F^{(q-1)}\big(1 + (q-1)(F^{(2)} - 1)\big)\,. \tag{31}$$

In Fig. 6 this relation is indicated by the dashed line, which shows a small curvature. In some cases the NB relation is rather well satisfied, in others not (see also [82,4]). On the theoretical side the NB form has been derived in a simplified description of the QCD parton shower with or without hadronisation [1]. Therefore the similarity of both results (28) and (31) is not surprising. It is an interesting problem to reduce the number of free parameters in the relations (28).

Linked pair approximation. Further insight into the correlation structure has been gained by the observation that the multiparticle correlations can be expressed by the two particle correlations through the "linked pair"-ansatz [57,83] and that the NB distribution is a special case of it [82].

One starts from the n-particle cumulant correlations $K_q(y_1, \ldots y_q)$ or the reduced correlation $k_q(y_1, \ldots, y_n) = K_q(y_1, \ldots, y_q)/\rho(y_1)\ldots\rho(y_q)$. These correlations (for $n = 2 : K_2(y_1, y_2) = \rho_2(y_1, y_2) - \rho_1(y_1)\rho_1(y_2)$) express the nontrivial correlations of a given order; they vanish if any variable becomes statistically independent. In general, K_q can be expressed as a sum of terms involving all distributions $\rho_1(y_1)\cdots\rho_q(y_1, \cdots, y_q)$. The linked pair approximation consists in expressing all k_n by products of 2-particle cumulants k_q, for example,

$$\begin{aligned} k_3(y_1, y_2, y_3) = & \frac{a_3^2}{3}\big(k_2(y_1, y_2)k_2(y_1, y_3) \\ & + k_2(y_1, y_2)k_2(y_2, y_3) + k_2(y_1, y_3)k_2(y_2, y_3)\big), \end{aligned} \tag{32}$$

with one free parameter a_q for each q. For special values of these parameters one obtains the NB. The linked pair approximation is supported by various data including those which are not fit by the NB, but a satisfactory theoretical explanation of this simple structure is not yet known.

4.6. Intermittency and phase transition

So far we have discussed how intermittency can originate from selfsimilar cascade processes. Selfsimilarity is also realised in statistical systems at the critical point of a second order phase transition. The important aspect of this system is that the correlation length ξ diverges and therefore observables show scale invariance for finite length scales $L \ll \xi$. Intermittency effects have been studied for the Ising model which describes the magnetisation of a spin lattice, both by computer simulations [65,67] and analytical calculations [66]. We first give a short outline of the analytical result (for a more detailed introduction see also Hwa [84]) and then discuss a possible application to heavy ion collisions [68], where a phase transition from the quark gluon plasma to a hadron gas could occur.

Ising model. One considers a d-dimensional lattice with R sites in each dimension. With each site we associate a spin variable $s_i = \pm 1,\ i = 1 \cdots R^d$. The Hamiltonian is given by

$$H = -E \sum_{i,j} s_i s_i - B \sum_i s_i \,, \tag{33}$$

where E is the interaction energy of any pair of nearest neighbours and $\sum_{ij}$ goes over all pairs; B is the magnetic field strength.

The intermittency properties at the critical point have been derived analytically by Satz [66]. The lattice is first subdivided into $M = (R/L)^d$ blocks of size L^d. If the temperature approaches the critical value T_c the correlation length diverges and the "blocked" system behaves as in the original model. Then one can introduce the block spin $S_I = \pm 1$ for block I as

$$\sum_{i \epsilon I} s_i / L^d = Q(L) S_I \,, \tag{34}$$

with a scale dependent factor $Q(L)$ and the Hamiltonian can be written in terms of the block spins S_i insted of s_i in (33) with rescaled $E(L)$ and $B(L)$ parameters. For the scale factor one obtains $Q(L) = L^\kappa$ where κ is related to the critical exponent which governs the behaviour of the correlation function of two spins separated by the distance r at $T_c : \Gamma(r, T_c) \sim r^{-2\kappa}$.

Now one defines the moments for the block spins as

$$f_q(L) = < \frac{1}{M} \sum_{\text{blocks}} (\sum_{i=1}^{L^d} s_i/L^d)^q / (\frac{1}{M} \sum_{i=1}^{R^d} s_i/L^d)^q > \,, \tag{35}$$

where the average is taken over all spin configurations. Inserting now, for $T = T_c$, (34) into (35) one obtains the intermittency power law

$$f_q(L) = f_q(l = 1)(1/L)^{\alpha_q} \,, \tag{36}$$

$$\alpha_q = q\kappa \quad (q \text{ even}), \qquad \alpha_q = (q-1)\kappa \quad (q \text{ odd}). \tag{37}$$

For κ one finds 1/8 and 1/2 for 2 and 3 dimensions, respectively.

Quark gluon plasma phase transition. The important point abstracted from the previous considerations is the linear behaviour of the intermittency slopes α_q on q for a second order phase transition. For cascade processes one rather obtains a stronger rise of the slopes, as one can deduce, for example, from Fig. 6 for e^+e^- data which are fitted by the parton cascade model.

Białas and Hwa [68] pointed out therefore that the study of q-dependence of slopes could distinguish between the two possibilities – cascade process or second order phase transition. This assumes that the exponents are not significantly altered in the final evolution phase with $T < T_c$ away from the critical temperature where the above reasoning applies. This seems plausible if long hadronic decay chains are unimportant. If there is a first order phase transition no selfsimilarity is expected and therefore no intermittency.

A compilation of published slope values from linear fits of $\log F^{(q)}$ vs. $\log \delta y$ in limited δy ranges indicate a tendency towards a linear relation ship $\alpha_q \sim (q-1)$ for heaviest nuclei [68]. Recently we determined slope ratios α_q/α_2 from plots $\log F^{(q)}$ vs. $\log F^{(2)}$ as in Fig. 6 using data from the full δy ranges in which a power law is valid. Then all collision process from e^+e^- to heavy nuclei show a rather universal pattern, an exception being the sulfur-emulsion data from the KLM collaboration, in particular for $d = 2$, so the experimental situation is not yet fully conclusive [78]. It will be very interesting to study this new idea further.

5. Summary and Outlook

We summarise what has been learned and could be studied further on the problems listed in the introduction.

(a) The moment analysis has turned out to be a sensitive tool in the study of selfsimilarity in the multiparticle production as well as of hadronisation phenomena. A full exploration of scaling properties requires a 3d analysis in the appropriate variables, only in this case scale invariant cascade models yield a power law of moments (intermittency). Results from e^+e^- annihilation on 1d and 2d analyses are in good agreement with models based on the **QCD parton cascade** with low cut off ($Q_0^2 \lesssim 1\ GeV^2$) at high energies (LEP) and possibly also at lower energies (PETRA). A low hadronisation scale is also suggested by the early onset of KNO scaling.

(b) In the study of hadronisation it appears to be an interesting research project to investigate further the role of **parton hadron duality**, which works for inclusive particle spectra and for jet multiplicities in e^+e^- annihilation. Which observable quantities can be described by the corresponding parton model calculations disregarding hadronisation?

(c) Another challenge are the observed **similarities of the different collision processes** with respect to KNO scaling, rising moments in 1d analysis, stronger increase in 2d analysis, great similarity of all slope ratios α_q/α_2. The 2d moments in μp and hh collisions actually show a power behaviour in a large range of scales which provides the best evidence for selfsimilarity so far. These phenomena suggest that not only in the e^+e^- annihilation process but also in the other, even soft processes, selfsimilar cascades play an important role.

(d) There is the interesting suggestion that also a phase transition from the **quark gluon plasma** to a hadron gas could lead to intermittency, but with characteristics of higher moments different from the case of cascade processes. Whereas most existing data involving nuclei are not very different in this respect from the others (except for some effects in the collisions of the heaviest (sulfur) nuclei) this remains an interesting problem for further investigations.

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