LARGE N EXPANSION FOR THE WILSON COEFFICIENTS IN $K \to \pi\pi$ DECAYS.

P. BIAŁAS

Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Cracow, Poland

AND

K. ZALEWSKI

Institute of Nuclear Physics
Kawiory 26a, 30-055 Cracow, Poland

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The effective Hamiltonian governing the decays $K \to 2\pi$ depends on the Wilson coefficients $z_1, \ldots, z_6, y_1, \ldots, y_6$. We express these coefficients as convergent series in the parameter $x = 1/N$, where N is the number of colours. Analytic formulae for the (N-dependent) coefficients of these series are given. The first approximation reproduces the results of Bardeen, Buras and Gérard. Two more expansion terms are calculated and the corresponding approximations to $z_1, \ldots, y_6$ are compared with the exact results.

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The effective Hamiltonian used to describe the decay $K \to \pi\pi$ depends on the Wilson coefficients $z_1(\mu), \ldots, z_6(\mu), y_1(\mu), \ldots, y_6(\mu)$ [1]. Since in the following many statements apply to both z_i and y_i, we shall use the notation $\zeta_i(\mu)$ to denote $z_i(\mu)$ or $y_i(\mu)$. In the formulae describing the $K \to \pi\pi$ decay the coefficients y_i occur multiplied by

$$\tau = -\frac{V_{\rm td}V_{\rm ts}^*}{V_{\rm ud}V_{\rm us}^*}, \tag{1}$$

where V_{ij} are the elements of the Cabibbo-Kobayashi-Maskawa matrix. Since the absolute value $|\tau|$ is of the order of 0.001 [2], the coefficients y_i contribute little. The purpose of the present paper is to study the nature of the $^1/_N$ expansion, where N is the number of colours, in order to understand why the simple high N approximation used by Bardeen, Buras and Gérard [3] (further quoted BBG) gives a good approximation to the Wilson coefficients. Therefore, we will use the BBG formalism suitable for the case $m_t \ll M_W$, in spite of the fact that now it is known that $m_t > 89$ GeV [4].

The coefficients ζ_i are assumed to satisfy the initial conditions [1]

$$\zeta_i(M_W) = \delta_{i2} \tag{2}$$

and the renormalization group evolution equations

$$\mu^2 \frac{d}{d\mu^2}\zeta(\mu^2) = \frac{\alpha_s N}{4\pi}\gamma^T \zeta(\mu^2), \tag{3}$$

where ζ denotes the vector $(\zeta_1, \ldots, \zeta_6)^T$. The anomalous dimension matrices are

$$\gamma = \begin{pmatrix} -\frac{3}{N^2} & \frac{3}{N} & 0 & 0 & 0 & 0 \\ \frac{3}{N} & -\frac{3}{N^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{N^2} & \frac{3}{N} & 0 & 0 \\ 0 & 0 & \frac{3}{N} & -\frac{3}{N^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{N^2} & -\frac{3}{N} \\ 0 & 0 & 0 & 0 & 0 & -3+\frac{3}{N^2} \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3N^2} & \frac{1}{3N} & -\frac{1}{3N^2} & \frac{1}{3N} \\ 0 & 0 & -\frac{2}{3N^2} & \frac{2}{3N} & -\frac{2}{3N^2} & \frac{2}{3N} \\ 0 & 0 & \frac{n_f}{3N^2} & -\frac{n_f}{3N} & \frac{n_f}{3N^2} & -\frac{n_f}{3N} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{n_f}{3N^2} & \frac{n_f}{3N} & -\frac{n_f}{3N^2} & \frac{n_f}{3N} \end{pmatrix} F_i(\mu^2), \tag{4}$$

where the second term is due to the penguin diagrams. The factor N in Eq. (3) is extracted, because together with α_s it gives a factor $\alpha_s N$, which tends to a finite limit for $N \to \infty$. The factors $F(\mu^2)$ distinguish between the z and the y coefficients. In the simplest approximation assuming exact GIM cancellation [5] we have for the z coefficients

$$F_z(\mu^2) = \theta(m_c^2 - \mu^2) \tag{5}$$

and for the y coefficients.

$$F_y(\mu^2) = \theta(m_t^2 - \mu) - O_2 \times F_z(\mu^2). \tag{6}$$

In the formulae O_2 is a projective operator acting on the matrix before him. It is equal identity on the second row of the matrix on which it acts and zero on all the remaining rows, $\theta(x)$ is the step function equal one for $x \geq 0$ and zero otherwise.

Equation (3) can be solved by diagonalizing the γ matrices [5]. This, however, is a cumbersome procedure. BBG noticed that replacing matrices (4) by their large N approximations

$$\gamma = \begin{pmatrix} -\frac{3}{N^2} & \frac{3}{N} & 0 & 0 & 0 & 0 \\ \frac{3}{N} & -\frac{3}{N^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3N} & 0 & \frac{1}{3N} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} F_i(\mu^2), \tag{7}$$

one very simply obtains results, which almost coincide with the results of the full calculation. Note that the rules for going from (4) to (7) are not simple power counting. The 2×2 submatrix in the upper right corner is kept exact, because it governs the evolution of the large coefficients over the large interval from M_W^2 to M^2. In the penguin matrix only the terms $O(N^0)$ coupling the large coefficients ζ_2 to the penguins are kept. From the 4×4 submatrix in the lower right corner only the term $O(N)$ is left.

In the present note we substitute in the matrix (4) x for every $^1/_N$ everywhere except in the 2×2 submatrix in the upper left corner. We find the coefficients ζ_i as convergent power series in x. The first two terms of our expansion reproduce the results of BBG. We calculate two more terms and discuss the rapidity of convergence.

Let us note first that the equations for ζ_1 and ζ_2 decouple and do not contain x:

$$\mu^2 \frac{d\zeta_1}{d\mu^2} = -\frac{3\alpha}{N}\zeta_1 + 3\alpha\zeta_2 \tag{8}$$

$$\mu^2 \frac{d\zeta_2}{d\mu^2} = 3\alpha\zeta_1 - \frac{3\alpha}{N}\zeta_2 \tag{9}$$

where

$$\alpha = \frac{\alpha_s}{4\pi} = \frac{1}{b_f \ln \frac{\mu^2}{\Lambda^2}}; \qquad b_f = \frac{11N - 2n_f}{3}. \tag{10}$$

The solution of (8,9) satisfying the initial conditions (2) is

$$\zeta_1 = \frac{c_+ - c_-}{2}, \qquad \zeta_2 = \frac{c_+ + c_-}{2} \tag{11}$$

with

$$c_{\pm} = \left[\frac{\alpha(M_W^2)}{\alpha(\mu^2)}\right]^{-\frac{3}{Nb_f} \pm \frac{3}{b_f}} \tag{12}$$

In the following the coefficients ζ_1, ζ_2 are considered known functions of μ^2 and of order $O(x^0)$. The BBG approximation, therefore, is an approximation only for the coefficients $\zeta_3, \ldots, \zeta_6$.

Let us introduce the matrix σ obtained from matrix γ by replacing all its elements in the first two rows by zeros. We also introduce a vector f consisting of the elements of the second row of the penguin part of γ. Then the equations for $\zeta_3, \ldots, \zeta_6$ are contained in

$$\mu^2 \frac{d\zeta}{d\mu^2} = \alpha N(\sigma^T \zeta + f^T \zeta_2). \tag{13}$$

Substituting the expansions

$$\sigma^T = \sigma^{(0)} + \sigma^{(1)} x + \sigma^{(2)} x^2, \tag{14}$$

$$f^T = f^{(1)} x + f^{(2)} x^2, \tag{15}$$

$$\zeta_i = \sum \zeta_i^{(n)} x^n, \tag{16}$$

into (13) and equating to zero the coefficients of the subsequent powers of x, we obtain equations of the form

$$\mu^2 \frac{d\zeta^{(n)}}{d\mu^2} = \alpha N \sigma^{(0)} \zeta^{(n)} + h^{(n)}, \qquad n = 0, 1, \ldots \tag{17}$$

where

$$h^{(n)} = \sigma^{(1)} \zeta^{(n-1)} + \sigma^{(2)} \zeta^{(n-2)} + f^{(n)} \zeta_2, \qquad n = 0, 1, \ldots \tag{18}$$

with

$$\zeta^{(-2)} = \zeta^{(-1)} = 0, \qquad f^{(0)} = f^{(3)} = f^{(4)} = \ldots = 0 \tag{19}$$

The BBG approximation consists in putting $\sigma^{(1)} = \sigma^{(2)} = 0$ and $f_2 = 0$. Since $\zeta_{i\geq 3}^{(0)} = 0$, this leaves the first two equations (17) unchanged, while the higher n equations with initial conditions (2) have the solution $\zeta_{i\geq 3}^{(n\geq 2)} = 0$. Thus, the BBG approximation coincides with the first approximation. Here and in the following the n-th approximation means the approximation up to and including the terms of order $O(x^n)$. Actually BBG make one more approximation by replacing $\alpha_s N$ by its $N \to \infty$ limit, which happens to improve the approximation for ζ_6.

Starting with the $n = 0$ equation (17) and working upward in n, we find at each step that h_n is a known function. Substituting

$$\sigma_0^{ij} = -3\delta_{6j}\delta_{6i}, \tag{20}$$

one finds

$$\zeta_i^{(n)}(\mu^2) = -N \int_{\mu^2}^{M_W^2} \frac{\alpha h_i^{(n)}(Q^2)}{Q^2} dQ^2 \qquad i = 3,4,5 \tag{21}$$

$$\zeta_i^{(n)}(\mu^2) = -N \int_{\mu^2}^{M_W^2} \frac{\alpha h_i^{(n)}(Q^2)}{Q^2} \left[\frac{\alpha(\mu^2)}{\alpha(Q^2)}\right]^{\frac{3N}{b_f}} dQ^2 \qquad i = 6 \tag{22}$$

In order to prove the convergence of the series (16) let us note that $\alpha(\mu^2)$ is a decreasing function of μ^2. Moreover the largest absolute value of an element of $\sigma^{(1)}, \sigma^{(2)}$ is $(^{11}/_3)N$. Therefore

$$\left|\zeta^{(n)}(\mu^2)\right| < A \int_{\mu^2}^{M_W^2} \left|\zeta^{(n-1)}(Q^2)\right| + \left|\zeta^{(n-2)}(Q^2)\right| dQ^2, \tag{23}$$

where $|\zeta^{(n)}(\mu^2)|$ is the absolute value of the largest component of $\zeta_{i\geq 3}^{(n)}$. The coefficient

$$A = \frac{11N\alpha(M^2)}{3M^2} \left[\frac{\alpha(M^2)}{\alpha(M_W^2)}\right]^{\frac{3N}{b_f}}, \tag{24}$$

where $M^2 \leq \mu^2$ is arbitrary, depends neither on μ nor on n. Using inequality (23) to eliminate from its right hand side all the components of $\zeta^{(n>2)}$ we obtain finally

$$\left|\zeta^{(n)}(\mu^2)\right| < \sum A^k |\zeta| \int_{\mu^2}^{M_W^2} d\mu_1^2 \int_{\mu_1^2}^{M_W^2} d\mu_2^2 \ldots \int_{\mu_{k-1}^2}^{M_W^2} d\mu_k^2$$

$$= |\zeta| \sum \frac{[A(M_W^2 - \mu^2)]^k}{k!} \tag{25}$$

where $|\zeta| = \max(|\zeta^{(1)}|, |\zeta^{(2)}|)$. In the sum

$$\mathrm{E}(\frac{n-1}{2}) \leq k \leq n-2, \tag{26}$$

where E(x) denotes the biggest integer not exceeding x. In general there is more than one term for each k. The total number of terms, however, does not exceed 2^n and for $E\big((n-1)/2\big) > A(M_W^2 - \mu^2)$ the terms with the smallest k are the biggest. Thus we get the estimate

$$|\zeta^{(n)}(\mu^2)| < \frac{2^n[A(M_W^2 - \mu^2)]^{E(\frac{n-1}{2})}}{(E(\frac{n-1}{2}))!} \tag{27}$$

which implies the convergence of series (16) for $i = 3, 4, 5, 6$ and all x.

TABLE I

	z_3^n	z_4^n	z_5^n	z_6^n
$n = 1$	0.0000	−0.0500	0.0000	−0.0596
$n = 2$	0.0586	0.0081	0.0500	0.0090
$n = 3$	−0.0090	−0.0210	−0.0081	0.0159
first	0.0000	−0.0167	0.0000	−0.0197
second	0.0065	−0.0157	0.0055	−0.0189
third	0.0062	−0.0165	0.0053	−0.0183
exact	0.0064	−0.0165	0.0052	−0.0183

In Table I we list the coefficients z_i for $i = 3, 4, 5, 6$ and $n = 1, 2, 3$. As mentioned $z_{i\geq 3}^{(0)} = 0$. We also compare the first, second and third approximation to $\zeta_{i\geq 3}$ with the exact results obtained by solving numerically equation (13). In the calculations we have used the following parameters: $M_W = 80.6$ GeV, $m_c = 1.35$ GeV, $\Lambda = 200$ MeV, $\mu = 0.8$ MeV and $n_f = 4$. The choice of BBG for the number of flavours: $n_f = 4$ deserves a comment. The z penguins evolve only in the mass range $\mu^2 \leq Q^2 \leq m_c^2$, where $n_f = 3$, but this evolution is strongly influenced by the initial (at m_c^2) value of z_2, which evolves all the way from M_W^2, where $n_f = 6$. Thus $n_f = 4$ is some reasonable average. In the present approximation the values of z_i do not depend on m_t. Note, however that for $m_t > M_W$ the whole theory changes, because then W exchange cannot be considered a point interaction and the initial conditions (2) are no longer plausible [6]. Then $\zeta^{(0)} \neq 0$ and the simple relation with the BBG approximation is lost. Therefore, strictly speaking, these results for the z_i coefficients are valid for models with $m_t \ll M_W$. We find that the third approximation is good for all the coefficients, while the first (BBG) approximation is good within 10 per cent for the bigger coefficients z_4 and z_6, while it introduces an error of the order of 30 per cent of the bigger coefficients for the smaller coefficients z_3 and z_5, which it puts equal zero. The coefficients z_3 and z_5 are well reproduced by the second approximation while z_4 in the second approximation is actually much worse than in the first one. We see no hint of an effective expansion parameter $(sN)^{-1}$ with s constant.

In the Table II we list the coefficients $y_{i \geq 3}$ and the corresponding approximations as in Table 1. The parameters have been chosen as before, except that $n_f = 5$ corresponding to the evolution from M_W^2 to m_c^2. The additional parameter is $m_t = 40$ GeV as in BBG. For the y_i coefficients the convergence of our expansion is poorer, as was to be expected, since the evolution of the penguins is over a larger mass interval. The third approximation is good within about 10 per cent except for y_3, where the error is almost 25 per cent. The BBG approximation is particularly bad for y_6, where the error exceeds 50 per cent. In order to get some idea about the effects of changing the t-quark mass, we recalculated the coefficients $y_{i \geq 3}$ with $m_t = 80$ GeV. The result shown in Table III. are qualitatively similar to those from Table II.

TABLE II

$m_t = 40$ GeV				
	y_3^n	y_4^n	y_5^n	y_6^n
$n = 1$	0.0000	−0.1361	0.0000	−0.3348
$n = 2$	0.2518	0.1747	0.1361	0.2872
$n = 3$	−0.2520	−0.4311	−0.1747	0.1840
first	0.0000	−0.0453	0.0000	−0.1116
second	0.0280	−0.0260	0.0151	−0.0800
third	0.0186	−0.0419	0.0087	−0.0729
exact	0.0242	−0.0405	0.0091	−0.0718

TABLE III

$m_t = 800$ GeV				
	y_3^n	y_4^n	y_5^n	y_6^n
$n = 1$	0.0000	−0.1514	0.0000	−0.4030
$n = 2$	0.2900	0.2167	0.1515	0.3728
$n = 3$	−0.3200	−0.5413	−0.2167	0.2084
first	0.0000	−0.0505	0.0000	−0.1343
second	0.0322	−0.0264	0.0168	−0.0930
third	0.0204	−0.0464	0.0088	−0.0852
exact	0.0276	−0.0446	0.0095	−0.0835

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