

ANOMALIES AND EXTENDED BRS-TECHNIQUE*

W. KUMMER

Institut für Theoretische Physik, Technische Universität Wien
Wiedner Hauptstrasse 8-10, A-1040 Vienna, Austria

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Anomalies related to an external global symmetry of the gauge-fixed action may well possess relevance in physical observables. Their gauge-(in-)dependence is most suitably controlled by a combination of the BRS-technique which includes a BRS-transformation of the gauge-parameters, with the external symmetry coupled to an external gauge field. For an external symmetry which commutes with BRS, anomalies are shown to be essentially gauge-independent (to one loop order). A nontrivial example is the ghost number anomaly of the bosonic string. However, *e.g.* the anomalies of superconformal transformations do not fall into this category.

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1.

In a quantized gauge theory the Green functions already depend on gauge-parameters. Only physical observables, like the S -matrix, are gauge-independent. Actually, in the latter the gauge-dependence of wave-function renormalization constants for external lines, of "polarizations" and of the (amputated) Green functions can be shown to neatly compensate each other. It must be emphasized that not gauge-invariance, but precisely this gauge-independence is a prerequisite for the physical observable, although not a sufficient condition. Both properties are related, but sometimes in a very sophisticated manner, so that gauge invariance does not necessarily guarantee gauge-independence. A quantized gauge-theory must not be anomalous, *i.e.* must not break the symmetry at the quantum level. In the absence of such "internal" anomalies, however, some "external" (global) symmetry of the action may develop an (external) anomaly in the conservation equation

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of its Noether current. The chiral symmetry with the triangle anomaly [1] is an ancient famous example. As an additional insertion in S -matrix elements it may even lead to physically observable consequences — like the $\pi^0 \rightarrow 2\gamma$ decay rate [2]. Therefore, also such amplitudes should be gauge-independent. In fact, a related proof has been given long ago [3] based upon the elegant “extended” BRS-technique [4].

2.

Recently also the gauge-(in)dependence of the ghost current's anomaly in string theories has attracted interest [5,6]. In the action $L = \int_x (\mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{gb}})$ of the bosonic string ($\alpha = 0, 1$) in a flat background

$$\begin{aligned}\mathcal{L}_{\text{inv}} &= \frac{1}{2} \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^i, \\ \mathcal{L}_{\text{gb}} &= s(\bar{b}_{\alpha\beta} K^{\alpha\beta}) = B_{\alpha\beta} K^{\alpha\beta} - \bar{b}_{\alpha\beta} (sK^{\alpha\beta}),\end{aligned}\quad (1)$$

the general gauge condition $K^{\alpha\beta}(g, \hat{g}, p) = 0$, involving a background field \hat{g} and gauge parameter(s) p , is taken care of by the auxiliary field $B_{\alpha\beta}$. In the “standard” gauge $K^{\alpha\beta} = g^{\alpha\beta} - \hat{g}^{\alpha\beta}$. The harmonic gauge corresponds to

$$K^{\alpha\beta} = n^\alpha \hat{\nabla}_e \sqrt{-g} (g^{e\beta} - \hat{g}^{e\beta}) + \hat{g}^{\alpha\beta} (\sqrt{-g} - \sqrt{-\hat{g}}), \quad (2)$$

where n^α is an “axial” gauge parameter and the covariant derivative $\hat{\nabla}$ is expressed in terms of \hat{g} . In gauges with first derivatives it is useful to move ∂_e as in (2) to $B_{\alpha\beta}$. From the invariance of $\int \mathcal{L}_{\text{inv}}$ under diffeomorphisms and Weyl-scaling, (1) retains the BRS-invariance

$$\begin{aligned}sg^{\alpha\beta} &= -c^\alpha{}_{|\lambda} g^{\lambda\beta} - c^\beta{}_{|\lambda} g^{\lambda\alpha} + c^\lambda g^{\alpha\beta}{}_{|\lambda} - cg^{\alpha\beta}, \\ s\bar{b}_{\alpha\beta} &= B_{\alpha\beta}, \\ sB_{\alpha\beta} &= 0,\end{aligned}\quad (3)$$

with all other fields $y = (X^i, c^\alpha, c)$ transforming like scalars:

$$sy = c^\lambda y_{|\lambda}. \quad (4)$$

The conservation of Faddeev-Popov (F.P.) ghost number (c is now the pair (c^α, c))

$$\tau^{(c)} = \int_x \left(c \frac{\delta}{\delta c} - \bar{b} \frac{\delta}{\delta \bar{b}} \right) \quad (5)$$

in (1) is broken by quantum mechanical corrections. In the standard gauge the well-known result [5,6] for the Noether current of (5) is

$$\partial_\mu \mathcal{J}^\mu = -\frac{3}{4\pi} \sqrt{-\hat{g}} R(\hat{g}). \quad (6)$$

Not surprisingly, this need not be true in other gauges, because \mathcal{J}^μ is not BRS-invariant. As shown, however, in [6] the action (1) may be effectively replaced by

$$\begin{aligned} L^{\text{eff}} &= \hat{L}_{\text{inv}} + \int_x \left(B_{\alpha\beta} \hat{K}_{e\sigma}^{\alpha\beta} h^{e\sigma} - \bar{b}_{\alpha\beta} \hat{K}_{e\sigma}^{\alpha\beta} \hat{\sigma}^{e\sigma} \right), \\ h^{e\sigma} &= g^{e\sigma} - \hat{g}^{e\sigma}, \\ \hat{K}_{e\sigma}^{\alpha\beta} &= \partial K^{\alpha\beta} / \partial g^{e\sigma} \Big|_{g=\hat{g}}, \end{aligned} \quad (7)$$

where $\hat{\sigma}^{\alpha\beta}$ is given by the r.h.s. of the first Eq. (3) with $g^{\alpha\beta} \rightarrow \hat{g}^{\alpha\beta}$. (7) shows a linearized BRS-invariance

$$s^{\text{eff}} = \int_x \left(B \frac{\delta}{\delta \bar{b}} + \hat{\sigma} \frac{\delta}{\delta h} \right), \quad (8)$$

which commutes with a $U(1)$ -invariance

$$\tau = \frac{(c)}{\tau} + \int_x \left(h \frac{\delta}{\delta h} - B \frac{\delta}{\delta B} \right). \quad (9)$$

The generalization of the ghost current, the “Rebhan-Kraemmer” (RK) current corresponding to (9) exhibits a gauge-independent anomaly [6].

3.

A general argument, yielding a compact proof of the gauge independence of an “external” anomaly seems to be missing. The generally used technique [7] either considers internal and external anomalies separately, or, even in an application of the “extended” BRS-technique [3] only makes statements for vanishing sources of the internal fields. The main line of our argument is this [8]: With the shorthand notation $\Phi_A = (\phi, c, \bar{b}, B)$ for the fields, the right and left F.P.-ghost, and the auxiliary field B , the generating functional $W(j, k, \bar{\varphi}, \beta, z)$ is written as

$$W = \int_{(\phi)} \exp i \left(\tilde{L} + \int_x j_A \Phi_A \right), \quad (10)$$

$$\tilde{L} = \tilde{L}_{\text{inv}} + \int_x [(\tilde{s} + \tilde{t} + d)\tilde{b}\tilde{K} + k_A(\tilde{s} + \tilde{t})\Phi_A]. \quad (11)$$

In (11) we have generalized the original BRS-operation in the gauge-breaking term by adding not only a BRS-variation of the gauge-parameter ($d = z \frac{\partial}{\partial \beta}$, $z^2 = 0$ for one parameter) [4], but also by an (anticommuting) symmetry operator

$$\tilde{t} = \int_x \tilde{t}_A(\Phi, \tilde{\varphi}) \frac{\delta}{\delta \Phi_A} + \int_x \tilde{t}_a(\tilde{\varphi}) \frac{\delta}{\delta \tilde{\varphi}_a}. \quad (12)$$

\tilde{t} is related to a certain "external" global symmetry τ ($[\tau, s] = 0$) of the original L by the introduction of an external ghost \tilde{u}^i as $\tilde{t}_A = \tilde{u}^i(x)\tau_A^i$, and the original symmetry τ is "gaugified" i.e. whenever a derivative acts on a field participating in a symmetry τ , this derivative is replaced by a covariant one $\partial_\mu \rightarrow \tilde{D}_\mu = D_\mu(\tilde{\varphi})$, involving a new external gauge field $\tilde{\varphi}$. In order to maintain $\tilde{t}^2 = 0$ for nonabelian fields, \tilde{t}_a with $\tilde{\varphi}_a = (\tilde{\varphi}, \tilde{u}^i)$ acts in an appropriate manner also on \tilde{u}^i . Gaugifying L to $\tilde{L}(\Phi, \tilde{\varphi})$ may include even $(s\Phi_A)$ as well, if, as in the example of strings above, $L_{gb} \rightarrow \tilde{L}_{gb}$ participates in the symmetry. Then besides $\tilde{t}\tilde{L}_{\text{inv}} = 0$ even $\tilde{t}\tilde{b}\tilde{K} = 0$, but this is not necessary for our argument. We require only $\tilde{s}, \tilde{t} = 0$, i.e. $(\tilde{s} + \tilde{t})^2 = 0$ for the specific external symmetry to be true even after this "gaugification" has been completed. Hence also sources k_A are introduced for $\tilde{s} + \tilde{t}$.

Changing the variables ϕ_A by $\delta\phi_A = \delta\lambda(\tilde{s} + \tilde{t})\phi_A = \delta\lambda\hat{s}\phi_A$ leaves (11) invariant, if the simultaneous variation of the external fields $\tilde{\varphi}_a$ by \tilde{t} is subtracted out (the grading of Φ_A and j_A is denoted by the same symbol $A = 0, 1$)

$$0 = \int_{(\phi)} \int_x \left[(-1)^{A+1} j_A \hat{s}\Phi_A + d(\hat{s}\tilde{b}\tilde{K}) + \tilde{t}_a \frac{\delta}{\delta \tilde{\varphi}_a} \right] \exp i \left(\tilde{L} + \int_x j_A \Phi_A \right). \quad (13)$$

This presupposes, of course an invariant measure and an invariant regularization. Replacing $d \int_x \hat{s}\tilde{b}\tilde{K} = d(\tilde{L} + \int_x j_A \phi_A)$ the "symmetry-extended" Slavnov-Taylor identity follows:

$$\int \left[(-1)^{A+1} j_A \frac{\delta}{\delta k_A} + \tilde{t}_a \frac{\delta}{\delta \bar{\varphi}_a} \right] W + dW = 0. \quad (14)$$

The usual Legendre transform to the 1-p-i-functional $\Gamma(\underline{\Phi}, k, \bar{\varphi}, p, z) = Z - \int_x j_A \underline{\phi}_A$, $W = \exp i Z$, $\underline{\phi}_A = \delta Z / \delta j_A$, $j_A = (-1)^{A+1} \delta \Gamma / \delta \underline{\phi}_A$ yields ($\underline{\phi}_A \rightarrow \phi_A$ from now)

$$\mathcal{B}(\Gamma) = \int_z \left(\frac{\delta \Gamma}{\delta \Phi_A} \frac{\delta \Gamma}{\delta k_A} + \tilde{t}_a \frac{\delta \Gamma}{\delta \bar{\varphi}_a} \right) + d\Gamma = \mathcal{A}, \quad (15)$$

where in the r.h.s. of this "Lee-identity" already a candidate anomaly \mathcal{A} has been included, which appears if no covariant regularization is employed. Defining

$$\mathcal{B}_X = \int_z \left(\frac{\delta X}{\delta \Phi_A} \frac{\delta}{\delta k_A} + \frac{\delta X}{\delta k_A} \frac{\delta}{\delta \Phi_A} + \tilde{t}_a \frac{\delta}{\delta \bar{\varphi}_a} \right) + d \quad (16)$$

and using the identity $\mathcal{B}_X \mathcal{B}(X) = 0$ leads to a consistency relation [7,9] for the anomaly:

$$\mathcal{B}_\Gamma \mathcal{A} = 0. \quad (17)$$

(15) and (17) determine the **internal** (BRS-identity) and **external** (Ward identities) symmetries of the theory and their anomalies in a comprehensive manner. To lowest loop order and after expansion in z (17) becomes

$$\mathcal{B}_L \mathcal{A} = (\mathcal{B}^0 + z \mathcal{B}^1)(\mathcal{A}^0 + z \mathcal{A}^1) = 0. \quad (18)$$

The term of $\mathcal{O}(z)$ expresses the gauge-dependence of \mathcal{A}^0 :

$$\frac{\partial \mathcal{A}^0}{\partial p} = -\mathcal{B}^1 \mathcal{A}^0 - \mathcal{B}^0 \mathcal{A}^1. \quad (19)$$

Eqs. (15)-(17) may be simplified, because the BRS-transformation of \bar{b} is linear and $sB = 0$ so that k_3 and k_4 , the sources for $(\hat{s}\bar{b})$ and $(\hat{s}B)$, may be eliminated if $t\bar{b} = tB = 0$. Repeating the argument yields

$$\mathcal{B}(\Gamma) = \int_z \left(\frac{\delta \Gamma}{\delta \Phi_A} \frac{\delta \Gamma}{\delta k_A} + B \frac{\delta \Gamma}{\delta \bar{b}} + \tilde{t}_a \frac{\delta \Gamma}{\delta \bar{\varphi}_a} \right) + d\Gamma = \mathcal{A}, \quad (20)$$

where the sum $a = 1, 2$ extends over ϕ and c only. A further simplification is possible for linear gauges $\tilde{K} = \tilde{F}(\bar{\varphi}, p)\varphi$. Then the "equations of motion" for B and \bar{b} produce two relations between the derivatives in Γ and

$$\hat{\Gamma} = \hat{\Gamma}(\phi_a, \hat{k}_1, k_2, \bar{\varphi}, p, z) = \Gamma - \int_z (B\bar{F}\varphi + d\bar{b}\bar{F}\varphi),$$

$$\hat{k}_1 = k_1 - F\bar{b} \quad (21)$$

fulfills a relation like (20), but without the term linear in B . In this case $\mathcal{B}^1 = 0$ in the nilpotent (for $\mathcal{B}(X) = 0$) operator defined by analogy to (16) and the content of (19) is that gauge-dependent terms may be removed by local counterterms $\int \mathcal{A}^1 dp$, the locality being guaranteed by the "action principle" [7]. For nonlinear gauges this statement must be somewhat restricted [8].

It is instructive to study the interplay between consistency equations for an external symmetry which affects only fields, not contained in the gauge condition — as the chiral fermions for the chiral symmetry. Then $t\bar{b}K = 0$ trivially and the external ghost \tilde{u}^α only resides in the last term of (11) as $\int \tilde{u}^\alpha t_{AB}^\alpha \phi_B$. For (10) this yields the additional identity

$$\left[\frac{\delta}{\delta \tilde{u}^\alpha} + (-1)^A k_A t_{AB}^\alpha \frac{\delta}{\delta j_B} \right] W = 0 \quad (22)$$

which implies (after Legendre-transform) that

$$\Gamma = \tilde{\Gamma}(\Phi, k, \bar{\varphi}_a) + \int_z k_A \tilde{u}^\alpha t_{AB}^\alpha \Phi_B, \quad (23)$$

where φ_a is the external gauge-field. The symmetry-extended Lee-identity (15) and its consistency equation for the anomaly (17) then decompose as

$$\check{B}(\tilde{\Gamma}) = \check{A}, \quad (24a)$$

$$\mathcal{D}^\alpha \tilde{\Gamma} = \check{A}^\alpha, \quad (24b)$$

and

$$\check{B}_F \check{A} = 0, \quad (25a)$$

$$\mathcal{B}_F \check{A}^\alpha = \mathcal{D}^\alpha \check{A}, \quad (25b)$$

$$\mathcal{D}^\alpha \check{A}^\beta - \mathcal{D}^\beta \check{A}^\alpha = f_{\alpha\beta\gamma} \check{A}^\gamma, \quad (25c)$$

where

$$\mathcal{D}^\alpha(x) = \tilde{D}_a^\alpha \frac{\delta}{\delta \bar{\varphi}_a(x)} + t_{AB}^\alpha \left(\Phi_B(x) \frac{\delta}{\delta \Phi_A(x)} - k_A \frac{\delta}{\delta k_B} \right) \quad (26)$$

contains the gauge-transform $\tilde{D}_a^\alpha = \partial_a^\alpha + t_{ab}^\alpha \tilde{\varphi}_b$ with respect to the external field, but also a term which is characteristic for a local version of the external symmetry. $f_{\alpha\beta\gamma}$ are the structure constants of t . $\tilde{B}(\tilde{I})$ and \tilde{B}_F are defined by analogy to (15) and (16) but without the term proportional to \tilde{t} , they are the "extended" but not "symmetry extended" operators. The simple calculation leading to (24) and (25) trivially implies the decomposition

$$\mathcal{A} = \tilde{\mathcal{A}} + \int_z \tilde{u}^\alpha \tilde{\mathcal{A}}^\alpha, \quad (27)$$

i.e. the external ghost appears only linearly. (25c) means that $\tilde{\mathcal{A}}^\alpha$ is the "consistent" anomaly, because for vanishing "internal" fields $\phi = k_A = 0$, (26) coincides with the gauge-derivative in the classical Wess-Zumino consistency condition [9], and (24b) represents the associated "anomalous Ward-identity". From the BRS-formalism it is clear that there is no necessity to search for a "covariant" anomaly, more important for a physical observable is the gauge-dependence which is controlled by (25b). *E.g.* for an anomaly with $\tilde{\mathcal{A}} = 0$ and $\delta\tilde{\mathcal{A}}^\alpha/\delta k = 0$ (25b) becomes at one loop order

$$B_L \tilde{\mathcal{A}}^\alpha = (\tilde{s} + d) \tilde{\mathcal{A}}^\alpha, \quad (28)$$

i.e. the consistent anomaly in that case is BRS-invariant ((28) at $z = 0$) and gauge-independent (28) at $\mathcal{O}(z)$). It should be stressed, however, that our formalism is very general and covers much more than this special situation.

4.

Our string example falls into the range of this theorem; actually the "Lee-identity" is even simpler here because of the linearity of (8) [10]. Gauging of the $U(1)$ symmetry (9) means

$$\tilde{t} = \int_z \tilde{u}(x) \left(c \frac{\delta}{\delta c} - \bar{b} \frac{\delta}{\delta \bar{b}} + h \frac{\delta}{\delta h} - B \frac{\delta}{\delta B} \right) + \int_z \tilde{u}_{|\mu} \frac{\delta}{\delta \tilde{\varphi}_\mu}, \quad (29)$$

and the derivatives on (c, h) and (\bar{b}, B) must be replaced by $\partial_\mu \rightarrow \partial_\mu + \tilde{\varphi}_\mu$ and $\partial_\mu \rightarrow \partial_\mu - \tilde{\varphi}_\mu$, respectively, i.e. $\sigma \rightarrow \tilde{\sigma}$ in (8). The gauge condition is effectively linear according to (7) and thus the general argument of section 3 applies although the simplification from $t\bar{b} = tB = 0$ is obviously not true here. This is offset by the linearity of (8) which allows $k_A = 0$ in (11) and leads to a trivial linear Lee-identity $(\tilde{s} + d)\mathcal{A} = 0$ [10]. In Ref. [6] this result was obtained by a special argument.

For the chiral anomaly $t\bar{b}K = 0$ because the Dirac-fields do not appear in a reasonable gauge-condition. Less trivial applications may occur, however, if the gauge-condition involves the scalar fields (*c.f.* the t'Hooft gauge) which in turn, may be affected by the external symmetry [8].

An interesting application of our argument occurs in the generalization of chiral symmetry to the supersymmetric (SUSY) Yang-Mills theory. The chiral fermion fields Ψ_{\pm} are here a component of a chiral superfield

$$\phi_{\pm} = (S_{\pm}, \Psi_{\pm}, F_{\pm}), \quad (30)$$

which also contains complex scalars S_{\pm} and auxiliary fields F_{\pm} . A chiral transformation

$$\delta\phi_{\pm} = \pm i\delta_{\alpha}\phi_{\pm} \quad (31)$$

affects all these components in the same manner. As can be seen easily, it also commutes with the SUSY-BRS-transform

$$s\phi_{\pm} = ic_{\pm}\phi_{\pm}, \quad (32)$$

where $c_{+} = c_{-}^{\dagger}$ is the (chiral) F.-P. ghost in the appropriate representation of Φ_{\pm} . Since the requirements for our arguments are met, this SUSY generalization of chiral symmetry has an essentially gauge-independent anomaly, which implies here that the anomaly is independent of the way in which we have fixed the components of the vector superfield. *E.g.* in the Wess-Zumino gauge, the ordinary gauge-field and the "gaugino" may be used to calculate the anomaly.

The situation is different for superconformal transformations which comprise besides the conformal ones also chiral transformations and the SUSY-transformations themselves. The corresponding anomaly, in turn, is composed of the chiral and of the conformal anomaly and the anomalous relation may be written in terms of superfields [11]. However, chiral transformations are generated here by the so called \mathcal{R} operator ($\Theta_{\alpha}, \alpha = 1, 2$ are Weyl-spinor Grassmann variables spanning the super-part of superspace)

$$\mathcal{R} = \theta^{\alpha} \frac{\partial}{\partial \theta^{\alpha}} + \bar{\theta}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \quad (33)$$

as

$$\delta\phi_{\pm} = i\delta\beta\mathcal{R}\phi_{\pm} \quad (34)$$

with

$$\mathcal{R}(S_{\pm}, \Psi_{\pm}, F_{\pm}) = (0, \pm\Psi_{\pm}, \pm 2F_{\pm}), \quad (35)$$

so that $[\mathcal{R}, s] \neq 0$ may be shown. As suggested by a simple toy example [8] in such cases not only our argument for gauge-independence does not apply, but even gauge independence is lost. The calculation of anomalies in different SUSY-gauges can thus be expected to yield different results. Thus we do not see a necessity to insist on a “SUSY Adler-Bardeen theorem” which would require the coefficients of the conformal and of the chiral anomaly to coincide. It is interesting that discrepancies precisely on that point are the origin of a long-standing controversy [12].

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