ON QUANTUM (q-) OSCILLATORS*

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(*Received September 2, 1991*)

To the memory of my mother

An algebraic procedure for obtaining oscillator realizations of quantum (q-) oscillators is described for both bosonic and fermionic q-oscillators. The existence of canonical q-transformations is demonstrated and some interesting aspects of such transformations are discussed.

PACS numbers: 03.65. Fd

1. Introduction

Quantum Lie algebras first emerged in the sphere of quantum inverse scattering problems and Yang–Baxter equations [1]. It is known that the Jacobi identity is an associativity condition for a Lie algebra. The quantum Yang–Baxter equation plays a similar role for a new algebraic structure which in a certain sense is a generalization of a Lie algebra. This new structure is often referred to as a q-deformation of a Lie algebra, with the deformation parameter $q = e^s$, such that the usual Lie algebra is reproduced in the limit $s \to 0$, *i.e.*, $q \to 1$. Drinfeld showed [2] that these deformed structures are essentially connected with quasi-triangular Hopf algebras. Extensive developments regarding the nature, structure and representations of these deformed algebras have also been made by Jimbo and Woronowicz [2] while Manin has given an appealing geometrical approach [2]. There are versions of deformed Kac–Moody and Virasoro algebras [3], the realization of quantum $SU(2)_q$ algebra in terms of q-oscillators has been extensively studied [4] and there exist q-oscillator realizations of many other

* Presented at the XXXI Cracow School of Theoretical Physics, Zakopane, Poland, June 4–14, 1991.

quantum algebras [5]. Here we shall discuss the harmonic oscillator realizations of both bosonic [6] and fermionic q-oscillators and show how canonical q-transformations [7] can naturally exist. The aim is to show how all this can be cast into one theoretical framework which is purely algebraic. The essence of this framework is to set up functional equations resembling finite difference equations and then try to solve them exactly. Accordingly, the plan of the paper is as follows. In Section 2 we discuss the basic elements of our methodology in the context of the harmonic oscillator realization of bosonic [6] and fermionic q-oscillators. In particular, we shall show how the usual harmonic oscillator realizations of fermionic q-oscillators occur quite naturally in this approach. In Section 3 we demonstrate how canonical q-transformations fit into this scheme, underlining the power of this algebraic procedure. Section 4 comprises of our conclusions.

2. The methodology for harmonic oscillator realizations

We shall now describe the harmonic oscillator realizations for q-oscillators. First let us recall the case for bosonic q-oscillators [6]. The equations characterizing the q-deformed bosonic oscillator system are (q real)

$$aa^\dagger - qa^\dagger a = q^{-N}\,, \qquad N^\dagger = N\,, \tag{1}$$

$$[N, a] = -a\,, \qquad aN = (N+1)a\,, \tag{2}$$

$$[N, a^\dagger] = a^\dagger\,, \qquad a^\dagger N = (N-1)a^\dagger\,, \tag{3}$$

$$a^\dagger a = [N]\,, \qquad aa^\dagger = [N+1]\,, \tag{4}$$

where $a, a^\dagger$ and N are the annihilation, creation and number operators, respectively, and $[x] = (a^x - a^{-x})/(q - q^{-1})$. One can verify that Eq. (4) is a solution of (1) for both real and complex q. We shall confine ourselves to real q [6].

Ordinary bosonic oscillators $\widehat{a}$, $\widehat{a}^\dagger$ are described by

$$[\widehat{a}, \widehat{a}^\dagger] = 1\,, \qquad \widehat{N} = \widehat{a}^\dagger\widehat{a} = \widehat{a}\widehat{a}^\dagger - 1 \tag{5a}$$

$$[\widehat{N}, \widehat{a}] = -\widehat{a}\,, \qquad [\widehat{N}, \widehat{a}^\dagger] = \widehat{a}^\dagger\,, \tag{5b}$$

where $\widehat{N}$ is the usual number operator. We want to find the solutions for a, $a^\dagger$ and N satisfying Eqs (1)–(3) together with

$$[\widehat{N}, N] = 0\,, \qquad [\widehat{N}, a] = -a\,, \qquad [\widehat{N}, a^\dagger] = a^\dagger\,. \tag{6}$$

From (6) one has

$$N = \Phi(q, \widehat{N})\,, \qquad a = \widehat{a}f(q, \widehat{N})\,, \qquad a^\dagger = f(q, \widehat{N})\widehat{a}^\dagger\,, \tag{7}$$

where Φ and f are some arbitrary functions and f is real. Substituting (7) into (1) we get

$$(\hat{N}+1)f^2(q,\hat{N}+1) - q\hat{N}f^2(q,\hat{N}) = q^{-\Phi(q,\hat{N})} = q^{-N}\,. \tag{8}$$

Further from (2) and (3) one concludes

$$q^{-N}a = aq^{-N+1}\,, \tag{9a}$$
$$q^{-N}a^{\dagger} = a^{\dagger}q^{-N-1}\,. \tag{9b}$$

Putting Eq. (8) into (9) one obtains the functional equation

$$\left(\frac{1}{a}+q\right)F(q,\hat{N}) - F(q,\hat{N}-1) - F(q,\hat{N}+1) = 0\,, \tag{10}$$

where $F(q\hat{N}) = \hat{N}f^2(q,\hat{N})$. The same equation is also obtainable from Eq. (9b).

To solve Eq. (10) for $F(q,\hat{N})$ note that

$$F(q,\hat{N}) \xrightarrow{q\to 1} \hat{N}\,. \tag{11}$$

Thus one has the following system of equations to solve:

$$\left(\frac{1}{a}+q\right)F(q,\hat{N}) - F(q,\hat{N}-1) - F(q,\hat{N}+1) = 0\,, \tag{12a}$$

$$F(q,\hat{N}+1) - qF(q,\hat{N}) = q^{-\Phi(q,\hat{N})} = q^{-N}\,, \tag{12b}$$

$$F(1,\hat{N}) = \hat{N}\,, \qquad \Phi(1,\hat{N}) = \hat{N}\,. \tag{12c}$$

The functional equations (12a), (12b) are analogous to finite difference equations and there will appear initial values in their solutions. Also it is easily seen that the conditions (12c) satisfy the Eqs (12a) and (12b). The solution to Eq. (12a) has been elaborately described in Ref. [6] and is

$$F(q,\hat{N}) = q^{\hat{N}}F(q,0) + [\hat{N}]q^{-\Phi(q,0)} \tag{13}$$

for arbitrary $F(q,0)$ and $\Phi(q,0)$. Moreover, note that if $F \equiv \tilde{F}(q,\hat{N})$ is a solution of (12a), then $F = \tilde{F}(q,-\hat{N})$ is also a solution. So the general solution is [6]:

$$F(q,\hat{N}) = \frac{q^{\hat{N}}\Phi_1(q) - q^{-\hat{N}}\Phi_2(q)}{(q-q^{-1})}\,, \tag{14}$$

where $\Phi_{1,2}$ are arbitrary functions with the restriction that one has $\Phi_{1,2}(1) = 1$. For reasons stated in Ref. [6] Φ_i may be taken to be functions of q only. Then using $F = \widehat{N} f^2(q, \widehat{N})$ we have [6]

$$a = \widehat{a}\sqrt{\frac{q^{\widehat{N}}\Phi_1 - q^{-\widehat{N}}\Phi_2}{\widehat{N}(q - q^{-1})}}, \qquad a^\dagger = \sqrt{\frac{q^{\widehat{N}}\Phi_1 - q^{-\widehat{N}}\Phi_2}{(q - q^{-1})}}\,\widehat{a}^\dagger$$
$$N = \widehat{N} - \left(\tfrac{1}{s}\right)\ln \Phi_2 . \tag{15}$$

Solutions (15) satisfy all the fundamental relations. Choosing $\Phi_1 = \Phi_2 = 1$ gives known realizations [9].

Let us now consider the harmonic oscillator realization of fermionic q-oscillators. We shall explicitly show that the application of the above formalism leads to the known realizations [9]. Here there are no arbitrary functions as these are fixed by the constraint on the number operator, *viz.* $M^2 = M$.

The q-deformed fermionic oscillator is described by the relations:

$$bb^\dagger + qb^\dagger b = q^M , \qquad M = M^\dagger = M^2 , \tag{16}$$
$$[M, b] = -b , \qquad bM = (M + 1)b , \tag{17}$$
$$[M, b^\dagger] = b^\dagger , \qquad b^\dagger M = (M - 1)b^\dagger , \tag{18}$$
$$b^\dagger b = [M] , \qquad bb^\dagger = [1 - M] , \tag{19}$$

where b, $b^\dagger$ and M are the annihilation, creation and number operators, respectively. As before q is real. The standard fermionic oscillator is defined through the relations:

$$\{\widehat{b}, \widehat{b}^\dagger\} = 1 , \qquad \widehat{M} = \widehat{b}^\dagger\widehat{b} = 1 - \widehat{b}\widehat{b}^\dagger = \widehat{M}^2 , \tag{20a}$$
$$[\widehat{M}, \widehat{b}] = -\widehat{b} , \qquad [\widehat{M}, \widehat{b}^\dagger] = \widehat{b}^\dagger . \tag{20b}$$

The analogous of Eq. (6) are:

$$[\widehat{M}, M] = 0 , \qquad [\widehat{M}, b] = -b , \qquad [\widehat{M}, b^\dagger] = b^\dagger . \tag{21}$$

This means that

$$M = \Psi(q, \widehat{M}) , \qquad b = \widehat{b}\theta(q, \widehat{M}) , \qquad b^\dagger = \theta(q, \widehat{M})\widehat{b}^\dagger , \tag{22}$$

where Ψ and θ are some functions to be subsequently determined and we take θ to be real. Putting (22) into (16) one obtains

$$(1 - \widehat{M})\theta^2(\widehat{M} + 1) + q\widehat{M}\theta^2(\widehat{M}) = q^{\Psi(q, \widehat{M})} . \tag{23}$$

From (17) and (18) we have further

$$q^M b = bq^{M-1}\,, \tag{24a}$$
$$q^M b^\dagger = b^\dagger q^{M+1}\,. \tag{24b}$$

Now proceeding exactly as before, *viz.* substituting (23) in (24a) we have

$$2q\xi(q,\,\widehat{M}) - q^2\xi(q,\,\widehat{M}-1) - \xi(q,\,\widehat{M}+1) = 0\,, \tag{25}$$

where

$$\xi(q,\,\widehat{M}) = \theta^2(q,\,\widehat{M})\,,$$
$$\xi(1,\,\widehat{M}) = \widehat{M}\,, \qquad \xi(1,\,M) = M\,. \tag{26}$$

Note that (23) can be written as

$$(1-\widehat{M})\xi(q,\,\widehat{M}+1) + q\widehat{M}\xi(q,\,\widehat{M}) = q^{\Psi(q,\,\widehat{M})} = q^M\,. \tag{27}$$

Here a short digression is necessary. Owing to the fact that $M = M^2$ and $\widehat{M} = \widehat{M}^2$, the eigenvalues of these operators are 0 and 1. The solution for $\xi(q,\,\widehat{M})$ can be obtained as:

$$\xi_1(q,\,\widehat{M}) = -(\widehat{M}-1)q^{\widehat{M}}\xi(q,\,0) + \widehat{M}q^{\widehat{M}-1}q^{\Psi(q,\,0)} \tag{28}$$

for arbitrary $\xi(q,\,0)$ and $\Psi(q,\,0)$. It can be readily checked that if $\xi_1(q,\,\widehat{M})$ is a solution of (25) then $\tilde{\xi}_1(q,\,-\widehat{M})$ is also a solution.

We rewrite (28) in the form

$$\xi_1(q,\,\widehat{M}) = \widehat{M}q^{\widehat{M}}A(q) + q^{\widehat{M}}B(q)\,, \tag{29a}$$

where

$$A(q) = q^{-1}q^{\Psi(q,\,0)} - \xi(q,\,0)\,, \qquad B(q) = \xi(q,\,0)\,. \tag{29b}$$

Similarly

$$\tilde{\xi}_1(q,\,-\widehat{M}) = \widehat{M}q^{-\widehat{M}}\tilde{A}(q) + q^{-\widehat{M}}\tilde{B}(q) \tag{30a}$$

with

$$\tilde{A}(q) = \tilde{\xi}(q,\,0) - q^{-1}q^{\tilde{\Psi}(q,\,0)}\,, \qquad \tilde{B}(q) = \tilde{\xi}(q,\,0)\,. \tag{30b}$$

Therefore, the general solution of Eq. (25) is

$$\begin{aligned}\xi(q,\,\widehat{M}) &= \xi_1(q,\,\widehat{M}) + \tilde{\xi}_1(q,\,-\widehat{M})\\ &= q^{\widehat{M}}(\widehat{M}A + B) + q^{-\widehat{M}}(\widehat{M}\tilde{A} + \tilde{B})\,.\end{aligned} \tag{31}$$

From (26) the condition $\xi(1, \widehat{M}) = \widehat{M}$ implies

$$A + \tilde{A} = 1\,, \qquad B + \tilde{B} = 0\,. \tag{32}$$

Hence the solution (31) takes the form

$$\xi(q, \widehat{M}) = \widehat{M} q^{-\widehat{M}} + \left(q^{\widehat{M}} - q^{-\widehat{M}}\right)(\widehat{M} A + B)\,. \tag{33}$$

We now demonstrate that the restriction on the number operator ($\widehat{M} = \widehat{M^2}$) will determine $\xi(q, \widehat{M})$ uniquely *i.e.*, there will be no dependence on the arbitrary functions $A(q)$ and $B(q)$. Substituting the solution (33) into (27) and using the identities

$$q^{\widehat{M}} = 1 - \widehat{M} + q\widehat{M}\,, \qquad q^{-\widehat{M}} = 1 - \widehat{M} + q^{-1}\widehat{M}\,, \tag{34}$$

we find

$$q^{M} = q^{\widehat{M}}\left[q^{-1} + (A + B)(q - q^{-1})\right] \tag{35}$$

so that

$$M = \widehat{M} + \left(\tfrac{1}{s}\right) \ln \overline{F}(q)\,, \tag{36}$$

where

$$\overline{F}(q) = q^{-1} + \{A(q) + B(q)\}(q - q^{-1})\,. \tag{37}$$

Now imposing the condition $M^2 = M$ and remembering that $\overline{F}$ is a function of q only, the restriction on $\overline{F}$ becomes

$$\ln \overline{F}(q) = 0 \qquad\qquad i.e.\ \overline{F}(q) = 1 \tag{38a}$$

which in turn leads to

$$A(q) + B(q) = \frac{(1 - q^{-1})}{(q - q^{-1})}\,. \tag{38b}$$

We thus have

$$M = \widehat{M}\,. \tag{39}$$

Using (34) and (38b) the general solution (33) simplifies to

$$\xi(q, \widehat{M}) = \widehat{M} = \widehat{M^2} \qquad\qquad i.e.\ \theta(q, \widehat{M}) = \widehat{M}\,. \tag{40}$$

Thus the harmonic oscillator realization of fermionic q-oscillators is obtained as

$$b = \hat{b}\theta(q, \widehat{M}) = \widehat{b b^{\dagger} b} = \hat{b}; \qquad b^{\dagger} = \hat{b}^{\dagger}\,. \tag{41}$$

Thus we have arrived at the usual realization for fermionic q-oscillators [9].

In this section we have seen that in the usual oscillator realization for bosonic q-oscillators there are present arbitrary functions of the deformation parameter q. However, for fermionic q-oscillators such dependence on arbitrary functions is absent owing to the fact that the number operator M satisfies $M^2 = M$. In the next section we show that there are harmonic oscillator realizations of q-oscillators where arbitrary functions of q play a nontrivial role.

3. Canonical q-transformations

The power and utility of the formalism just described will now be demonstrated in the context of canonical q-transformations [7, 8].

For bosonic q-oscillators these transformations can be written in the form [7]:

$$\begin{pmatrix} a' \\ a'^{\dagger} \end{pmatrix} = \begin{pmatrix} \tilde{u}(\hat{N}+1) & \tilde{v}(\hat{N}) \\ \tilde{v}^*(\hat{N}+1) & \tilde{u}^*(\hat{N}) \end{pmatrix} \begin{pmatrix} a \\ a^{\dagger} \end{pmatrix}, \tag{42}$$

where $(a,\, a^{\dagger})$, $(a',\, a'^{\dagger})$ satisfy

$$aa^{\dagger} - q^2 a^{\dagger} a = 1\,. \tag{43}$$

The fundamental relation (43) is equivalent to (1) under the identification $a \to q^{N/2}a$, $\quad a^{\dagger} \to a^{\dagger}q^{N/2}$ and the functions $\tilde{u}(\hat{N})$ and $\tilde{v}(\hat{N})$ can be determined exactly. The transformations (42) act on the two dimensional quantum space of vectors $(a,\, a^{\dagger})$ satisfying (43) and preserve this property for $(a',\, a'^{\dagger})$. Thus we can interpret the transformations (42) as an element of the q-deformed SL(2,R) group. However, this q-deformed group is not related to the quantum group $\mathrm{SL(2,R)}_q$ as the quantities $\tilde{u}$, $\tilde{v}$, $\tilde{u}^*$, $\tilde{v}^*$ in (42) are commuting operators while the elements of the $\mathrm{SL(2,R)}_q$ matrix has nontrivial commutation relations. It can be shown that [7]

$$\tilde{u}(\tilde{N}) = \left[\frac{\hat{N}(q^2-1)}{\Phi_1 q^{2\hat{N}} - \Phi_2}\right]^{1/2} u(\hat{N}), \quad \tilde{v}(\hat{N}) = v(\hat{N})\left[\frac{\hat{N}(q^2-1)}{\Phi_1 q^{2\hat{N}} - \Phi_2}\right]^{1/2},$$

where $\Phi_1(q)$, $\Phi_2(q)$ are the same arbitrary functions as in (14) and the equations to be solved to determine $u(\hat{N})$, $u^*(\hat{N})$ and $v(\hat{N})$, $v^*(\hat{N})$ are

$$U(\hat{N}+1) - q^2 U(\hat{N}) + V(\hat{N}) - q^2 V(\hat{N}+1) = 1\,, \tag{44}$$

$$u(\hat{N})v^*(\hat{N}+1) = q^2 v^*(\hat{N})u(\hat{N}+1)\,, \tag{44a}$$

$$u^*(\hat{N})v(\hat{N}+1) = q^2 v(\hat{N})u(\hat{N}+1)\,, \tag{44b}$$

with

$$U(\widehat{N}) = \widehat{N}u^*(\widehat{N})u(\widehat{N}), \qquad V(\widehat{N}) = \widehat{N}v^*(\widehat{N})v(\widehat{N}). \tag{44c}$$

Thus here also we arrive at functional equations resembling finite difference equations. The solutions to these equations have been elaborately given in Ref. [7] and the canonical transformations (42) can be written as (for $\Phi_1 = \Phi_2 = 1$)

$$\begin{aligned} a' &= a\{|\bar{u}(\widehat{N})|e^{i\alpha}\} + \{q^{2\widehat{N}}W^{1/2}|\bar{u}(\widehat{N})|e^{i\beta}\}a^\dagger \\ a'^\dagger &= \{|\bar{u}(\widehat{N})|e^{-i\alpha}\}a^\dagger + a\{q^{2\widehat{N}}W^{1/2}|\bar{u}(\widehat{N})|e^{-i\beta}\} \end{aligned} \tag{45}$$

with

$$|\bar{u}(\widehat{N})| = \left[\frac{1 - q^{2\widehat{N}}W}{\{1 - q^{4\widehat{N}-2}W\}\{1 - q^{4\widehat{N}+2}W\}}\right]^{1/2}. \tag{46}$$

Here $W(q)$ is an arbitrary function of q and $\alpha(q)$, $\beta(q)$ are arbitrary phase factors. It can be easily shown that (for α and β independent of $\widehat{N}$) the limit $q = 1$ gives the usual SL(2,R) canonical transformations of the ordinary harmonic oscillator where $\alpha(1)$, $\beta(1)$ and $W(1)$ are parameters of the SL(2,R) transformations [7].

The next natural question to ask is whether canonical q-transformations can be set up with two bosonic q-oscillators. The answer is yes and certain interesting aspects of such transformations have come to light [8]. We briefly discuss this below.

Consider two bosonic q-oscillators a_1, a_2 satisfying (no sum over i)

$$a_i a_i^\dagger - q^2 a_i^\dagger a_i = 1. \tag{47}$$

One then carries out the following linear transformations on a_i, $a_i^\dagger$

$$\begin{aligned} a_1' &= a_1X_1(\widehat{N}_1, \widehat{N}_2) + Y_1(\widehat{N}_1, \widehat{N}_2)a_1^\dagger; \quad a_1'^\dagger = X_1^*a_1^\dagger + a_2Y_1^*, \\ a_2' &= a_2X_2(\widehat{N}_1, \widehat{N}_2) + Y_2(\widehat{N}_1, \widehat{N}_2)a_1^\dagger; \quad a_2'^\dagger = X_2^*a_2^\dagger + a_1Y_2^*. \end{aligned} \tag{48}$$

Certain points are worth mentioning regarding the philosophy of the approach in this case. Initially, except for the fundamental relation (47), nothing else is specified, *i.e.*, relations between the operators $(a_i,\ a_j)$, $(a_i^\dagger,\ a_j^\dagger)$ and $(a_i,\ a_j^\dagger)$, $i \neq j$, are not given. They are determined later.

Substituting (48) into (47) again leads to slightly more complicated functional equations resembling finite difference equations and these can be solved exactly [8]. The answers are:

$$a_1' = a_1 \left\{ \sqrt{\frac{q^{\widehat{N}_2}}{[\widehat{N}_1]} \tilde{f}_1(\bar{N}) e^{i\alpha_1}} \right\} + \left\{ q^{2\overline{N}} \widetilde{W}_1^{1/2} \sqrt{\frac{q^{\widehat{N}_1}}{[\widehat{N}_2]} \tilde{f}_1(\overline{N}) e^{i\beta_1}} \right\} a_2^\dagger ,$$

$$a_2' = a_2 \left\{ \sqrt{\frac{q^{\widehat{N}_1}}{[\widehat{N}_2]} \tilde{f}_2(\bar{N}) e^{i\alpha_2}} \right\} + \left\{ q^{2\widehat{N}} \widetilde{W}_2^{1/2} \sqrt{\frac{q^{\widehat{N}_2}}{[\widehat{N}_1]} \tilde{f}_2(\bar{N}) e^{i\beta_2}} \right\} a_1^\dagger , \quad (49)$$

where

$$\tilde{f}_i(\overline{N}) = \frac{(1 - q^{2\overline{N}} \widetilde{W}_i)[\bar{N}]}{\{1 - q^{4\overline{N}-2} \widetilde{W}_i\}\{1 - q^{4\overline{N}+2} \tilde{W}_i\}} ,$$

$$\overline{N} = \widehat{N}_1 + \widehat{N}_2 ,$$

and $\widetilde{W}_i(q)$ are arbitrary functions of q while α_i, β_i are phase factors. The relations for $i \neq j$ are (for $\Phi_1 = \Phi_2 = 1$ and all phase factors α, $\beta = 2\pi m$):

$$[a_1, a_2] = [a_1^\dagger, a_2^\dagger] = 0 ,$$

$$a_1 a_2^\dagger - q^2 a_2^\dagger a_1 = \sqrt{\frac{q^{1-\overline{N}}}{[\widehat{N}_1 + 1][\widehat{N}_2]}} a_1 a_2^\dagger ,$$

$$a_2 a_1^\dagger - q^2 a_1^\dagger a_2 = \sqrt{\frac{q^{1-\overline{N}}}{[\widehat{N}_1][\widehat{N}_{2+1}]}} a_2 a_1^\dagger .$$

We therefore see that canonical q-transformations with two q-oscillators can be set up, but these oscillators cannot be strictly independent. This indicates that relations between different q-oscillators are not *a priori* fundamental.

4. Conclusions

An algebraic procedure has been described which gives oscillator realizations for quantum oscillators in a systematic way. The procedure entails setting up equations similar to finite difference equations and then solving them. In this way harmonic oscillator realizations for both bosonic and fermionic quantum oscillators can be obtained. Canonical q-transformations can also be set up in this formalism in a rather elegant way for the single q-oscillator as well as for two bosonic q-oscillators. For the single q-oscillator, the picture obtained is consistent with Manin's geometrical approach. For two q-oscillators there are indications that relations between different q-oscillators are *not* fundamental. Using this algebraic procedure an exact prescription for q-bosonization has also been recently obtained [10].

The author would like to thank the organizers of the XXXI Cracow School of Theoretical Physics for their hospitality.

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