ISOPERIMETRIC INEQUALITIES IN THE PHYSICS OF BLACK HOLES

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There are reported results on the following problems: *(i)* on the validity of the Penrose inequality; *(ii)* on the hoop conjecture; *(iii)* on the physical criteria for the formation of trapped surfaces; *(iv)* on estimates of a total scalar curvature. A special emphasis is put on the last two points.

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1. Introduction

Originally the name "isoperimetric inequalities" referred to the following problem: amongst all closed curves of a fixed length that lie on a plane, identify those that surround a surface of the largest area. A problem of this type was known to ancient Greek mathematicians.

The generalization of that question on higher dimensional spaces is straightforward — closed surfaces instead of curves and volume instead of an area, and so on; nowadays, however, the isoperimetric problem is understood even in a wider sense. Namely, if one fixes a property X (*e.g.*, length, total mean curvature, total curvature) of a geometric object, the question is what is the extremal value of a property Y (*e.g.*, a volume, a radius) of that object. Isoperimetric inequalities appear as solutions of the isoperimetric problem. For example, the inequality $L^2 - 4\pi S \geq 0$, holds for all curves on a plane, but the equality is satisfied only by circles.

Many physical questions can be understood as isoperimetric problems. A comprehensive review is contained in the book of Polya and Szegö [1]. One

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of the advantages of that approach is a possibility to get some information about an object, independent of many of its details; it is desirable in view of our lack of knowledge about detailed properties of many physical objects.

In the case of black holes, one would like to solve say, their existence problem, without reference to a particular dynamics that governs the motion of matter. (In this place let me say that the first to connect isoperimetric inequalities with black holes was G. Gibbons [2].) For instance, one can ask about a minimal amount of matter (or, more strictly, of a scalar curvature) inside a fixed volume that inevitably leads to the formation of black holes. No doubt, a detailed answer to this question would require the knowledge of the self-interaction of matter; but also, even having the relevant information, we most likely would not be able to use it. The global Cauchy problem, that should be solved then, is still almost untractable in general relativity (although, let us mention the impressive works of Christodoulou [3] on the formation of trapped surfaces in self-gravitating scalar fields, and of Christodoulou and Klainermann [4] on the global existence of almost flat geometries).

For this reason, in known to me attempts to diagnose the presence of black holes in gravitating systems, one usually restricts to the analysis of initial data. Most of them constitute criteria for the formation of trapped surfaces on a Cauchy surface; they may be understood as various versions of the hoop conjecture [5] (this point requires an explanation, that is given in one of the subsequent sections).

Trapped surfaces, defined by R. Penrose [6] are objects that can develop only in strongly curved geometries; their existence signals the existence of black holes, modulo the Cosmic Censorship [7]. The last statement does not strongly persuade that the notion of trapped surfaces is important, since the Cosmic Censorship itself requires justification and a proof. I prefer to think that trapped surfaces are concepts that would be useful in the process of proving the Cosmic Censorship; their geometric properties are of sufficient interest to study conditions under which they can appear.

The second group of problems that are naturally isoperimetric, to say, is that related to the Penrose inequality [8]. Penrose wished to find a criterion that should be satisfied in any collapse which does not lead to the formation of naked singularities. In other words, he wanted to find a sufficient condition for the breakdown of the principle of Cosmic Censorship [8]. His condition compares the total mass of the system with area of the apparent horizon.

The last group of questions that might be understood as isoperimetric ones, address the following issue: fixing volume (radius, *etc.*), how much mass can be packed into a body?

I would like to warn the reader that although it is possible to formulate all results in the spirit of isoperimetric inequalities (like in the book of Burago and Zalgaller [9]), I will adopt a style of exposition more familiar to physicists. The order of the remaining part of this review is following. The next section comprises a list of definitions of various quantities that are used later.

Section 3 describes attempts of many authors to formulate inequalities that relate various quantities describing black holes. Amongst them, I will describe shortly the status of the hoop conjecture as well as recent results on the validity of the Penrose inequality. As parenthetical (but inherently related) remarks, I comment on some quasilocal definitions of the gravitational energy and on measures of size.

Section 4 contains various criteria for the formation of trapped surfaces in a quite general class of Cauchy data of Einstein equations. There are also proofs that, in particular classes of geometries and assuming a maximal slice, only a finite amount of energy can be compacted into a finite volume. This supports the once expressed Einstein's view that "matter cannot be concentrated arbitrarily" [10]. The first part of Sec. 4 deals with nonspherical geometries and bases mainly on my own work [11,12]. The second part, on spherically symmetric geometries, has been done in collaboration with Piotr Bizoń and Niall O'Murchadha [13-15].

Section 5 reports results concerning a longstanding conjecture of Arnowitt, Deser and Misner [16]. That conjecture, proven in [15,17], implies that the ADM mass is not additive for strong gravitational fields.

Section 6 presents some applications of results obtained in Sec. 4 as well as a short description of known examples of the so called "bag of gold".

Section 7 describes a set of criteria for the formation of trapped surfaces by spherically symmetric inhomogeneities in a cosmological expanding model.

2. Definitions and notation

Most of the definitions below can be found in the books of Choquet–Bruhat *et al.* [18] , Wald [19] and Hawking [20].

1. Σ — a 3-dimensional Cauchy hypersurface (Cauchy surface)
2. t — a unit normal in a given metric $h_{\mu\nu}$ of space-time to Σ. (We adopt a convention in which Greek indices μ, ν change from 0 to 3, while Latin indices i, j, k change from 1 to 3. The signature is $(-+++)$.)
3. D_i — a covariant derivative in the metric $h_{\mu\nu}$.

$$K_{ij} = D_i t_j$$

— the extrinsic curvature tensor of Σ: in Gaussian coordinates $K_{ij} = \Gamma^0_{ij}$, where Γ^0_{ij} is a Christoffel's symbol for the metric $h_{\mu\nu}$.

4. *A maximal Cauchy slice, maximal Cauchy hypersurface* — a hypersurface Σ with the vanishing trace of the extrinsic curvature,

$$\mathrm{tr}K = (h^{ij} - t^i t^j)K_{ij} = h^{ij}K_{ij}.$$

5. *A momentarily static* Cauchy surface — a hypersurface with the vanishing extrinsic curvature,

$$K_{ij} = 0.$$

6. S — a two-dimesional closed surface on a Cauchy surface Σ.
7. (a) *A total rest mass* contained in a volume V enclosed by a surface S:

$$M(S) = \int_V T_{\mu\nu} t^\mu t^\nu dV.$$

(b) *An energy density*

$$\rho = T_{\mu\nu} t^\mu t^\nu.$$

(c)*A momentum density*

$$J_\mu = T_{\mu\nu} t^\nu + \rho t_\mu.$$

Above $T_{\mu\nu}$ is the energy-momentum tensor.
8. $R^{(3)}$ — a scalar curvature in a prescribed metric of a Cauchy surface.
9. *A total scalar curvature* contained within S:

$$R^{(3)}(S) = \int_V R^{(3)} dV;$$

V is a volume enclosed by S.

10.

$$K(S) = \int_V K_{ij} K^{ij} dV$$

— a measure of the amount of the extrinsic curvature in S.
From the hamiltonian constraint of the Einstein equations it follows that on the maximal Cauchy slice

$$M(S) = \frac{R^{(3)}(S) - K(S)}{16\pi}.$$

11. ∇_i — a covariant derivative in the (induced) metric

$$g_{ij} = h_{ij} - t_i t_j$$

of a Cauchy surface.

12. *A mean curvature* at a point p of a surface S as embedded in a Cauchy surface Σ:
$$p(p) = \nabla_i n^i.$$
13. *A convergence in the future* (or: a mean curvature of a space-like two-surface S as embedded in space-time) at a point p of a surface S
$$P(p) = \nabla_i n^i + \mathrm{tr}K - n^i n^j K_{ij}.$$
14. *A total mean curvature* of a space-like two-surface S:
$$H(S) = \int_S p dS.$$
15. (a) *A trapped surface* — a closed space-like 2-surface S such that at each point $P(p)$ is nonpositive.
(b) *An apparent horizon* — a closed space-like two-surface with everywhere vanishing $P(p)$.
(c) *An external apparent horizon* — an outermost apparent horizon that surrounds a region with trapped surfaces.

Comments. One can check [6] that the rate of change of a two-dimensional surface element (that is a part of a two-dimensional space-like closed surface S with n being a normal vector directed outward) of a light front outgoing from S is given by

$$d/dl d^2S = P(p)d^2S = [\nabla_i n^i + \mathrm{tr}K - n^i n^j K_{ij}]d^2S,$$

where l is an affine parameter along null geodesics.

Thus, in physical terms, trapped surface is a closed 2-surface with the property that narrow beams of light orthogonal to it at any point decrease in area, at least initially, when propagating outwards; the intensity of light increases, when moving outwards.

Let us point out that the conventions (3) and (13) would give the rate of the change of a volume element of a 3-dim. Cauchy slice equal to the (plus) trace of extrinsic curvature, $\partial_t dV = \mathrm{tr}K dV$.

Remark. The above definitions differ from normally used in literature. A *trapped surface* in the sense of the above definition, is called an *outer trapped surface* by Hawking and Ellis (p. 319 in [19]). An *apparent horizon* is called *a marginally trapped outer trapped surface* (p. 321 in [19]; note, however, that the notion of an *apparent horizon* used by Hawking and Ellis coincides with the given above, if the apparent horizon is smooth.

A (closed) trapped surface is defined by Hawking and Ellis (p. 262 in [19]) as a closed 2-surface that simultaneously satisfies two conditions, that the convergence of both outgoing and ingoing light rays is negative. Thus, not only $P(p)$ is negative but also $\widehat{P}(p)$ defined as follows

$$\widehat{P}(p) = -\nabla_i n^i + \mathrm{tr}K - n^i n^j K_{ij}.$$

I found this definition as too restrictive (only a subset of geometries that possess *outer trapped surfaces* may contain *trapped surfaces* in the sense of [19]). For instance, in those momentarily static geometries that possess *outer trapped surfaces, trapped surfaces* are absent, since if $P(p) < 0$ then $\widehat{P}(p) = -P(p) > 0$.

To make things more confusing, let us note that on p. 2 in [19] *(closed) trapped surfaces* are clearly understood as closed 2-surfaces that possess non-positive $P(p)$ and $\hat{P}(p)$ (not necessarily strictly negative, as on p. 262). Also, R. Penrose says on p. 662 in [7] that "a trapped surface ... is a compact space like 2-surface T (normally $\cong S^2$) having the property that both systems of null normals to T are converging in future directions". That requires only the weak negativity of $P(p), \widehat{P}(p)$.

Later on I will understand *a closed trapped surface in the sense of Hawking and Ellis* as such a surface that both $P, \widehat{P}$ are weakly negative. To avoid any ambiguity, let me say that because all results on the formation of trapped surfaces (presented in Sec. 4) refer to outer trapped surfaces, I decided to use only (with the exception made during reporting results on the Penrose inequalities) the notions *"a trapped surface"* and *"an apparent horizon"*, in the sense defined in 15.(a).

16. *A minimal surface* — a closed space-like 2-surface S with everywhere vanishing mean curvature $p(p)$ (see (2.12)).
17. *An averaged trapped surface* — a closed 2-surface S with a negative total mean curvature $H(S)$.
18. *The weak energy condition*-if the energy-momentum tensor $T^{\mu\nu}$ satisfies the inequality $T^{\mu\nu}t^{\mu}t^{\nu} \geq 0$ for any time like vector t, then it is said to satisfy the weak energy condition (p. 89 in [19]; but note that Penrose gives a different definition (p. 63 in [6]).

3. A review of miscellaneous isoperimetric inequalities

There is a number of quantities that potentially can be used to formulate isoperimetric inequalities. Of all definitions of the energy of the gravitational field, only the total (ADM) gravitational mass [21] is proven to be well posed – positive under reasonable physically conditions and vanishing if (and only if) the geometry is flat [22,23].

About the validity of the existing definitions [24-27] of a gravitational quasi-local mass much less is known. In fact, it is necessary to point out that none of the existing definitions is proven to be satisfactory, even in the regime of weak fields. Also, not all measures of a size of bodies are appropriate. I will review briefly some of the proposed definitions.

Definitions of gravitational mass (energy)

Let us begin with displaying down the requirements that a definition of a quasilocal mass might satisfy. I will follow (but I omit one of his conditions) Eardley [28], who argues that "the best we can do is to associate a mass-energy $M(S)$ with a 2-surface S; we will take S to be space like, topologically spherical and embedded in an asymptotically flat space-time." (Eardley clearly adapts his requirements to the expected properties of the Hawking mass which is given by a surface integral; but note, that if we have a volume-dependent definition, then we can always assign the mass inside a volume to its boundary. Some of the properties listed below must not be satisfied then; but, as Eardley pointed out himself, probably no mass can satisfy all the requirements one would like to have.)

"*(i)* A point p in space time must have zero mass, in that $M(S) = 0$ if S shrinks to p, for instance if S is a shrinking geometric sphere in a given slice Σ.

(ii) A metric 2-sphere S in Minkowski space time should have $M(S) = 0$. (...)

(iii) If Σ is an asymptotically flat slice, r any standard radial in Σ, and $S(r)$ any large coordinate sphere in Σ, then we should have $\lim M(S(r)) = m$ as $r \to \infty$. Here m is the ADM mass of the slice (...).

(iv) If Σ is an asymptotically null slice, that is, a slice that crosses I^+ (or I^-) in some cut C, r any standard radial coordinate, and $S(r)$ any large coordinate sphere in S, then we should have $\lim M(S(r)) = M_{\rm B}(C)$ as $r \to \infty$. Here $M_{\rm B}(C)$ is the Bondi mass (...) .

(v) If S is an apparent horizon (...) then we should have $M(S) = \sqrt{S/(16\pi)}$.

(vi) If S' is "bigger" than S in the sense that $S \supset \Sigma'$ for some achronal hypersurface Σ' with outer boundary S', $M(S') \geq M(S)$."

3.1. The total (ADM) mass [21] ; see also [29].

It is defined as the limiting (for a coordinate sphere S_R of a coordinate radius R going to infinity) value

$$m = \lim_{R\to\infty} \int_{S_R} d^2S\, n^j[\partial^i g_{ij} - \partial_j g_{ii}] \tag{3.1}$$

As remarked above, the ADM mass is proved to be a well posed quantity, in the sense of being always positive if a geometry is nonflat, the weak energy condition is satisfied and some (reasonable) boundary properties of a metric are assumed. It can vanish only in the flat space-time. In Sec. 5 we will demonstrate, that m is not additive in strongly curved geometries. This means any mass that satisfies point *(vi)* above cannot satisfy point *(iii)* of the properties listed above.

3.2. The total rest mass

$$M = \int_V d^3V T_{\mu\nu} t^\mu t^\nu; \tag{3.2}$$

is entirely satisfactory in spherically symmetric space-times, although it does not fulfill the requirements *(iii)–(v)*. M is manifestly additive and positive (assuming nonnegative energy density), as well as finite for any smooth self-gravitating configuration. In general space-times, M is not appropriate, since it does not measure the contribution of gravitational waves. In the next Section we use M to derive criteria for the formation of trapped surfaces.

3.3. The Bartnik mass [27].

In loose terms, it is defined in the following way. Take a piece δV of a 3-dimensional Cauchy slice with a physical geometry; each piece can be smoothly extended to a full Cauchy slice by joining with it an arbitrary (but with a nonnegative energy density) asymptotically flat manifold, called "end". To each "end" corresponds a certain value of the ADM mass. Bartnik then defines the quasi-local mass of δV as the infimum of all ADM masses corresponding to those possible extensions of δV, that do not contain trapped surfaces.

Bartnik has proven that in the spherically symmetric geometries that do not contain trapped surfaces, his quantity satisfies the above conditions. More generally, in the so-called quasi-spherical foliations without shear (in analytical terms it means that $\nabla_i \beta^i = 0$, where $\beta's$ are coefficients in the metric element $ds^2 = u^2 dr^2 + (\beta^1 dr + r d\theta)^2 + (\beta^2 dr + r \sin\theta d\phi)^2)$, that do not contain trapped surfaces, Bartnik has shown that his mass is strictly positive and monotonic [30]. There are serious difficulties in proving that the mass is strictly positive in general geometries. Nevertheless, the Bartnik mass is probably the best candidate for the quasi-local energy in the regime of weak gravitational fields.

3.4. The Hawking mass [25].

One of the advantages of this definition is its easy computability. For this reason I will discuss it in detail. The Hawking mass $m_{\rm H}(S)$ inside a sphere S on the maximal slice is given by [31]

$$m_{\rm H}(S) = \left(\tfrac{S}{16\pi}\right)^{1/2}\Big[1 - \tfrac{1}{16\pi}\int_S d^2S\big(-K_{ij}K^{ij} + p^2\big)\Big]; \tag{3.3}$$

that expression is bounded from below by

$$\widehat{m}_{\rm H}(S) = \left(\tfrac{S}{16\pi}\right)^{1/2}\Big[1 - \tfrac{1}{16\pi}\int_S d^2Sp^2\Big]. \tag{3.4}$$

In spherically symmetric geometries we can use isotropic coordinates in which the line element reads

$$ds^2 = f^4(dr^2 + r^2 d\Omega^2), \tag{3.5}$$

where f is a conformal factor that satisfies the Lichnerowicz equation

$$f^{-5}\Delta f = -\frac{R^{(3)}}{8}. \tag{3.6}$$

In (3.2) $R^{(3)}$ is a scalar curvature of a maximal slice, assumed to be nonnegative.

Take a centered sphere of a coordinate radius R. Its area S, normal unit vector and mean curvature p are given by the following formulae:

$$S = 4\pi f^4 r^2, \tag{3.7a}$$

$$(n_i) = (0, 0, f^2). \tag{3.7b}$$

$$p = \nabla_i n^i = \frac{2(1 + 2f'r/f)}{f^2 r}, \tag{3.7c}$$

Using (3.4) and the above metric, we get

$$\widehat{m}_{\rm H}(S) = -2r(f'^2r^2 + f'fr) = -2r^2 f'(f'r + f). \tag{3.8}$$

One has [15]

$$f'r + f \geq 1, \tag{3.9}$$

assuming that $R^{(3)}$ is nonnegative.

From the maximum principle we have $f' \leq 0$, with the equality attained only at 0 and infinity (but then $\lim_{r\to\infty} f'r^2 = -m/2$, where m is the ADM mass).

Thus, we may infer that the Hawking mass assigned to a sphere centered around the center of symmetry is:

(i) equal to zero for a sphere of a zero diameter;

(ii) strictly positive for any other sphere;

(iii) equal to the ADM mass m for any sphere that lies outside a distribution of matter of compact support.

Let us remark that the property *(i)* was proven to hold for any metric sphere (in the limit of a radius tending to 0) in nonspherical geometries [31].

It is easy to see that, for a momentarily static Cauchy slice ($K_{ij} = 0$) the Hawking mass is monotonic in the absence of trapped surfaces and that it is not monotonic in the region which contains trapped surfaces. To prove that, let us differentiate the equation (6) with respect to r. We obtain then

$$\begin{aligned} \partial_r m_H(S) &= -2\partial_r(r^2 f')(2f'r + f) \\ &= -2(\Delta f)(f + 2f'r) = pr^3 f^8 \frac{R^{(3)}}{8}. \end{aligned} \tag{3.10}$$

In order to get the last equality, I used the Eq. (3.6) and the definition of mean curvature (3.7c). A surface S is said to be trapped if $p(S) \leq 0$; thus, if S is not trapped then $m_{\mathrm{H}}(S)$ cannot decrease, while if S is trapped then $m_{\mathrm{H}}(S)$ may decrease.

It is a known fact that the Hawking mass may be well defined only on surfaces of constant mean curvatures. Even in flat space it is negative, when calculated on convex closed surfaces that are not spheres. In fact, one of isoperimetric inequalities [9] states that

$$[1 - \tfrac{1}{16\pi} \int_S d^2 S p^2] \leq 0$$

for any convex surface and with equality only for a sphere. The fact that Hawking mass is not monotonic should be anticipated (see Sec. 5), since it is asymptotic to the ADM mass. In spherically symmetric geometries the Hawking mass satisfies all properties (except for the sixth one) listed above.

Now I will comment on some existing proofs on the positivity of the Hawking mass. Christodoulou and Yau [32] have proved that the Hawking mass is positive for any surface (that lies on a maximal Cauchy slice) and satisfies the following two conditions:

(i) its mean curvature is constant;

(ii)

$$\delta_{\text{ChY}} \equiv \int_S [\nabla_i n \nabla^i n - n^2 p^i_j p^j_i - n^2 R(n)] d^2 S \geq 0. \qquad (3.11)$$

In (3.11), n is a normal deformation of the surface such that

$$\int_S n d^2\, S = 0,$$

p_{ij} is the second fundamental form of S and $R(n)$ is the Ricci tensor in the direction of n.

The condition (3.11) is stronger than just the positivity of the second area variation (but at a minimal surface the expression at the left hand side of (3.11) is equal to the second area variation). Even in the spherically symmetric case (3.11) is not satisfied by all spheres.

Take, for instance, a spherical geometry generated by a conformal factor

$$f = 1 + (2k+1)a - ar^k, \quad (r \leq 1, a > 0)$$

(k is an integer) and

$$f = 1 + \frac{2ka}{r} \quad (r > 1).$$

Such a conformal factor is of class C^1 and for all spheres of a coordinate radius r smaller than 1 we have

$$\delta_{\text{CHY}} = \frac{16\pi k a r^k [(1 + (2k+1)a)(2-k) - 2ar^k(k+1)]}{(1 + (2k+1)a - ar^k)^2}.$$

Thus, for $k \geq 2$ and for centered spheres that are contained within the coordinate sphere of a unit radius, the condition (3.11) is never satisfied, even for very small values of the parameter a; the proof of Christodoulou and Yau does not work even for (generic) weak gravitational fields.

The last work of Christodoulou [3] on the evolution of spherically symmetric massless scalar field coupled to gravity contains also a proof that (in that particular case) the Hawking mass is positive when calculated on spheres. To summarize, apart from the spherically symmetric case, the validity of the Hawking quasi-local mass is not proved, even in the limited sense when an energy is assigned only to surfaces of constant mean curvature.

Definitions of size

Following this short review of quantities proposed to measure the gravitational energy, it is natural to discuss various propositions of quantities which measure size.

3.5. Proper radius, maximal proper radius

Proper radius can be defined naturally in spherically symmetric geometries or conformally flat systems, where one can distinguish either a center of symmetry or the point (or domain) where the conformal factor is maximal. Both cases can be given a physical sense and a relevant procedure can be proposed that allows to measure the radius.

Given a distinguished point, one can define the proper radius (or the maximal proper radius in the case of nonspherical bodies or nonspherical, but conformally flat geometries) as a geodesic distance from that point (the center of symmetry) to a boundary of a sphere (or as a maximal geodesic distance to a boundary). In the isotropic coordinates (3.5) the proper radius reads

$$L(S) = \int_0^R \sqrt{g_{rr}}dr = \int_0^R f^2 dr, \tag{3.12}$$

where R is the coordinate radius.

In the general case, however, there are some conceptual problems. One can define the radius of a body Ω as, for example, half of the length of a largest geodesic that joins two points on the surface. That definition is good in flat space, but in strongly curved geometries it can be misleading, because then some geodesics may be repelled from the inner part of the boundary into outside. Therefore, there may be more than one geodesic, (possibly none of them contained in Ω), that connect a particular point with other points.

Technically, a way to avoid such difficulties could be the restriction to bodies (or pieces of space) that are globally convex. A size of a particular domain Ω can be then defined as a largest proper radius of a smallest convex set that contains Ω. I doubt, however, that this definition would be a good measure of a size. In strongly curved geometries the notion of global convexity is not well defined, unless we demand some strong conditions, for instance, that all sectional curvatures are positive (see, *e.g.*, Th. 5.3 in [33]); and that, in turn is unacceptable from the physical point of view. The Schwarzschild geometry, for example, does possess sectional curvatures that are negative. In Sec. 4, fortunately, I work only with conformally flat geometries, and the first definition may be applied.

3.6.

Rad(S) – a quantity described in Sec. 4.1.. It reduces to a proper radius in spherically symmetric geometries and is bounded by a largest proper radius in some nonspherical geometries. Its status is not quite clear, although a convincing, in my opinion, evidence is given in favour of its interpretation as a measure of size.

3.7.

A circumference — a measure used in the formulation of the hoop conjecture [5] (see below). It can be defined as follows. Take a surface S of a body, with an induced internal geometry. Thus, the notion of a geodesic on S is well defined; the largest circumference of a body is defined as the length of a largest closed geodesic on S. (Let us mention that there may appear problems related to the existence of closed geodesics; they do not have to exist in some cases.) That definition is not good in strongly curved geometries. Even in spherically symmetric geometries, a circumference of a large body (*i.e.*, having a large proper radius) may be very small. See a discussion below on the formulation of the hoop conjecture.

3.8.

Obvious quantities to measure a size of a body are its volume and the area of its boundary. The area of a boundary is not a good size measure of Ω, since, if trapped surfaces are present inside Ω, then the boundary of Ω may shrink even to a point (as, for instance, in "bag of gold" configurations — see Sec. 6). This may happen even in spherically symmetric geometries. The volume does not appear in inequalities describing black holes.

3.9.

A torus radius rad(Ω) of Schoen and Yau [34] (see also [35] , for a more accessible presentation). This measure is defined as the radius of a largest torus that can be inscribed into Ω. In [14] an example is presented which shows that in strongly curved spherically symmetric geometries the torus radius is not good in the sense that large bodies may have small values of the torus measure.

The Penrose inequality

As already pointed out in the introduction, Penrose discussed a possibility of breaking the Cosmic Censorship and he came to the conclusion, that the Cosmic Censorship will be broken if the following is not true.

The Penrose inequality I.

Let Σ be an asymptotically flat Cauchy hypersurface with a non-negative energy density, m — the ADM mass and S be an area of a surface trapped in the sense of Hawking-Ellis (see 2.15). Then the following is true:

$$m \geq \sqrt{S/16\pi}. \tag{3.13}$$

That formulation was publicized by Penrose (*e.g.*, [7]). In another paper [6] Penrose formulated a version that differs from the above in that it replaces trapped surfaces by an apparent horizon (see also [34]). That is:

The Penrose inequality II.

Let Σ be an asymptotically flat Cauchy hypersurface with a non-negative energy density, m – the ADM mass and S be an area of an external apparent horizon. Then the following inequality is true:

$$m \geq \sqrt{S/16\pi}. \tag{3.13a}$$

(One can recognize in (3.13a) the condition *(v)* on the Eardley's list.) A careful inspection of the line of reasoning of Penrose shown in [7,8] leads one to the conclusion, that this formulation of the above inequality is not necessary for the validity of the Cosmic Censorship. One of the assumptions that are implicit in the Cosmic Censorship hypothesis, is that the event horizon surrounds the external apparent horizon; this does not mean, however, that its area is greater than the area of the latter, what was assumed by Penrose.

Horowitz [36] was the first, to my knowledge, to notice this fact. But we should point out that no one was able to find any counter-example against the above versions. Horowitz stated also another form of the Penrose inequality, which, in my opinion is both plausible and necessary for the Cosmic Censorship to hold.

The Penrose inequality III. [36]. Assume an asymptotically flat Cauchy hypersurface, with non-negative energy density and nonsingular outside an apparent horizon (in the sense of 2.15c). Let S be the minimum area of all surfaces that surround the apparent horizon. Then the following inequality holds

$$m \geq \sqrt{S/16\pi}, \tag{3.14}$$

with equality if and only if the space-time is the Schwarzschild solution. G. Gibbons [2] conjectured an inequality for charged matter with total charge q.

Conjecture. (Isoperimetric inequality for charged black holes.)

$$m \geq q^2[4\sqrt{S/(16\pi)}]^{-1} + \sqrt{S/16\pi}, \tag{3.14a}$$

for Einstein-Maxwell data which are regular outside a singly connected apparent horizon and with equality only in the Reissner-Nordstrom case.

The inequality (3.14) is proven only in special cases. R. Bartnik gave a simple proof [30] for a class of quasi-spherical foliations with divergence-free shear and with matter fields satisfying the weak energy condition. His proof concerns geometries in which the region containing (outer) trapped surfaces is surrounded by a minimal surface, *e.g.*, for momentarily static geometries, in which minimal surfaces coincide with apparent horizons.

Ludwigsen and Vickers [37] assumed that a surface S is trapped in the sense of Hawking-Ellis (see 2.15), which is a stronger property than assumed in the preceding formulation (a trapped surface in my sense is not necessarily trapped according to the mentioned definition, while surfaces trapped in the sense of Hawking-Ellis are always trapped in my sense). They assumed also a kind of an asymptotic convexity of S as well as a global property of space-time and an functional inequality (the formula (23) in [8] and the Eq. (33) in [37]) that was proven later by Tod [38] , and under these conditions they proved the Penrose inequality (3.13). Their conditions are quite strong; they imply the existence of outer trapped surfaces in the past history of a self-gravitating system (see the remark below (8) in [37]). From the analysis of the spherically symmetric case, one can draw the conclusion that the proof of Ludvigsen and Vickers is true for geometries that are quickly shrinking (in spherically symmetric geometries matter should fall in the direction to the center, at least in regions close to minimal surfaces).

An interesting and basically simple proof has been proposed by Jezierski [39]. Its inherent part is the use of the Jezierski-Kijowski radial foliations [40] , whose existence was proven thanks to an important technical result of Chruściel [41]. In [39] inequality (3.14) is proven with S being related to (or equal to, *e.g.*, in spherically symmetric geometries) the area of a minimal surface (thus, it proves the Penrose inequality for those Cauchy data that contain trapped surfaces inside a minimal surface), assuming a kind of a gauge condition. Jezierski's proof contains a gap, in that there is no proof that his gauge condition is satisfied (although a perturbative argument can be used to show, that it is satisfied in almost spherical geometries).

Remark. There is one more reason, apart from the wish to check the validity of the Cosmic Censorship, to be interested in the Penrose inequality. Namely, it can be interpreted also as a diagnostic tool, to determine whether in a given gravitating system trapped surfaces are present. An external observer can compare the area of constant mean curvature surfaces with the ADM mass, and he can conclude that if both sides of (3.14) are comparable, then his geometry may contain trapped surfaces (and presumably, also black holes - modulo all reservations that can be stated in describing the mutual relation between the existence of black holes and trapped surfaces). From this point of view it is valuable, that the Penrose inequality is formulated in terms of quantities that could be detected from the outside, without entering the dangerous internal part of a strongly curved geometry.

Tod [38] proved the Penrose inequality (3.14) for static black holes, with the ADM mass being replaced by the Penrose's quasi-local mass.

The hoop conjecture

The hoop conjecture was formulated by K. Thorne in 1972 [5a]. Let us quote a relevant part of [5b]: *"Black holes with horizons form when and only when a mass m gets compacted into a region whose circumference in every direction is*

$$C \le 4\pi m."$$

This passage is followed by "(Like most conjectures, this one is sufficiently vague to leave room for many mathematical formulations.)"

Thus, the hoop conjecture may be divided into two parts. One, which may be called "the Compactness Conjecture", says that matter should be compressed in all three spatial directions. (Its validity is established for a class of nonspherical geometries [12].) The second, analytical part, basically was not formulated, although the above expression suggests that the right quantities would be the ADM mass and a measure of the size of the surface a body. Let us remark, that although in the above formulation "horizons" clearly mean event horizons [19], in attempts to prove the hoop conjecture one usually deals with apparent horizons and trapped surfaces — surrogates of the event horizon (modulo reservations that are partly expressed in the introduction).

The work done in [11-15,42,43] may be regarded as an attempt to prove the hoop conjecture, but it is necessary to say that the authors were forced to reject the idea of finding criteria for the formation of trapped surfaces in terms of quantities that are asymptotic or external (as m and C are). I do not say more on that, because the next section is devoted to a detailed description of results obtained in the mentioned papers, also in the relation to the hoop conjecture [12].

Bonnor [44] has found a static solution with a circumference twice smaller than its ADM mass, thus contradicting the above stated analytic version of the hoop conjecture. One can argue, however, that it is not a real counter-example, since Bonnor's solution possesses a nonzero electric charge; thus long-range electrostatic forces are present. In nature this does not happen on the cosmic scale. One can try to prove the hoop conjecture for distributions of matter of compact support, by which I mean the absence of long-range material forces.

Flanagan [45] attempted to establish the hoop conjecture in terms of a modified measure of size (a modified "circumference", which is not a geodesic on the surface of a body), and using a mass different than the ADM mass (possibly the Hawking mass).

Special cases in support of those modifications are shown in [45]. Another way to establish the form of a hoop conjecture was chosen by Shapiro and Teukolsky [46]. They computed the minimum circumferences outside the matter and they tested the standard version of the hoop conjecture, with the ADM mass. Their numerical results support both the compactness conjecture and the (modified by them) analytical form.

There should exist a connection between the "only if" part of the Shapiro–Teukolsky version and the Penrose inequality II. Indeed, if the Gauss curvature of a surface S is bounded from below, then one can estimate a circumference by the area S of a surface S:

$$C \geq \sqrt{S}. \tag{3.15}$$

(The above estimation follows from Th. (9) on p. 250 in [9] and certainly can be improved.) Hence, if the Shapiro–Teukolsky version of the hoop conjecture is valid, then having a configuration with the apparent horizon S that encloses it, we conclude that

$$4\pi m \geq C. \tag{3.16}$$

Because of (3.15), also the following inequality

$$m \geq \sqrt{S/(16(\pi)^2)}.$$

is true; that is almost the Penrose inequality II, up to the factor $\sqrt{1/\pi}$ (and this factor may be improved, according to the above remark).

Thus, the breakdown of the Penrose inequality II might imply that the "only if" part of the hoop conjecture (as formulated by Shapiro and Teukolsky) is false; if there exist cases such that at the horizon the ratio $m/\sqrt{S/(16\pi)}$ is smaller than $\sqrt{1/\pi}$, then both conjectures will be broken.

A more detailed description of attempts to prove the hoop conjecture is presented in [45].

The results of Schoen and Yau

Let us define, following [34], $\underline{\mathrm{Rad}}(S)$ as the torus measure of a volume V inside S (see also subsection 3.9.), on an asymptotically flat Cauchy hypersurface Σ. Let ρ and J mean the energy and momentum density (see Sec. 2.7). The main result of Schoen and Yau gives a sufficient condition for the formation of apparent horizons and may be formulated as follows.

Theorem 1. [34] Let Λ be the minimal nonnegative value of $\rho - \sqrt{J_\mu J^\mu}$ in V. If

$$\Lambda(\underline{\mathrm{Rad}}(S))^2 \geq \frac{3(\pi)^2}{2}, \tag{3.17}$$

then Σ contains an apparent horizon.

Schoen and Yau have also proven a result that shows that it is not possible to pack an infinite amount of matter into a finite volume that is placed on a maximal Cauchy slice Σ. Namely, keeping the notation as above, they proved that

$$\underline{(\mathrm{rad}(S))}^2 \Lambda \leq \frac{4\pi^2}{3}. \tag{3.18}$$

There are several factors that limit the validity of the above results. As I have pointed out in the subsection 3.6, there exists an example which shows that the torus radius is not always a good size measure. The ratio of $\underline{\mathrm{Rad}(S)}$ to the proper radius $L(S)$ may be very small for some configurations [14]. Nevertheless, there exist examples of geometries that satisfy the conditions of Th. 1, so its content is not empty [47].

Next, the Theorem 1 is likely to be valid as a criterion for the formation of trapped surfaces inside stars, since then everywhere inside a star the value of Λ is expected to be nonzero; thus, Th. 1 says that if a star is made very big, then it should contain an apparent horizon. However, one can imagine a different situation in which black holes can emerge. Namely, assume a set of stars falling into a common centre. Then, as long as the stars do not coalesce, $\Lambda = 0$ for any surface surrounding them; hence, Th. 1 does not give any information about this case, although apparent horizons are likely to be created.

Resuming this short discussion: *(i)* the validity of the torus radius measure requires an examination; *(ii)* the results of Schoen and Yau may be used to diagnose a formation of apparent horizons inside a single star, but not (for instance) in a galaxy.

Trapped surface conjecture

We will quote the Seifert's conjecture in extenso [48] : "Trapped Surface Conjecture (TSC): any mass that is concentrated within a region of sufficiently small diameter can be surrounded by a trapped surface." The sentence following the above statement in [48] says that "Today, we do not even have a precise formulation of this conjecture. (What is 'mass'. What is 'diameter'?)"

Thus, TSC may be regarded as a specification of the hoop conjecture; the event horizon (that basically relates to the hoop conjecture) is replaced by trapped surfaces. Further, the fact that we do not have suitable (quasilocal) mass and size definitions is pointed out clearly. The next section is devoted to the proof of the trapped surface conjecture (TSC), in terms of the total rest mass and the proper radius, in classes of geometries that may be identified as not having gravitational radiation.

4. The proof of the trapped surface conjecture

This section reports mainly the results of the author's paper [12]. The subsection 4.2 bases on the author's work made together with Piotr Bizoń and Niall O'Murchadha, and subsection 4.3 on the paper [43]. Most of the material discussed below proves that the compression of matter leads to the formation of trapped surfaces. (But let us remark, that pure gravitational radiation might also form trapped surfaces [49].) The Theorems 4, 7 and 10 support the once expressed Einstein's opinion that in a finite volume only a finite amount of matter can be compacted [10].

A part of this section deals with averaged trapped surfaces. The notion of an average trapped surface has been introduced by Hartle and Wilkins [50], but their and mine results are disconnected.

4.1. Convex Geometries

In [12] it is assumed that Cauchy data are momentarily static (with one exception, to be mentioned later). The geometry of the Cauchy slice is asymptotically and conformally flat, so that the line element reads

$$ds^2 = f^4 dx^i dx^j \hat{g}_{ij}, \tag{4.1}$$

where $\hat{g}$ is a metric in the 3-dimensional Euclidean space. The Einstein constraint equations reduce then to the Hamiltonian constraint

$$\Delta f = -2\pi\rho f^5, \tag{4.2}$$

where Δ is a flat Laplacian. In what follows I assume that the constraints are satisfied, that is, the initial geometry is given, and my aim is to diagnose the (eventual) presence of trapped surfaces.

I assume that the level surfaces of the conformal factor f are convex with respect to the background, flat metric $\hat{g}$; it means that their Gauss curvature $\widehat{K}$ (in the flat geometry $\hat{g}$) is nonnegative. In flat space the nonegativity of $\widehat{K}$ of $\widehat{S}$ is sufficient to ensure that $\widehat{S}$ is globally convex, *i.e.*, that any two points inside a volume V enclosed by S can be joined by a geodesic that entirely lies in V. The Gauss curvature K of S with respect to the physical metric g is just a rescaling of $\widehat{K}$, so that $K \geq 0$ (and the opposite, if $K \geq 0$, then also $\widehat{K} \geq 0$); but that ensures only the local convexity with g. (Two sufficiently close points can be joined by a geodesic lying inside V.) In the rest of this subsection I will deal mainly with averaged trapped surfaces (see 2.17). The notation is that of Sec. 2. In [12] the following result is proven.

Theorem 1. Assume conditions stated above. A necessary condition for an equipotential convex surface S to be an averaged trapped surface (and thus: a trapped surface pointwise) is that

$$M(S) \geq \frac{\sqrt{S/\pi}}{2}. \tag{4.3}$$

I would like to point out that in Th. 1 we do not require the weak energy condition; ρ may be negative, although from (4.3) a negative matter density works against the formation of averaged trapped surfaces.

This inequality is sharp: in the spherically symmetric case it yields the exact [13,14] estimation

$$M(S) \geq R, \tag{4.4}$$

where R is the areal radius, $R =: \sqrt{S/(4\pi)}$ (in [12-15] we named R as "Schwarzschild radius").

Theorem 1 proves a part of the "only if" statement of the hoop conjecture (or, rather, of the compactness conjecture). If one starts with a given amount of matter and compresses it in at least two directions, then the condition (4.3) will always be met. But the compression in one dimension most likely will not produce a trapped surface, except in the case when a very long and thin body is compacted along its length. Then its surface becomes significantly smaller and the necessary condition could be satisfied. Let us point out that Theorem 1 shows that the notion of equipotential surfaces is nontrivial, in this sense that they may exist only in strongly curved geometries.

To proceed further, I will employ a foliation adapted to equipotential surfaces, to have the line element

$$ds^2 = f^4(\sigma)[\hat{g}_{\sigma\sigma}d\sigma^2 + \hat{g}_{ij}dx^i dx^j], \quad i,j = 2,3. \tag{4.5}$$

Here $\sigma \geq 0$, σ foliates the level surfaces of the conformal factor f and x^2, x^3 are quasi-angle variables. Such a choice is convenient from the technical point of view [11]. The sufficient condition is the following.

Theorem 2. Assume conditions of Th. 1 and, in addition, the weak energy condition. If the content of matter $M(S)$ inside S exceeds the quantity Rad(S), then there must exist averaged trapped surfaces inside S.

The radius Rad(S) (see also 3.6) can be bounded from above,

$$\mathrm{Rad}\big(S(\sigma)\big) \leq \int_0^\sigma f^2(s)\sqrt{\sup_{s=\mathrm{const}}\hat{g}_{\sigma\sigma}}ds + D(S(0)). \tag{4.6}$$

The quantity D(S(0)) can be bounded from above [11] by $\pi rad(0)/4$, where rad(0) is the radius of the disc on which the conformal factor achieves its maximal value. In the case of spherical symmetry rad(0) vanishes, of course. The quantity $Rad(S)$ equals to the proper radius in spherical geometries and appears to be bounded from above by the largest proper radius $L(S)$ of the volume enclosed by the surface S in all spheroidal geometries. *I conjecture that this is always true when the equipotential surfaces are convex.*

Theorem 2 is strict, since it gives the same coefficients as in the spherically symmetric case.

Assume that all equipotential surfaces inside S have the following property:

$$F \equiv \Big(\int_S \hat{p} d\hat{S}\Big)^2 - 4\int_S \widehat{K}\hat{n}_\sigma d\hat{S} \int_S \hat{n}^\sigma d\hat{S} \geq 0. \tag{4.7}$$

A few remarks are in order. The equality in (4.7) is achieved in the spherically symmetric case. The examination of foliations by spheroidal surfaces (both prolate and oblate), shows that the left hand side of (4.7) is the more positive the greater is eccentricity of surfaces. This gives incentive to make my second conjecture that *all convex (and, perhaps, sufficiently smooth) foliations in 3-dimensional Euclidean space satisfy the inequality 4.7.*

With the last property, Theorem 2 may be replaced by a stronger version.

Theorem 2'. Assuming the conditions of Th. 1, the weak energy condition and the property 4.7, if the content of matter $M(S)$ inside S exceeds $\mathrm{Rad}(S)$ (or $L(S)$, the largest proper radius of a volume enclosed by S, if the first conjecture is accepted) then S itself is an ATS.

Now follows a sufficient result for the existence of pointwise trapped surfaces, *i.e.*, with $P(p) \leq 0$ (see 2.13) at each point p of a surface S.

Theorem 3. Under the conditions of Th. 2, if at S the following inequality is satisfied

$$M(S) \geq \mathrm{Rad}(S) + \int_S [n^\sigma (n_\sigma p)_{\max} - p] dS, \tag{4.8}$$

(no summation over σ) then S is pointwise trapped.

Thus, if the content of matter inside S is made infinitely large for fixed $\mathrm{Rad}(S)$, then S will always become a trapped surface. In this sense massive singularities on a momentarily static Cauchy surface are always trapped, according to Seifert's conjecture. The point is, however, that in the class of geometries discussed the ratio $M(S)/\mathrm{Rad}(S)$ is bounded from above.

Theorem 4. If the initial (nonsymmetrical in time) data of Einstein's equations are conformally flat on the maximal Cauchy slice, with the level surfaces of the conformal factor being convex inside S and satisfying the condition (4.7), then

$$\begin{aligned} M(S) &\leq \mathrm{Rad}(S) + \int_{\widehat{S}} f^2 \hat{p} d\widehat{S} \\ &\leq 2\mathrm{Rad}(S). \end{aligned} \tag{4.9}$$

All the above inequalities are exact, since they are saturated by some spherically symmetric geometries ([13-15]). Above we pointed out that Th. 1 essentially proves the "only if" part of the hoop conjecture. Theorem 2' and 3, in turn, essentially proves the "if" part of the hoop conjecture, with the restriction that although trapped surfaces will form when matter is compressed in all dimensions, in addition, *the compression should be sufficiently "round"*. This reservation is related to the (possible) existence of the bound expressed in Th. 4. All the results are valid for compact and noncompact distributions of matter and for geometries that are convex (and satisfy the condition (4.7)) or almost convex (in a sense that could be precisely defined).

4.2. Spherically symmetric geometries

The formation of trapped surfaces in spherically symmetric geometries has been solved completely by P. Bizoń, N. O' Murchadha and the author ([13,14]; see also an improvement in [15]). Below I present hitherto proven results using theorems of the preceding section. In the case of spherical symmetry we notice that

$$\mathrm{Rad}(S) = L(S), \tag{4.10}$$

where $L(S)$ is a proper radius of the centered ball enclosed by S. Moreover, the concepts of averaged trapped surfaces and pointwise trapped surfaces are equivalent. It is remarkable also that spheres satisfy the property 4.7.

Theorem 5. Assuming conditions of Th. 1 in 4.1, if

$$M(S) < R(S), \tag{4.11}$$

where $R(S)$ is the Schwarzschild radius of S, then S is not trapped.

Th. 5 follows directly from Th. 1, while the next theorem is consequence of Th. 2'.

Theorem 6. Assuming conditions of Th. 2' in 4.1, if

$$M(S) \geq L(S) \tag{4.12}$$

then S is trapped.

Theorem 7. Under conditions as in Th. 4 in 4.1, the amount of matter inside S is bounded from above

$$M(S) \leq L(S) + R(S) \leq 2L(S). \tag{4.13}$$

This estimation is valid, similarly as Th. 4 (from which it follows), also for nonsymmetrical in time Cauchy data, in contrast with Ths. 5 and 6, that are true only in momentarily static geometries. The next two inequalities have no analogues in 4.1.

Theorem 8. Under conditions as in Th. 1, a necessary condition for a sphere S to be trapped is

$$M(S) \geq \frac{L(S)}{2}. \tag{4.14}$$

The theorem 8 is stronger that the theorem 5, since $L \geq R$, but both theorems are strict [13]. Both theorems do not require the weak energy condition, as opposite to all remaining statements in this subsection.

Theorem 9. Let Cauchy data be nonsymmetrical in time, $K^{ij} \neq 0$, asymptotically flat and satisfy the weak energy condition. Then, if

$$M(S) - P(S) \geq \frac{7L(S)}{6}, \tag{4.15}$$

where $P(S) = \int_{V(S)} J_i n^i dV$ is a total radial momentum transfer inside S, there exists a trapped surface inside S. (For notation see 2.7. I assume that $J_i n^i$ is positive if matter moves outwards.)

Let us point out that in Ths 7, 9 we require the weak energy condition but we do not require the strong energy condition. Theorems 5–8 are exact, as shown in [13], but the last result is not exact. In [14] an example is discussed in which a value of the coefficient at $L(S)$ is achieved of about 4 percents less than $^7/_6$.

4.3. Spheroidal geometries

These are special cases of convex geometries, with level surfaces of the conformal factor f being either oblate or prolate spheroids [51]. Such geometries satisfy, as shown by direct calculation, the condition (4.7). Moreover,

$$\mathrm{Rad}(S) \leq \sup L(S). \tag{4.16}$$

We should remark that the proof of (4.16) requires a set of subtle differential inequalities. For oblate spheroids, it has been done in [43]; for prolate spheroids it has been done by the author (unpublished). Thus, all results of Sec. 4.1 are true in the class of spheroidal geometries. In [43] the result of Th. 1 has not been proven, but instead there is a different necessary condition for the existence of averaged trapped surfaces (ATS).

Theorem 10. Assume the geometry of oblate spheroids on asymptotically flat Cauchy slice (no weak energy condition). If

$$M(S) < \frac{\sup L(S)}{2}, \tag{4.17}$$

then a spheroid S is not ATS.

In [43] a sufficient condition is proven for the formation of pointwise trapped surfaces; it is a more detailed specialization of Th. 3. It implies that strongly nonspherical spheroids cannot be trapped; that follows from the existence of the bound expressed in Th. 7. (But, as pointed out in [43], there might exist trapped surfaces that are not spheroids.)

5. Estimations of the mass

The theorems 4 and 7 of Sec. 4 estimate the total rest mass M inside a surface S by $2\mathrm{Rad}(S)$; in spherically symmetric geometries $\mathrm{Rad}(S) = L(S)$, and in that case this estimation was conjectured by Arnowitt, Deser and Misner [16,52,53]. They have proved that massive spherical shells (momentarily static) satisfy the identity

$$m = M - \frac{M^2}{2L}. \tag{5.1}$$

In this case the ADM mass m is explicitly positive, assuming that the energy density is nonnegative. From (5.1) it follows that $M \leq 2L$, since otherwise m would become negative. Therefore, also m is bounded from above. Thus, momentarily static configurations generated by massive shells are not plagued by infinities; in a sphere of a finite proper radius both total rest mass and the ADM mass are finite. Arnowitt, Deser and Misner wished to use the fact that $m \geq 0$ to prove that gravitational forces cancel infinities. They have argued that spherical massive shells are the only configurations that satisfy the equality in (5.1), and that more general spherical configurations should satisfy the inequality

$$m \leq M - \frac{M^2}{2L}. \tag{5.2}$$

Arguing as above, Arnowitt, Deser and Misner convincingly suggested that not only m is finite, but also M, if one fixes the proper radius L. The estimation (5.2) has been proven in [15]. In [17] a more general result has been proven.

Theorem 11. Let Σ be an asymptotically flat and spherically symmetric Cauchy hypersurface with the (spherically symmetric) extrinsic curvature K_{ij} vanishing outside a body (which is assumed to be of compact support). If $|q| \leq m$, q being the total electric charge, then

$$m \leq M - \frac{M^2}{2L} + \frac{q^2}{2L}. \tag{5.3}$$

The inequality (5.3) is strict, as being saturated by a massive and charged spherical shell.

Let us remark, that the condition that the extrinsic curvature vanishes outside a body is quite restrictive, since it means that although there might be some radial flux of matter, but its total radial momentum has to vanish (Hint: solve the momentum constraints of the Einstein equations.). In contrast with that, the estimations of Ths 4 and 7 from the previous section hold even for moving matter. The inequality $M \leq 2\text{Rad}(S)$ has been proven earlier also in [43].

Nevertheless, the above inequalities are of sufficient interest in themselves. From (5.2) it follows that if the ratio $M/(2L)$ tends to zero, then the ADM mass m vanishes; thus, adding matter into a configuration of a fixed diameter, one observes that the ADM mass becomes (eventually) less. In this sense the ADM mass is not additive. That fact was shown by Arnowitt, Deser and Misner in the case of spherical shells (notice, that in (5.1) m decreases for $M > L$), and in more general spherical configurations by Bartnik [54].

One comment more on this point, because it is one of the reasons why the conditions for the formation of trapped surfaces (see Sec. 5) are expressed in terms of M and not m. Namely, (5.2) implies that there is no universal (and finite) constant C such that

$$M \leq Cm. \tag{5.4}$$

Although the inequality $M \geq m$ is always (and trivially) true in spherically symmetric geometries, M and m are not equivalent, because there is no universal C in (5.4); that is, no inequality $m \leq M \leq Cm$ holds. Thus, the results of Sec. 4 cannot be expressed in terms of the asymptotic mass m.

6. Applications of isoperimetric inequalities

The criteria of Sec. 4 were used in [55,56] to prove the absence of trapped surfaces if the Cauchy data for Einstein–Yang–Mills–Higgs equations are of the type of the magnetic monopole of Bogomolny–Prasad–Sommerfield [57].

More interesting, perhaps, is the numerical proof that the BPS-gravitational configuration can form a "bag of gold" [58,59]. In the pioneering work on the appearance of "bag of gold", Wheeler discussed the possibility, that densely packed gravitational fields could be surrounded by surfaces of area that becomes the less, the more gravitational energy inside. Eventually, the area of the smallest surface will shrink to a point, which is a real curvature singularity.

An example of a material system that has a similar property is a massive spherical shell configuration with $M/(2L)$ very close to 2 and a fixed value of the radius L, with that difference that then the ADM mass goes to zero and the configuration just disappears (no nonzero mass is seen from outside).

In [55] an example is presented of a field configuration that can form the "bag of gold". The numerical solution of [55] possesses a nonzero (negative) ADM mass; thus, the gravitational BPS monopole disappears but the external observer still feels its (repelling!) interaction. The solution is, however, only continuous and possesses a nonintegrable curvature singularity at a surface shrinking to a point.

In the paper [56] a simpler (than in [55]) proof is presented that "naively self-dual" BPS monopoles cannot create trapped surfaces. There are shown also examples of BPS monopoles that are not "naively self-dual" and that create geometries that contain trapped surfaces. Later Bizoń [60] has discovered a numerical solution that possesses the same property; he has found a static "coloured black hole", related to the Bartnik's static solution of Einstein-Yang-Mills equations [61].

7. Trapped surfaces in expanding open universes.

Assume we have an open k=0 Friedmann–Lemaître universe with spherically symmetric inhomogeneities on a space like slice, that do not change (initially) a rate of the expansion of a volume of the slice. I give a set of necessary and sufficient criteria for the formation of trapped surfaces due to those inhomogeneities on the surface of initial data. A bound for the size of a perturbed trapped region is found, which depends on the cosmological energy density. Those results are proven in [62].

The background quantities (denoted by a hat) are related to the perturbed ones as follows:

$$g_{ab} = \phi^4(r)\hat{g}_{ab} \tag{7.1}$$

$$\hat{g}_{ab} = a(t)\delta_{ab} \tag{7.2}$$

$$\widehat{K}_{ab} = da/dt\delta_{ab} =: \beta(t)\hat{g}_{ab} \tag{7.3}$$

$$K_{ab} = \phi^4(r)\widehat{K}_{ab} + \delta K_{ab} = g_{ab}\beta + \delta K_{ab}. \tag{7.4}$$

In (7.3), $a(t)$ is a scalar function determined by the Friedmann equation and in (7.4) β is a scalar function (the Hubble function) describing the rate of change of the 3-metric.

I demand that although the extrinsic curvature may be perturbed, the value of it's trace cannot be changed, $K_a^a = g^{ab}K_{ab} = \hat{g}^{ab}\widehat{K}_{ab}$; from this follows that $g^{ab}\delta K_{ab} = 0$, that is

$$\delta K_{ab} = (n_a n_b - g_{ab}/3)K(r). \tag{7.5}$$

Intuitively, the motivation for (7.5) is that we do not wish to produce caustics among time-like curves orthogonal to Σ. We denote the perturbation of ρ by $\delta\rho$ and of the current J_b by δJ_b. The cosmological background satisfies the following the relation:

$$(K_a^a)^2 - K_{ab}K^{ab} = 6\beta^2 = 16\pi\hat{\rho} \quad (7.6).$$

In [62] the following results are proven.

Theorem 1. [A sufficient condition]
Assume that spherical perturbations of homogeneous cosmological (k=0) Cauchy data satisfy the conditions:
(i) $\delta\rho$ is nonnegative;
(ii) $K_a^a = \text{const}$, *i.e.*, the rate of expansion of the volume is nonperturbed;
(iii) $\delta J_b = 0$.

If at a sphere S, its radius L and the mass of the perturbation $\delta M = \int_V \delta\rho dV$ satisfy

$$\delta M > L + S\sqrt{\hat{\rho}/(6\pi)} \quad (7.7)$$

then S is trapped.

Theorem 2. [A necessary condition]
Assume that perturbed initial data satisfy the conditions *(ii)* and *(iii)* of Th. 1. If

$$\delta M < \frac{L}{2} + S\sqrt{\hat{\rho}/(6\pi)} \quad (7.8)$$

then S is not trapped.
Remarks.

1. Note that here we do *not* require,in contrast to the theorem 1, that $\delta\rho \geq 0$.
2. Another form of the necessary condition for S to be trapped is $\delta M > R_0 + S\sqrt{\hat{\rho}/(6\pi)}$, where R_0 is the areal (Schwarzschild) radius, $R_0 = \sqrt{S/(4\pi)}$.
3. The sufficient condition is saturated by perturbations of the form of a massive spherical shell (see an explicit solution in [15], while examples saturating the necessary condition can be found in [14].

   Thus the estimates in both theorems are sharp.

Theorem 3. (Absence of large trapped surfaces)
Assume conditions of Theorem 1. If a background energy density $\hat{\rho}$ satisfies at a sphere S of a proper radius L and the Schwarzschild radius R_0 the inequality

$$\sqrt{\hat{\rho}/(6\pi)} \geq \frac{1}{4\pi R_0} + \frac{L}{8\pi R_0^2}, \quad (7.9)$$

then S cannot be trapped.

Let me remark, that large $\hat{\rho}$ means that the rate of expansion of the Universe is large (see the relation (7.6). Therefore a part of Th. 3 is intuitively obvious — the quicker the geometry is expanding, the more difficult it should be to have the negative expansion, *i.e.*, to enforce initially parallel photons to create caustics. It comes as a surprise, however, that there is an upper bound for the size of trapped spheres, while no lower bound exists; it is easier, in a sense, to form small trapped spheres than large ones.

Theorem 4. (A sufficient condition for moving perturbations)

Assume that spherically symmetric perturbations satisfy the conditions *(i)* $\delta\rho \geq 0$); *(ii)* $K_a^a = \text{const}$. If at a sphere S

$$\delta M - \int_V \delta J_b n^b dV \leq \frac{7L}{6} + \sqrt{(8/3)\pi\hat{\rho}}L^2, \tag{7.10}$$

then there is a trapped surface inside s.

Theorem 5. Under the above conditions, the mass of the inhomogeneity $\delta M = \int_V \delta\rho dv$ inside a sphere S cannot exceed the sum of its proper radius L and the areal radius R_0,

$$\delta M = \int_V \delta\rho dv \leq L + R_0. \tag{7.11}$$

To get the above results, one should apply techniques similar to those used in [13-15] where the formation of trapped surfaces in asymptotically flat space-times was investigated. The results are similar to those of Sec. 4, with the significant difference, that instead of dealing with the total rest energy and/or the total radial momentum, we work with the energy and momentum carried by spherical perturbations only. The Theorem 3 has no analogue in asymptotically flat space-time, or rather we would say that in asymptotically flat space-times the inequality (7.9) can never be satisfied, since then $\hat{\rho} = 0$. In asymptotically flat spacetimes no upper bound exists for the size of trapped surfaces. From Ths 1, 2, 3 and 4 one can get all statements proven in Sec. 4.2. by putting $\hat{\rho} = 0$.

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