THE S-MATRIX AND ITS POLE STRUCTURE FOR THE RELATIVISTIC LEE MODEL*

E. J. J. KIRCHNER, TH. W. RUIJGROK

Instituut voor Theoretische Fysica
Princetonplein 5, P.O. Box 80.006
3508 TA Utrecht, The Netherlands

(Received September 10, 1990)

The $N\theta$-sector of the Lee model is studied in the framework of a relativistic theory, in which also the recoil of the N- and V-particle are taken into account. For several values of the mass ratios the s-wave phase shift and total cross section are calculated as functions of the energy. The Riemann surface of the scattering amplitude as a function of the complex Mandelstam variable s is investigated. It is found to have an infinite number of sheets. The poles of the S-matrix on the first (physical), second and third sheet are calculated and found to describe orbits of a peculiar form when the coupling constant is varied. In order to compare these results with nonrelativistic potential scattering the same quantities were first calculated for the δ-shell potential. It is concluded that even this simple relativistic field theory and the nonrelativistic field theory have very different characteristics.

PACS numbers: 11.20.Fm

1. Introduction

There is no proof of the consistency of quantum field theory and it will certainly not be given in this paper. It is, however, not unreasonable to hope that by studying simple models, some insight in this problem might be gained.

For that purpose the Lee model [1] $N+\theta \rightleftharpoons V$ was revived and extended to include a relativistic description of the N- and V-particle [2]. It was shown that for sufficiently strong coupling the V-particle turns into a tachyon. Also the phase shift for N-θ scattering changes so rapidly with energy that it

* Presented at XXX Cracow School of Theoretical Physics, Zakopane, Poland, June 2–12, 1990.

becomes inconsistent with the requirement of causality. Since causality is related to the analyticity of the S-matrix it seemed worthwile to find the analytical continuation of the scattering amplitude in this extended Lee model. The results are given in the next Section.

In order, however, to compare them with nonrelativistic scattering, the remaining part of this introduction will be devoted to a discussion of s-wave scattering by a δ-shell potential,

$$V(r) = aV_0\delta(r-a). \tag{1}$$

For arbitrary angular momentum the scattering amplitude $M_l(k, k')$ is obtained from

$$M_l(k,k') = \lim_{z \to k'^2a^2+i\varepsilon} M_l(k,k'|\,z), \tag{2}$$

where $M_l(k,k'|\,z)$ satisfies the Lippmann-Schwinger equation

$$M_l(k,k'|\,z) = W_l(k,k') - 8\pi ma^2 \int_0^\infty \frac{k''^2\,dk''}{a^2k''^2 - z} W_l(k,k'')\,M_l(k'',k'|\,z) \tag{3}$$

and

$$\begin{aligned} W_l(k,k') &= \frac{1}{2\pi^2}\int_0^\infty dr\, r^2 j_l(kr) j_l(k'r) V(r) \\ &= \frac{ag}{8\pi m} j_l(ka) j_l(k'a) \end{aligned} \tag{4}$$

with the dimensionless "coupling constant" defined by

$$g = \tfrac{4}{\pi} ma^2 V_0. \tag{5}$$

Because of the product form of $W_l(k,k')$ Eq. (3) can be solved immediately. Writing $M_l(k,k'|\,z') = j_l(ka)j_l(k'a)T_l(z')$ one obtains

$$T_l(z) = \frac{a}{8\pi m}\left[\frac{1}{g} + \int_0^\infty \frac{u^2 j_l^2(u)}{u^2-z}\,du\right]^{-1}. \tag{6}$$

For s-waves the integral is equal to

$$I(z) = \int_0^\infty \frac{\sin^2 u}{u^2 - z}\,du = \frac{\pi}{4\sqrt{-z}}\left(1 - e^{-2\sqrt{-z}}\right) \tag{7}$$

in agreement with Eq. (15.15) of reference [3], where the same model is discussed. From this explicit expression of $T_0(z)$ it is seen that the scattering amplitude is a meromorphic funcion in the complex plane, cut along the positive real axis and with at most one pole on the negative real axis. This pole really exists if

$$-\frac{1}{g} < \int_0^\infty \frac{\sin^2 u}{u^2}\, du = \frac{\pi}{2}$$

i.e. if $g < -2/\pi$. Its position — divided by $2ma^2$ — gives the energy of the bound state with zero angular momentum. The same situation, *i.e.* the right hand (unitary) cut and a number of poles on the negative real axis, is found for other models [4,5], although a special class of potentials, known as Yukawa-type potentials, has additional cuts along the negative real axis [6].

From the explicit form Eq. (7) it is seen that $z = 0$ is a square root branch point. Using the discontinuity of $I(z)$ across the cut (which can easily be obtained from the integral representation Eq. (7)), we choose the following representation for the analytical continuation of $I(z)$ to the second sheet (in this representation the Riemann surface on which the analytical continuation of $I(z)$ is defined has only two sheets):

$$I_2(z) = I(z) \mp \frac{i\pi}{\sqrt{z}} \sin^2 \sqrt{z} \quad \text{for } \operatorname{Im} z \gtrless 0\,. \tag{8}$$

This expression is used to find the poles of the scattering amplitude in the second sheet of the Riemann surface. They are defined as the solutions of $1/g + I_2(z) = 0$.

There is one real solution when g is in the interval $-2/\pi < g < 0$, which was called anti-bound state. For g decreasing below $-2/\pi$ this pole moves through the branchpoint $z = 0$ to the first sheet, to become a real bound state pole. All other poles in the second sheet lie in the complex plane, as is shown in Fig. 1. For $|g| \ll 1$ they are very far removed from the values of z on the upper rim of the cut, which corresponds to the energies in a scattering process. Only for $|g| \gg 1$ the poles approach the real axis and give rise to the well known resonances. For $|g| = \infty$ these poles turn into real bound states at $z_n = n^2\pi^2$ $(n = 1, 2, \ldots)$.

The structure of the pole orbits as shown in Fig. 1 is similar to what is found for other potentials. The purpose of showing this figure, therefore, is not to present a new result, but to be able to contrast it with the case of a simple field theory, which we will discuss now.

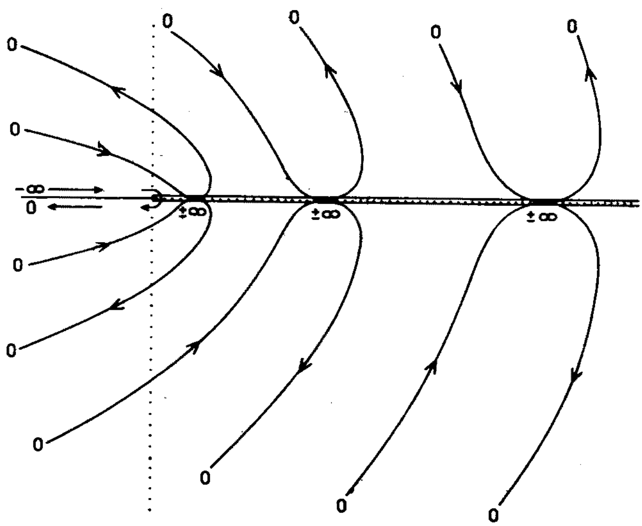

Fig. 1. Resonance and bound state poles for the δ-shell potential.

2. The relativistic Lee model

In the Lee model [1] the only allowed transition is between a V-particle with bare mass M_V and an (N, θ)-state with an N-particle with mass M_N and a θ-particle with mass m. No recoil of the N- and V-particle took place, because their energy did not depend on their momentum. This situation will be remedied by adopting a previously published theory [7], by which it is possible to calculate the relativistic scattering of two or more particles.

In this quasi potential theory the scatternig amplitude $M_{\beta\alpha}$ for a transition from the state α to the state β is given by

$$M_{\beta\alpha} = \lim_{s_0 \to s+i\epsilon} M_{\beta\alpha}(s_0) \quad \text{with} \quad s = P_\alpha^2 = P_\beta^2 . \tag{9}$$

The functions $M_{\beta\alpha}(s_0)$ must be solved from the following Lorentz invariant Lippmann-Schwinger equation

$$M_{\beta\alpha}(s_0) = V_{\beta\alpha} - \int_\gamma V_{\beta\gamma} L^{(3)}(\vec{v}_\gamma, \vec{v}_\alpha) L_\gamma^{(0)}(s_0) M_{\gamma\alpha}(s_0) \tag{10}$$

with equal velocities $\vec{v}_\alpha = \frac{\vec{P}_\alpha}{P_\alpha^0} = \vec{v}_\beta = \frac{\vec{P}_\beta}{P_\beta^0}$ for the initial and final states α and β. Also in the intermediate states γ the total velocity is conserved as can be seen from the form we choose for the propagator of these states, namely

$$L_{\gamma}^{(0)}(s_0) = \frac{2}{P_{\gamma}^2 \left(P_{\gamma}^2 - s_0\right)},$$

$$L^{(3)}(\vec{v}_{\gamma}, \vec{v}_{\alpha}) = \left(1 - \vec{v}_{\gamma}^2\right)\left(1 - \vec{v}_{\alpha}^2\right)\delta_3(\vec{v}_{\gamma} - \vec{v}_{\alpha}) \,. \tag{11}$$

Both expressions are invariant under Lorentz transformations.

The idea of velocity conservation has recently been used [8] to construct an effective Lorentz invariant field theory for heavy quarks. The theory formulated in this way is finally completely specified by making a choice for the matrix elements $V_{\beta\alpha}$ of the potential. In order to be as close as possible to the static Lee model, we take for the transition from $\alpha = \left(N, \vec{p}\right) + \left(\theta, \vec{k}\right)$ to $\beta = \left(V, \vec{q}\right)$ the following invariant function

$$V_{\beta\alpha} = \frac{gm^2}{\sqrt{(q+k)^2}} = V_{12}(q\,|p,k) = V_{21}(p,k|\,q)\,, \tag{12}$$

where g is the dimensionless coupling constant and the θ-mass is denoted by m.

With this choice of the potential Eq. (10) reduces to two coupled integral equations for the amplitudes $M_{12}(q\,|p,k)$ and $M_{22}(p',k'|\,p,k)$ for the transitions $\left(N,\vec{p}\right)+\left(\theta,\vec{k}\right) \to \left(V,\vec{q}\right)$ and $\left(N,\vec{p}\right)+\left(\theta,\vec{k}\right) \to \left(N,\vec{p'}\right)+\left(\theta,\vec{k'}\right)$ respectively.

Substitution of the first into the second equation gives the following equation for the elastic N-θ scattering amplitude:

$$M(p',k'|\,p,k) = W_{s_0}(p',k'|\,p,k) - \int dp_1\,dk_1\,\delta\left(k_1^2 - m^2\right)\theta\left(k_1^0\right)\delta\left(p_1^2 - M_N^2\right)$$
$$\times\,\theta\left(p_1^0\right)\frac{2\delta_3(\vec{v}_1)}{s_1(s_1 - s_0)}W_{s_0}(p',k'|\,p_1,k_1)\,M(p_1,k_1|\,p,k)\,, \tag{13}$$

where the "optical" potential W_{s_0} is defined by

$$W_{s_0}(p',k'|\,p,k)$$
$$= -\int dq_2\,\delta\left(q_2^2 - M_V^2\right)\theta\left(q_2^0\right)\frac{2\delta_3(\vec{v}_2)}{s_2(s_2 - s_0)}V_{21}(p',k'|\,q_2)\,V_{12}(q_2|\,p,k) \tag{14}$$

and $s_1 = (p_1 + k_1)^2$, $s_2 = q_2^2$, $\vec{v}_1 = \frac{\vec{p}_1 + \vec{k}_1}{p_1^0 + k_1^0}$, $\vec{v}_2 = \frac{\vec{q}_2}{q_2^0}$.

With V_{12} and V_{21} as given in Eq. (12) W_{s_0} becomes

$$W_{s_0}(s'|s) = -\frac{g^2 m^4}{\sqrt{s's}\,(M_V^2 - s_0)};$$

$$s' = (p' + k')^2; \quad s = (p + k)^2. \tag{15}$$

This is a separable potential (with only s-wave scattering) and for that reason Eq. (13) can be solved exactly, resulting in

$$M(s'|s) = \frac{g^2 m^4}{\sqrt{s's}} \frac{1}{g^2 m^2 B(s_0) - (M_V^2 - s_0)} \tag{16}$$

with

$$B(s_0) = \frac{\pi m^2}{2} \int_{s_+}^{\infty} \frac{\sqrt{\lambda(s, M_N^2, m^2)}}{s^2 (s - s_0)} ds,$$

$$s_\pm = (M_N \pm m)^2, \quad \lambda(s, M_N^2, m^2) = (s - s_+)(s - s_-). \tag{17}$$

The formulae for the phaseshift and total cross section, which can be derived from this amplitude, have been given in Ref. [2] and will not be repeated here.

Instead, a more detailed study will be made of the analytic properties of $M(s'|s)$ as a function of the complex energy variable s_0. As in the non-relativistic theory, bound states can be identified with poles on the real axis. This means that the value of s^*, which satisfies the equation

$$M_V^2 - s^* - g^2 m^2 B(s^*) = 0, \tag{18}$$

is equal to the square of the mass of the physical V-particle, $s^* = M_{V^*}^2$. From Eq. (17) it is seen that if s_0 is real and $s_0 > s_+$, the function $B(s_0 + i\varepsilon)$ has a non-vanishing imaginary part. Therefore, if Eq. (18) has a real solution at all, which will happen for sufficiently large g^2, the mass M_{V^*} will necessarily be less than the threshold value $M_N + m$ for decay into an (N, θ)-state.

In the original Lee model [1] divergences occurred, which could be hidden by introducing a renormalized coupling constant. Although in the present case all integrals converge, it is nevertheless possible to define a renormalized coupling constant g_R by calculating the transition matrix element of the interaction between the V^*-state and an (N, θ)-state, and putting the result equal to the expression in Eq. (12), but with g replaced by g_R. In this way it is found that

$$g_R^2 = \frac{g^2}{1 + g^2 m^2 \left(\frac{\partial B(s)}{\partial s}\right)_{s\,=\,s^*}} \tag{19}$$

which can be also inverted to give

$$g^2 = \frac{g_R^2}{1 - g_R^2 m^2 \left(\frac{\partial B(s)}{\partial s}\right)_{s\,=\,s^*}} . \tag{20}$$

From the wave function of V^* the probability for the bare V-particle to occur can be calculated. The result is

$$\begin{aligned} \Pr(V) &= \left[1 + g^2 m^2 \left(\frac{\partial B(s)}{\partial s}\right)_{s\,=\,s^*}\right]^{-1} \\ &= \frac{g_R^2}{g^2} = 1 - g_R^2 m^2 \left(\frac{\partial B(s)}{\partial s}\right)_{s^*} \end{aligned} \tag{21}$$

Since $\partial B(s)/\partial s > 0$ for $s < s_+$ this probability is positive and less than one, as it should. Also $B(s) > 0$ for $s < s_+$, so that according to Eq. (18) $s^* < M_V^2$; due to the coupling there is more binding and the V-particle gets lighter.

There is, however, also another point of view, in which the V^*-particle is considered as elementary, together with the N- and θ-particle, and in which it is illegitimate to ask questions about the occurence of the bare V-particle. In that case there is no need for $\Pr(V)$, as given in Eq. (21), to be less than one and g^2 and g_r^2 are allowed to become negative. Then Eq. (18) implies that s^* may be larger than M_V^2 and there might be even a second solution $s^{*\prime}$ to Eq. (18) with $s^{*\prime} < s_+$. Since in this case the number of stable bound states has increased by one, as compared to zero coupling case, this should be reflected in the behaviour of the phase shift. According to Levinson's theorem the decrease in phase shift from zero to infinite energy should then be equal to π. Fig. 2 shows that this is indeed the case. In the three pictures g^2 equals -20, -23.2 and -32 respectively. These values correspond to the situations in Fig. 4 of first one, then two poles in the first sheet on the negative real axis, and finally two poles moving into the complex plane of the first sheet.

The argument above shows that for some values of g^2 the scattering amplitude has two poles. These poles will move when g^2 is changed and

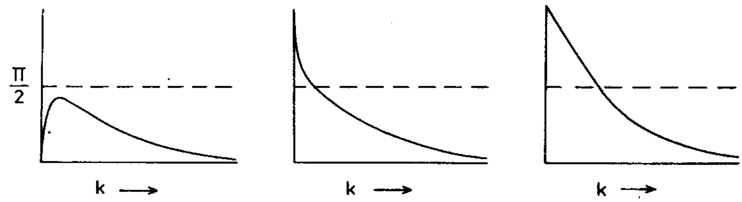

Fig. 2. Phase shift for $M_N = 2m$ and $M_V = 1.1m$, as function of k/M_V with $k = \frac{1}{2}\sqrt{\lambda/s}$.

although they may collide, they will not disappear. In order to follow their trace, however, it is necessary first to give the analytic continuation of

$$B_1(z) \equiv B(z) = \int_{s_+}^{\infty} \frac{f(s)}{s-z}\, ds\,,$$

$$\text{with } f(s) = \frac{\pi m^2}{2s^2}\sqrt{(s-s_+)(s-s_-)}\,. \tag{22}$$

$B_1(z)$ is analytic in the whole complex plane cut along the real axis from s_+ to $+\infty$, the unitary cut. The discontinuity across the u-cut is purely imaginary and is given by

$$\lim_{\varepsilon\to 0} \mathrm{Im}\ B_1(s \pm i\varepsilon) = \pm\pi f(s) \quad \text{for } s > s_+\,. \tag{23}$$

The Riemann surface on which the analytic continuation of $B_1(z)$ is defined, has an infinite number of sheets. Except for the function $B_1(z)$ on the physical sheet, the function $B_n(z)$ on the n^{th} sheet $(n \geq 2)$ has two cuts. In addition to the u-cut, there is a cut from s_- to $|z| = \infty$. We have chosen to put this cut along the negative real axis, and call it the l-cut. The function $B_{2k}(z)$ $(k = 1, 2, \ldots)$ is continuously connected to $B_{2k-1}(z)$ along the u-cut and to $B_{2k+1}(z)$ along the l-cut. These statements can easily be verified using the following representation of the functions $B_n(z)$:

$$B_{2k}(z) = B_1(z) \pm 2\pi k i f(z) \text{ for } \mathrm{Im}\ z \gtrless 0\,,$$

$$B_{2k+1}(z) = B_1(z) \pm 2\pi k i f(z) \text{ for } \mathrm{Im}\ z \gtrless 0\,. \tag{24}$$

In Fig. 3 a number of paths are drawn, which show how the sheets are connected through the l- and u-cuts. Here $f(z)$ is defined as that analytical continuation of $f(s)$, which has a cut from s_- to s_+ with a positive (negative) imaginary value on the upper (lower) part of the cut and with the real

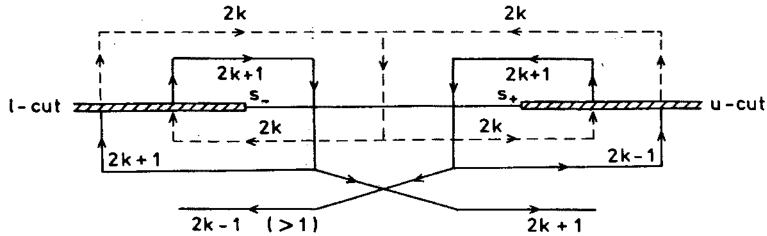

Fig. 3. Cut structure of the Riemann surface. The physical sheet has only the u-cut and not the l-cut.

positive (negative) values on the real axis for $z > s_+$ ($z < s_-$). It can be shown that the branch points s_+ and s_- are of the square root type and that $z = \infty$ is a logarithmic branch point. With the function $B(z)$ defined on the whole Riemann surface it is now possible to determine the position of the poles of the scattering amplitude, given by Eq. (16), on each sheet. They are solutions of the equation

$$M_V^2 - z - g^2 m^2 B_n(z) = 0\,. \tag{25}$$

By changing the integration variables in the integral $B_n(z)$ to $t = \sqrt{\dfrac{s - s_+}{s - s_-}}$ and decomposing the resulting integrand into rational functions, the function $B_n(z)$ could be calculated explicitly. The roots of Eq. (25) were then determined using a Newton-Raphson method for two real variables. With some confidence it can be said that in this way all roots were found. The results are collected in Figs 4 and 5. In both cases $M_N/m = 2$, but in Fig. 4 all values of M_V^2 are less than s_+, while in Fig. 5 $M_V^2 > s_+$. The curves shown are the paths followed by the poles when g^2 varies from $-\infty$ to $+\infty$. The value of g^2 for which $z = s_+$ is denoted by g_b^2 (b for bound). In Fig. 4 $g_b^2 < 0$ and in Fig. 5 $g_b^2 > 0$. The point $z = s_-$ is reached in all sheets for the same value of g^2, and is called g_a^2. Its value is positive in all cases that are shown. The expressions for g_a^2 and g_a^2 are

$$g_a^2 = \frac{M_V^2 - s_-}{m^2 B_1(s_-)}\,, \qquad g_b^2 = \frac{M_V^2 - s_+}{m^2 B_1(s_+)}\,.$$

From Eq. (18) it is easy to see that in all sheets the point $z = M_V^2$ corresponds to $g^2 = 0$. Moreover, in all sheets but the first, poles approach the origin when g^2 approaches zero, and pass through the l-cut towards another sheet.

A complicating factor for solutions in the third sheet is that there are two points (which we will call **transition points**) between s_- and s_+ in that sheet, where the corresponding value of g^2 jumps from $+\infty$ to $-\infty$ for

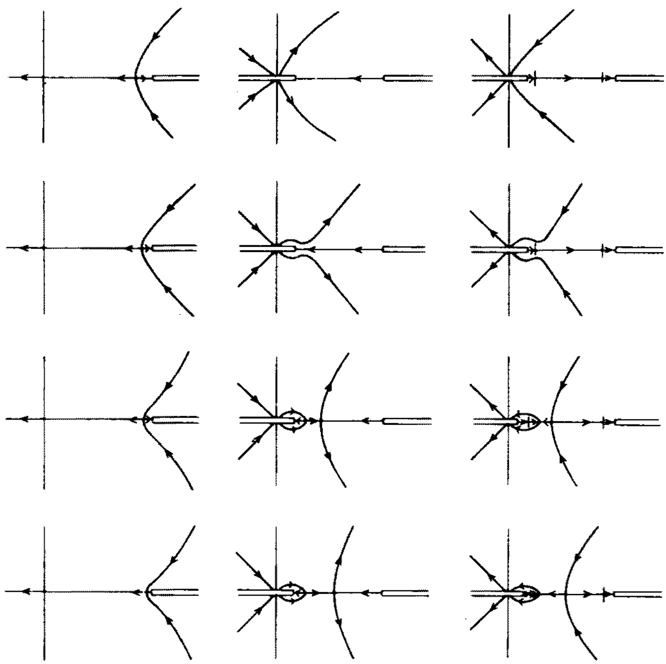

Fig. 4. Bound state and resonance poles in the complex energy plane, for $M_N/m = 2$ and $M_V^2 < s_+$. From top to bottom $M_V/m = 1.1$, 1.9, 2 and 2.5, while from left to right the first, second and third sheet are shown. The arrows indicate increasing values of g^2. The pictures are described in the text.

both points. Thus, there is a solution in the third sheet, which starts at the left transition point for $g^2 = -\infty$ and ends in the right transition point, where it arrives for $g^2 = +\infty$. The transition points are marked in Figs 4 and 5 with a small vertical bar.

The connectivity of the orbits through the cuts can be rather complicated. In addition to the single solution in the third sheet, which we have just described, in Fig. 4 for $M_V/m = 1.1$ and $M_V/m = 1.9$, four pairs of poles can be distinguished, each pair consisting of two poles, which at $g^2 = -\infty$ are very far apart. Since poles in the firs sheet which are not on the real energy axis imply acausality, we can expect that the large derivative of the phaseshift for these values of g^2 (Fig. 2) is just another manifestation of the same phenomenon. On increasing g^2 the members of each pair approach each other, collide and move apart as g^2 goes to plus infinity.

The first pair starts as remote poles in the upper and lower half of the first (physical) sheet. They collide on the real axis, somewhere between M_V^2 and s_+, at a value of g^2 for which g_R^2 diverges. One pole then moves to the

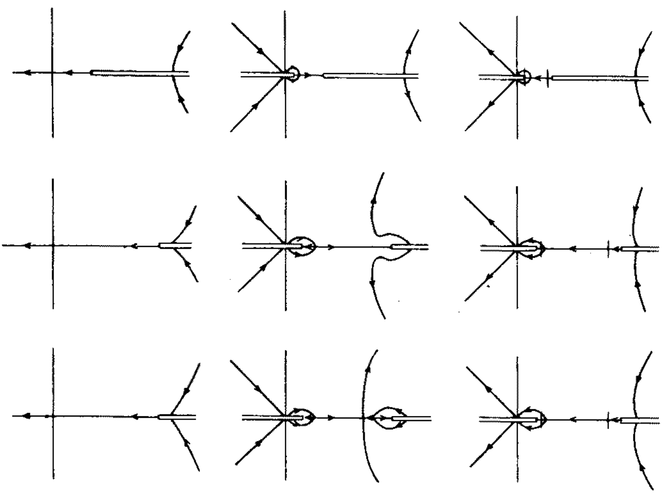

Fig. 5. Pole trajectories in the complex energy plane, as in Fig. 4 with $M_N/m = 2$ and $M_V^2 > s_+$. From top to bottom $M_V/m = 3.26$, 3.27 and 6.

left along the negative real axis, where there is no cut. The other pole moves to the right until it reaches the branch point s_+ for $g^2 = g_b^2 < 0$. There it disappears from the first sheet, only to reappear on the second sheet and to move back along the real axis passing through the cut in s_- for $g^2 = g_a^2$. On the third sheet it moves to the left transition point, where it arrives for $g^2 = +\infty$.

The second pair also starts very far apart, but now on the second sheet for Re $z < 0$. On increasing g^2 both poles move towards $z = 0$ (a massless particle on the second sheet), where each of them crosses the l-cut (without colliding) into the third sheet. Here they move apart again in the same half space Re $z < 0$. The third pair has a similar behaviour as the second pair, except that it starts in the third sheet for Re $z > 0$. After passing $z = 0$ both poles enter the second sheet again with Re $z > 0$. Of the fourth pair only the beginning of the path of one member is shown. For $g^2 = -\infty$ it starts at the right transition point and moves along the real axis towards $z = s_+$, where it arrives for $g^2 = g_b^2$. It then crosses the cut into the fourth sheet, where it will meet its counterpart.

For $M_V/m = 2$ and $M_V/m = 2.5$ only the second pair can still be recognized. The other three pairs, and probably more, have joined to form one big structure of colliding poles moving from one sheet to another.

When $M_V > M_N + m = 3m$, the structure develops further in a way that is shown better in Fig. 5 than can be told in a thousand words.

3. Conclusions

When comparing the analytical structure of the S-matrix for the relativistic Lee model and for potential scattering, the following observations should be made.

In both cases the right hand cut is present as it should in order to satisfy the requirement of unitarity. For a theory where particle production is allowed, like in the $V\theta$-sector of the Lee model, there should be additional branchpoints on the u-cut, at each point where a new channel opens. In the $N\theta$-sector this does not occur.

For potential scattering the form of the interaction (but not its strength) is usually reflected in fixed singularities on the negative energy axis. A Yukawa potential has branch points in the s-wave amplitude, while the square well and exponential interaction have fixed poles. The δ-shell potential and the Lee model, however, have no fixed singularities on the negative energy axis at all. This is probably caused by the fact that both are in effect theories with a separable potential. Still the Lee model is much richer than the δ-shell potential, because the s-wave scattering amplitude is a meromorphic function with an infinite number of sheets, each of which (except the first) having a branch point at $s_- = (M_N - m)^2$, which is characteristic for a relativistic theory. All other singularities are poles and move when the strength of the coupling is changing. In the theory of nonlinear systems this phenomenon, which is called the Painlevé property, is characteristic for its integrability [9]. It is not clear whether in the present context this analogy is more than just a coincidence.

The orbits of the moving poles in the Lee model are very much different from those of the δ-shell potential. In the former they move from one sheet to another in a very intricate way, whereas for the latter only the bound state pole moves from the second to the first sheet, while all other poles are restricted to the second sheet.

Another important difference between potential scattering and field theory is that for weak coupling the poles in potential scattering are always far away from the physical region, whereas in the Lee model, at least for $M_V > M_N + m$, even the weakest interaction causes the V-particle to be unstable, leading to a resonance in $N\theta$-scattering.

For the Lee model the motion of the two poles on the physical sheet causes unphysical effects in the two limits $g^2 \to +\infty$ and $g^2 \to -\infty$. For $g^2 \to +\infty$ only one pole remains on the physical sheet. Since, however, its position moves to minus infinity, it must be interpreted as a particle with imaginary mass, *i.e.* as a tachyon. This can only be prevented either by not letting g^2 take on too large values, or by changing the theory in such a way that also on the first sheet a branch point occurs at s_-. This will probably happen if crossing symmetry is incorporated into the theory.

It is usually stated that for negative g^2 the Hamiltonian becomes non-hermitian, leading to acausal effects. The same remark applies to the Lee model, because if g^2 is sufficiently large negative two poles appear in the first sheet off the real axis, which contradicts the causality condition. There are, however, negative values of g^2, for which the two poles are still on the real axis. The first has the characteristics of the original V-particle, whereas the second is more like an N-θ bound state. Due to Levinson's theorem this additional bound state will lead to an observable effect in the energy dependence of the phase shift.

Th.W.R. would like to thank Professor J. Namysłowski for drawing his attention to Ref. [8] and for explaining its contents.

REFERENCES

[1] T. D. Lee, *Phys. Rev.* **95**, 1329 (1954).
[2] Th. W. Ruijgrok, E. J. J. Kirchner, in: *Multiparticle Dynamics; Festschrift Léon Van Hove*, eds A. Giovannini, W. Kittel, World Scientific, Singapore 1990.
[3] K. Gottfried, *Quantum Mechanics*, W. A. Benjamin Inc, New York 1966.
[4] See *e.g.* H. M. Nussenzveig, *Causality and Dispersion Relations*, Academic Press, New York 1972.
[5] J. D. Bjorken, A. Goldberg, *Nuovo Cimento* **16**, 539 (1960).
[6] A. Klein, *J. Math. Phys.* **1**, 41 (1960).
[7] Th. W. Ruijgrok, in: *Quantum Theory of Particles and Fields*, eds B. Jancewicz, J. Lukierski, World Scientific, Singapore 1983, p. 117.
[8] H. Georgi, An effective field theory for heavy quarks at low energies, Preprint HUTP–90/A007. 1/90.
[9] See *e.g.* M. Tabor, *Nature* **310**, 277 (1984).