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ON THE EMERGENCE OF ASYMPTOTIC LOCALIZATION FOR SOME RANDOM DIFFUSION PROCESSES*

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Interpreting intermittency as the result of a cascade through a random medium, it is shown that the long time behaviour of the D-dimensional random heat equation generates intermittent patterns as well as a multifractal structure. Intermittent fluctuations are arranged in a hierarchical fashion. Moreover, multifractal analysis reveals that the cascading system organizes itself into a point-like set of "spikes" whose statistical properties are given. Such "spikes" are similar to localized asymptotic states. This allows one to sketch out possible applications to "non-thermal" transitions in multiparticle production.

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1. Introduction

Intermittency first appeared as an attempt to understand one of the possible routes towards fully developed turbulence in Hydrodynamics [1]. It means that from time to time there are highly irregular patterns emerging from a laminar flow. As a consequence, intermittency has been coined to designate phenomena where high peaks burst out from a flat background. Since then, this concept has been successfully applied to high energy physics [2], where the rapidity distribution of observed particles shows peaks and holes reminiscent of hydrodynamic structures. It is still a major issue to try to understand the origin of such fluctuations, and how to relate them to Quantum Field Theory such as QCD (Quantum ChromoDynamics).

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Nevertheless, it is known that intermittency is almost surely related to some kind of cascade. In Hydrodynamics, this is the well known Kolmogorov model, whereas in Particle Physics partonic evolution can be viewed as a quark–gluon cascade. Indeed random cascading models defined on the Cayley tree [3] describe most of the features of intermittency. It is therefore noteworthy to try to generalize such toy models. It will be shown that the random heat equation, *i.e.* Schrödinger's equation in euclidean time for a random potential is a natural extension of random cascading models.

Let us recall a few results about intermittency. Suppose that the state of a given system is specified by a density P and that phase space is partitioned into small cells whose volume is δ (for example the bin size of rapidity distributions in multiparticle production). Then, intermittency implies the following asymptotic behaviour

$$\frac{\langle P^q \rangle}{\langle P \rangle^q} \underset{\delta \to 0}{\propto} \left(\frac{1}{\delta} \right)^{\varphi_q}, \tag{1}$$

where the average is taken over samples of the system. Usually, the system is spatially homogeneous, *i.e.* $\langle P^q \rangle$ does not depend on the precise location of the cell, where the expectation value is taken. This will be assumed throughout the paper. The positive scaling exponent φ_q is called the intermittency index. It varies with q in a non trivial fashion.

It has been pointed out [4] that intermittent phenomena are linked to multifractal systems. It is even possible to establish the precise connection between the intermittency indices φ_q and the multifractal spectrum $f(\alpha)$ [5] for random cascading models [4]. Multifractal analysis enables one to tackle global features of the system. The cascade is most of the time dominated by a point-like set of high peaks whose structure is spin-glass like. This point-like set plays the role of an asymptotically *localized* attractor. There is a "non thermal" phase transition in the cascade. The most prominent characteristic of phase space is ergodicity breakdown. This is a consequence of the spin-glass like structure.

It will be shown that the random heat equation is the natural generalization of random cascading models. Furthermore, all mentioned properties of random cascading models, such as spin-glass like phase space and ergodicity breakdown are still true for the random heat equation.

This paper is arranged as follows. A few results about random cascading models will be recalled. This will help formulating the link with the random heat equation. Then intermittency and multifractals are introduced for the case of a time-dependent cascade. By analogy with random cascading models, intermittency in random media is studied. Intermittency indices are explicitly computed. The multifractal spectrum is thus obtained. Multifractal analysis hints that the end of the cascading evolution is dominated by a point-like set of "spikes" whose weights are computed. This is then spelt out for Levy laws whose Levy index is $\mu > 1$, yielding strong intermittency. Possible applications to multiparticle production are added to our calculations. In three appendices, we calculate the intermittency indices and the multifractal spectrum of the random heat equation.

2. Intermittency in time-dependent cascades

Let us introduce a few noteworthy results about random cascading models.

2.1. Random cascading models

These models are the building blocks used in the sequel to obtain general results about the random heat equation. The Cayley tree is a lattice depicted in Fig. 1, where there are λ branches at each node. As it stands there is no loop on this lattice. This is, therefore, a well suited approximation of a standard D-dimensional hypercubic lattice when the dimension D goes to infinity. This remark will be utilized at length later. As already mentioned, intermittency is specified by the behaviour of a random density P (a measure depending randomly on each sample) as the resolution δ goes to zero. The natural scale on the Cayley lattice is M^{-1} where M is the number of end-points, the random density P is obtained when assigning a realization W/λ to each branch of the tree; W being normalized random variable whose density $r(W)$ satisfies

$$\int r(W)dW = 1\,, \qquad \int r(W)W dW = 1\,. \tag{2}$$

The density P_m of each cell m at the end of the Cayley tree is given by the product of these random weights along the unique path (m) joining the root of the tree down to cell number m.

$$P_m = \prod_{\alpha \in (m)} \frac{W_\alpha}{\lambda}\,. \tag{3}$$

The normalized weights are independent, from (3) the intermittency indices are given by:

$$\varphi_q = \frac{\ln\langle W^q\rangle}{\ln\lambda}\,. \tag{4}$$

Thus the intermittency indices are only functions of the geometry of the tree and of the random variable W. Let us introduce the following notation

$$\frac{W_\alpha}{\lambda} = \mathrm{e}^{-\epsilon_\alpha}\,. \tag{5}$$

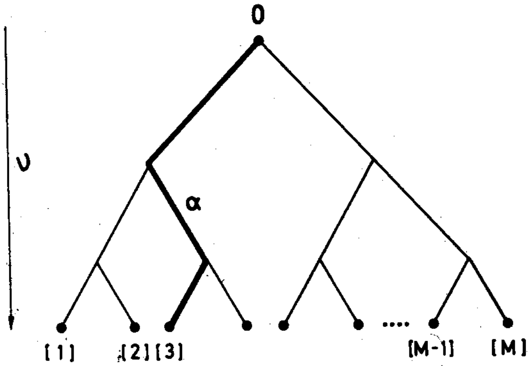

Fig. 1. The Cayley tree on which independent realizations of the random variable W are assigned. The number of steps is ν, the branching ratio λ (here equal to 2), and the different end-points are denoted by [1]......$[M]$. Path (3) corresponding to the end point [3] is depicted by a thicker line, the branch $\alpha \in (3)$ is an example.

The random weight W_α is then interpreted in terms of a Boltzmann weight $e^{-\epsilon_\alpha}$, the local exponent ϵ_α is an energy depending on each link. The density P_m can be seen as the probability that a particle evolving on the Cayley tree is at the m^{th} end point. This comes from equation (3) which is nothing but a Feynman–Kac formula giving the density P_m as the sum over all possible paths from the origin of the tree to the m^{th} cell. More precisely, identify the number of cascading steps ν (see Fig. 1) with a cascading time then

$$P_m = \sum_{\substack{\text{paths}\\ \text{from } 0 \text{ to } [m]}} \exp\left(-\sum_{\alpha=1}^{\nu} \epsilon_\alpha\right). \tag{6}$$

The path from the root of the Cayley tree 0 to $[m]$ is unique, thence formula (3). Therefore ϵ_α is a random potential in which evolves a test particle. The density P_m plays a rôle of the probability of presence at m.

As natural in statistical mechanics [8], the phase structure of random cascading models is deduced from a partition function

$$Z(q) = \sum_{m=1}^{M} P_m^q. \tag{7}$$

Notice that the partition function is normalized as $\langle Z(1)\rangle = 1$. One can also interpret q as an inverse temperature. This is partition function of the canonical ensemble describing an assembly of non interacting test particles evolving on the Cayley tree under the influence of the random potential ϵ_α. It happens that this is also what is required to get at the multifractal properties of the random measure P [5]. In a few words, P_m is a multifractal measure if and only if M goes to infinity there exists an exponent $\tau(q)$ such that

$$Z(q) \underset{M\to+\infty}{\propto} M^{-\tau(q)} \text{ almost surely}. \tag{8}$$

From now on, scaling behaviours are specified almost surely, therefore, we do not mention it anymore. To get at $\tau(q)$ it is possible to use the replica method [8, 9]. This amounts to computing the moments $\langle Z^p(q)\rangle$ as M gets large [9]. It is then straight forward to compute $\tau(q)$ as:

$$\tau(q) = -\lim_{M\to+\infty} \frac{\ln Z(q)}{\ln M} = -\lim_{M\to+\infty} \frac{\langle \ln Z(q)\rangle}{\ln M}. \tag{9}$$

as it is known [10] that $\tau(q)$ is a self averaging quantity. Suppose that one can swap the limit $p \to 0$ and $M \to +\infty$, this reads

$$\tau(q) = \lim_{p\to 0} \lim_{M\to+\infty} \frac{\langle Z^p(q)\rangle - 1}{p \ln M^{-1}}. \tag{10}$$

It is necessary to analytically continue $\langle Z^p(q)\rangle$ when the number of replicas p goes to zero. Results obtained using (10) agree with computations [8] based on an analogy with travelling wave equations. On general ground [11], it is proven that equation (8) entails that there exists a local Lipschitz–Hölder index α_q depending on q and a function $f(\alpha)$ such that the number of cells m whose density P_m behaves as

$$P_m \underset{M\to+\infty}{\propto} M^{-\alpha} \tag{11}$$

is given by

$$\mathcal{N}_\alpha \underset{M\to+\infty}{\propto} M^{f(\alpha)} \tag{12}$$

$f(\alpha)$ is the Legendre transform of $\tau(q)$, *i.e.*

$$\tau(q) = q\alpha - f(\alpha), \qquad \alpha = \frac{d\tau}{dq}, \tag{13}$$

$f(\alpha)$ is called the multifractal spectrum of the random measure P, it is equivalent to an entropy, and is interpreted as the fractal dimension of the subset of phase space whose density behaves as (11). From statistical mechanics, the multifractal spectrum satisfies

$$f(\alpha) \geq 0 \tag{14}$$

(positivity of the entropy). This is a natural requirement about the positivity of fractal dimensions which turns out to have dramatic consequences [12]. In the case of random cascading models, the multifractal spectrum $f(\alpha)$ is given by [4]

$$\alpha = 1 - \frac{d\varphi_q}{dq}, \qquad f(\alpha) = -q^2 \frac{d\epsilon_q}{dq}, \tag{15}$$

where ϵ_q is the free energy of the system:

$$\epsilon_q = \frac{1 + \varphi_q}{q} - 1. \tag{16}$$

It has been noticed [4] that if the distribution $r(W)$ is continuous, there always exists an index q_c such that $f(\alpha_{q_c}) = 0$. Therefore, above q_c the free energy ϵ_q remains frozen at its value ϵ_{q_c}. This is a "non thermal" phase transition where replica symmetry breaking occurs [8, 9]. This more or less means that the random distribution $\{P_m, m \in \mathbb{N}^*\}$ is dominated by a subset of cells whose dimension is formally zero, *i.e.* a finite number of "spikes". More about this transition and about explicit ergodicity breaking in Section 5. If we stick to the particle interpretation given at the beginning of this Section, these "spikes" are the result of a self-organization during the cascade. They correspond to regions of phase space where a test particle is *localized*. Particles belonging to the canonical ensemble will get clustered in these domains. It will be shown that these spikes are statistically uncorrelated, they behave as free "quasi-particles" whose masses are given by the weight of their surrounding neighborhoods. Explicit calculation of these weights as a function of the number of "quasi-particles" will be given in Section 5.

2.2. Time dependent intermittency

We endeavour to generalize the above considerations where the Cayley tree is replaced by some kind of phase space structure E while the number of cascading steps ν becomes a continuous time. In fact, we have in mind the generalization of the Feynman–Kac Formula (6) to this more general setting. Before embarking ourselves upon the precise definition of the random heat equation, let us comment on what intermittency in the final state of the time evolution of a random system would be.

Suppose that the probability density $P(y,t)$ of being at $y \in E$ at time $t > 0$ evolves from 0 to t according to

$$P(y, t) = G(t, 0)P(y, 0), \tag{17}$$

where G is the operator governing the time dependence and initial conditions are fixed by

$$P(y, 0) = P(y), \tag{18}$$

where P is a homogeneous random field, *i.e.* $P(y, 0)$ is drawn from a random variable P independently of $P(y', 0)$ if $y \neq y'$. $P(y, 0)$ is not supposed to vanish outside a finite domain but a large volume cut off will have to be introduced to compute multifractal properties. Then, intermittency occurs if a scaling law similar to (1) holds, *i.e.* if there exists an appropriate volume scale $\delta(t)$ and a scaling behaviour

$$\frac{\langle P^q(y, t)\rangle}{\langle P(y, t)\rangle^q} \underset{t \to +\infty}{\propto} \delta(t)^{-\varphi_q}, \tag{19}$$

where $\delta(t)$ goes to 0 as t gets large. The positive index φ_q is the intermittency index.

In order to probe phase space structure, it is natural to introduce a partition function $Z(q, t)$ describing a canonical ensemble of non interacting test particles evolving according to Eq. (17). These test particles have a probability of being at y at time t given by the random probability measure $\frac{P(y, t)}{\int_{M(t)} d^D y \langle P(y, t)\rangle}$. The region $M(t)$ is the volume cut off; in order to have normalized multifractal properties the volume of $M(t)$ is chosen to be $\delta^{-1}(t)$. Then the partition function is:

$$Z(q, t) = \frac{\int_{M(t)} d^D y P^q(y, t)}{\left(\int_{M(t)} d^D y \langle P(y, t)\rangle\right)^q}, \tag{20}$$

where the integration is understood as in (7), *i.e.* it can be replaced by a sum if phase space is discretized. Notice that the normalization $\langle Z(1, t)\rangle = 1$ has been performed. Furthermore, the index q retains its interpretation of the effective inverse temperature of the medium. Therefore, test particles flow through the medium towards a thermodynamically favoured final state at $t = +\infty$. This final state is a fractal set whose dimension is $f(\alpha_q)$. Following what has been said about random cascading models, the system is asymptotically multifractal if there exists a function $\tau(q)$ such that

$$Z(q, t) \underset{t \to +\infty}{\propto} \delta(t)^{\tau(q)/D}, \tag{21}$$

where D is the dimension of E. Formulae (10)–(13) are still valid and then interpretation is the same. If the evolution operator $G(t, 0)$ generates intermittency then the behaviour of $P(y, t)$ as t gets large is dictated by the multifractal spectrum $f(\alpha)$. It is possible to generalize random cascading models in such a way that most of the results are still true in their continuum version.

3. The random heat equation

3.1. Intermittency

The most natural extension of the Feynman–Kac formula [18] for a system in $E \simeq \mathbb{R}^D$ is the following (see Fig. 2)

$$P(y,t) = \frac{1}{Z_0} \int d^D y_0 \int_{y(0)=y_0}^{y(t)=y} [dy] P(y_0, 0) \times \exp\left[-\frac{1}{4}\int_0^t \left(\frac{\mathrm{d}\, y}{d\,\tau}\right)^2 d\tau - \int_0^t V(\tau, y(\tau))\mathrm{d}\tau\right], \tag{22}$$

where $V(t, y)$ is a random potential and

$$Z_0 = \int d^D y_0 \int_{y(0)=y}^{y(t)=y} [dy]\, exp\left[-\frac{1}{4}\int_0^t \left(\frac{dy}{d\tau}\right)^2 d\tau\right].$$

Realizations of the random potential $V(y, t)$ at different points and different non zero times are supposed to be independent. The path integral is taken over all paths ending at y at time t. The initial configuration is homogeneous in all phase space E (see (18)). From ordinary quantum mechanics, it is known that (22) is the solution of the random heat equation

$$\frac{\partial P}{\partial t} = (\Delta_y - V(y,t))P \tag{23}$$

parametrized by the initial conditions. As it stands this is not a well defined equation due to kinks of the random potential. Therefore solutions may not be continuous functions of space and time. In order to regularize the forward behaviour of the path integral (20), one has to discretize space on a lattice $a\mathbb{Z}^D$ where a is the lattice spacing. As shown in appendix A, the continuum

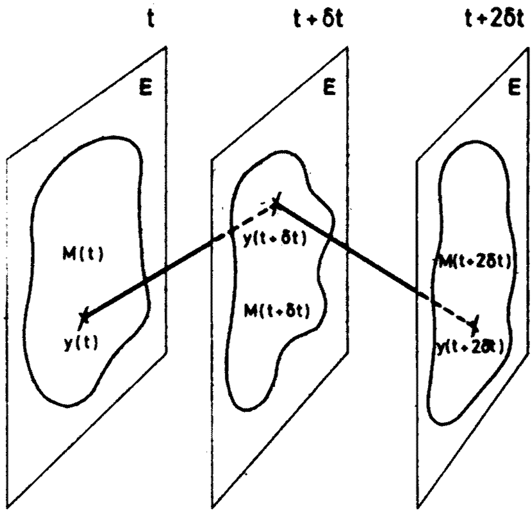

Fig. 2. The time evolution of the cut-off region $M(t)$ where the probability density $P(y,t)$ is taken to be non negligible. This domain evolves as time gets large. It is depicted by the non-hatched region. The lines joining points of $M(t)$ and $M(t+\delta t)$ (resp $M(t+\delta t)$ and $M(t+2\delta t)$ represent a path followed by a test- particle. On each link is assigned a random potential $V(y(t),t)\delta t$. Taking into account all paths from the original domain M_0 to a point y at time t gives $P(y,t)$ as a Feynman-Kac formula. The motion of test particles is considered once random potentials are fixed (quenched disorder). However thermodynamic quantities are averaged over the random potential V.

limit is obtained taking time t large on a lattice whose lattice spacing a is finite (small) before letting a going to zero.

Let us give a few physical hints about the Feynman-Kac formula (22). A similar equation to (22) models the behaviour of directed polymers in a random medium if one assumes that realizations of the random potential are independent (including the initial time) and provided that the initial condition is a delta function at y_0. This is nothing but the propagator $\mathcal{K}(y,t;\ y_0,0)$ whose scaling behaviour is given by:

$$\mathcal{K}(y,t;\ y_0,0) \underset{t\to+\infty}{\sim} t^{-(D-1)\nu} g\left(\frac{|y-y_0|}{t^\nu}\right) t^{\gamma-1} \exp(-ft)\,, \qquad (24)$$

where γ is the susceptibility exponent, f the polymer free energy, g a scaling function and ν is the exponent measuring polymer transverse displacements. The exponent ν is superuniversal, *i.e.* independent of D and has been recently computed [25] to be $\frac{2}{3}$. From the propagator (24), one can obtain the solution $P(y,t)$ by convolution with initial conditions. One cannot generally compute this integral as the function g is not precisely known.

Another way of representing the solution $P(y,t)$ of (22) is to use the spectral decomposition associated with the Hamiltonian operator $H = -\Delta + V$. To do so, one needs to know the generic form of the spectrum of $H = -\Delta + V$. First of all, in $D = 1$ dimension, the spectrum of the discretized Schrödinger operator $H = -\Delta + V(y)$ is pure point [13], *i.e.*, there are only eigenvalues and eigenstates (Anderson localization). This means that as t gets large, the dominant part of $P(y,t)$ will come from the lower band tail [15]. Furthermore, the eigenstates have exponential decay. Hence $P(y,t)$ will be localized around a few peaks (corresponding to the lowest energies). Phase space will be dominated by a subset whose fractal dimension is zero, *i.e.* $f(\alpha) = 0$. In higher dimensions, the situation is drastically different [14]. All that can be said is that there is an interval of energies $[-E_0, E_0]$ outside which the spectrum is pure point. Nevertheless, the presence of an absolutely continuous part of the spectrum could modify the above conclusion. In fact, the multifractal spectrum $f(\alpha)$ will be non trivial for continuous random potentials as soon as $D > 1$. It is suggested in the sequel that the long time behaviour of the random heat equation is dominated by a point-like set of high peaks ($f(\alpha) = 0$). These "spikes"play the role of asymptotically *localized* states. Particles evolving in time through the random medium converge towards these attractors at low temperature (see Section 5). Furthermore, the aggregation of particles in these regions of space tend to create "quasi-particles" which behave independently of each other.

Introduce the generating function of the random potential V

$$\langle \exp[-qV(y,\tau)] \rangle = e^{K(q)}, \tag{25}$$

which is space-time independent. It is shown in Appendix A that the normalized moments are:

$$\frac{\langle P^q(y,t) \rangle}{\langle P(y,t) \rangle^q} \underset{t\to+\infty}{\propto} \frac{\langle P^q(y,0) \rangle}{\langle P(y,0) \rangle^q} e^{(K(qt)-qK(t))}. \tag{26}$$

Intermittency will stem from the exponential behaviour of these moments. As mentioned in Section 2, the volume scale is defined once the volume $\delta(t)$ is evaluated. The volume scale is chosen to be

$$\delta(t) = \delta(0) \exp\left(+\frac{1}{\xi} K(t) \right), \tag{27}$$

where $\delta(0)$ is an unspecified constant. The constant ξ is arbitrary. This comes from the absence of scale in the model. ξ specifies the choice of length scale which is done when measuring intermittency indices. Notice that an intrinsic clock is given by the generating function $K(t)$. However, it is necessary to introduce a constant ξ to fix the length scale, *i.e.* intermittency requires the existence of a fundamental scale. As explained in appendix A, the generalized potential $K(t)$ is supposed to follow

$$K(t) \underset{t \to +\infty}{\sim} t^{\eta} B(t), \tag{28}$$

where $\eta > 1$ and $B(t)$ is bounded, furthermore $\lim_{t \to +\infty} B(t) = B > 0$. For instance Levy and Gaussian laws are within this category.

Following (26), the intermittency indices are

$$\varphi_q = \xi \lim_{t \to +\infty} \left[\frac{K(qt)}{K(t)} - q \right] = \frac{\varphi_2}{2^{\eta} - 2} (q^{\eta} - q), \tag{29}$$

where one swaps φ_2 for ξ. This will be made explicit for Levy laws in Section 4 (see also Fig. 3).

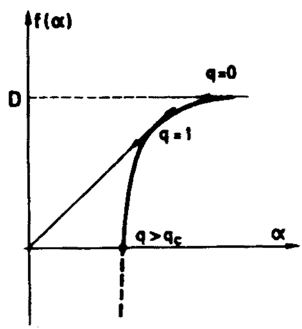

Fig. 3. Multifractal spectrum. The embedding dimension is obtained for $q = 0$, at $q = 1$ $f(\alpha)$ is tangent to the line $f(\alpha) = \alpha$. At q_c, there is a "non thermal" phase transition where ergodicity is broken. It is a signal that phase space is dominated by a point-like set of "spikes".

3.2. Multifractal spectrum

We will at first follow a naive approach which is justified in Appendix B. Notice that in order to evaluate $Z(q,t)$, one can try to use the replica method. Let us suppose that the integral $Z(q,t)$ is given by its expectation value for q sufficiently small. This can be backed up by the analysis on the Cayley tree [12] which shows that below the transition index q_c : $Z(q) = \langle Z(q) \rangle$. Therefore, taking the average of $Z(q,t)$ and introducing a time dependent volume cut off $\delta^{-1}(t)$,*i.e.* the volume of the domain $M(t)$ in which $P(y,t)$ is taken into account, yields

$$\langle Z(q,t) \rangle \underset{t \to +\infty}{\propto} \delta(t)^{-1+q} \exp[K(qt) - qK(t)] \tag{30}$$

and eventually for the function $\tau(q)$

$$\tau(q) = D\left[-1 + q(1+\xi) - \xi \lim_{t\to+\infty} \frac{K(qt)}{K(t)}\right]. \tag{31}$$

This can be expressed as a function of φ_q given by (29). Then taking the appropriate derivatives (see (13)), one retrieves the following formulae

$$\alpha(q) = D\left(1 - \frac{d\varphi_q}{dq}\right),$$
$$f(\alpha) = -Dq^2 \frac{d\varepsilon_q}{dq}, \tag{32}$$

where the free energy ε_q is given by (16). Hence, up to a factor D, the infinite dimensional case is recovered from mean field considerations. The overall factor D is present because in the continuum case the cascade is embedded in D-dimensional space whereas the end-points of the Cayley tree are naturally represented on the real line. Therefore fluctuations of solutions of the random heat equation as t gets large are multifractal. Notice that α_q is a decreasing function, its minimum is reached at q_c where $f(\alpha_{q_c}) = 0$. Therefore, the fractal set corresponding to the singularity α_{q_c} is a point-like set as its fractal dimension is zero. Furthermore, it is the most singular fractal set as α_{q_c} is minimum, *i.e.* $\forall q > q_c \quad \delta(t)^{\alpha_q - \alpha_{q_c}} \underset{t\to+\infty}{\longrightarrow} 0$. This entails (see (11)) that the density $P(y,t)$ is negligible outside this dominant fractal set in the asymptotic regime. Hence these "spikes" are asymptotically localized attractors towards which test particles converge when $q \geq q_c$ (low temperature). As shown in Section 5, where the dynamics of the convergence towards these attractors is analyzed, particles ending in the vicinity of different "spikes" follow independent paths, *i.e.* they share a negligible part of their trajectories. Therefore when considering the flow of a large but finite number of initial test-particles they tend to get clustered in independent "quasi-particles". These "quasi-particles" are the asymptotically localized states emerging from the self-organization of test-particles during their time evolution. There is a "non-thermal" transition between an initial state (at $t = 0$) which is randomly distributed and a final state ($t = +\infty$) where *localization* has taken place. ($q \geq q_c$).

4. Application to Levy laws [7, 17]

The simplest probability laws which generates intermittency are Levy laws L_μ. They depend on a single parameter μ, the Levy index, and their support is the positive real axis. A way of characterizing these random variables is to give their generating function

$$\langle \exp(-t\, L_\mu)\rangle = \exp(-C(\mu)t^\mu), \tag{33}$$

where $C(\mu) = \frac{\pi}{\mu^2 \sin \pi\mu \Gamma(\mu-1)}$ and $\mu \in]1, 2[$. They are stable laws, *i.e.* up to rescaling they are limiting distributions of a sum of random variables. The Gaussian case is recovered if $\mu = 2$ and $C(2) = \frac{-\sigma^2}{2}$. There is a significant difference between $\mu < 2$ (Levy laws) and $\mu = 2$ (Gaussian laws). The former laws are of very long range as their variance is infinite whereas σ^2 measures the width of the Gaussian law. Levy laws model distributions where large deviations from normal fluctuations are allowed. In the finite dimensional continuum models, the normalized moments read

$$\frac{\langle P^q(y,t)\rangle}{\langle P(y,t)\rangle^q} \underset{t\to+\infty}{\propto} \frac{\langle P^q(y,0)\rangle}{\langle P(y,0)\rangle^q} \exp[-C(\mu)[q^\mu - q]t^\mu]. \tag{34}$$

Observe that $C(\mu) < 0$ for $\mu > 1$ and therefore high moments diverge rapidly as t goes to infinity, a clear signal of strong fluctuations. From (34), the intermittency indices are given by

$$\varphi_q = \frac{\varphi_2}{2^\mu - 2}(q^\mu - q). \tag{35}$$

This is the result obtained on the Cayley tree. From (35), one can compute the multifractal spectrum (see Fig. 3)

$$f(\alpha) = D\left(1 - \left(\frac{q}{q_c}\right)^\mu\right), \qquad \alpha = D\left(1 + \frac{\varphi_2}{2^\mu - 2}(1 - \mu q^{\mu-1})\right), \tag{36}$$

where the transition index q_c is given by

$$q_c = \left(\frac{\varphi_2(\mu-1)}{2^\mu - 2}\right)^{-\frac{1}{\mu}}. \tag{37}$$

These expressions are meaningful for $\mu > 2$, they correspond to (28) where $\eta > 2$.

Notice that for $\mu < 1$, one cannot conclude from Appendix A that there is intermittency. This is drastically different from the infinite dimensional case where $\mu < 1$ leads to weak intermittency. Finally, the upper intermittent dimension is $D_{UI} = 1$ as D-dimensional cascades can be mapped onto the Cayley tree (D_{UI} is the dimension above which intermittency indices are equal to those obtained on the Cayley tree).

5. Non thermal phase transition

It has been several times mentioned that the long time behaviour of solutions of the random heat equation is dominated by a point-like set of "spikes". In Appendix B it is shown that this comes from a replica symmetry breaking which entails a drastic difference between the behaviours of the multifractal spectrum $f(\alpha)$ above and below q_c where the free energy ε_q gets frozen, *i.e.* remains constant. In this Section, we will indicate how this comes about, *i.e.* we study the time evolution of test particles through the medium given the inverse temperature q.

5.1. Replica symmetry breaking

In Appendix B, the moments of the partition function $\langle Z^p(q)\rangle$ are computed. They correspond to taking into account correlations of p test particles starting at the same point at $t = 0$. The result obtained is similar to what is calculated on the Cayley tree. One can apply the replica formalism [8, 9]. In order to get at the scaling exponent $\tau(q)$, the limit $p \to 0$ is performed in (B.16), the maximization problem is replaced by a minimization (following Parisi, see Ref. [19]).

$$\langle Z^p(q)\rangle \underset{\substack{t\to+\infty \\ p\to 0}}{\propto} \delta(t)^{-pq \underset{\{x(Q)\}}{\mathrm{Min}} \left[\int_0^1 dQ\, \varepsilon_{qx(Q)}\right]}, \tag{38}$$

where $x(Q)$ is a monotonic increasing function whose range lies between 0 and 1. The minimization is taken over functions $x(Q)$. This function measures the fraction of paths starting at a given point at $t = 0$ and sharing a common trajectory for a fraction Q of time t (the overlap Q).

$$\mathrm{Pro}\,[Q \le Q_0] = x(Q_0) \tag{39}$$

which can be also written for the density $\mathcal{P}(Q)$, the probability of having an overlap between Q and $Q + dQ$

$$\mathcal{P}(Q) = \frac{dx}{dQ}\,. \tag{40}$$

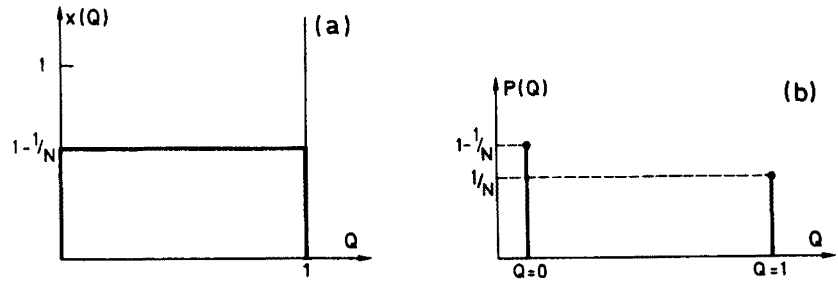

Fig. 4. Replica symmetry breaking: a) The overlap function $x(Q)$ measures the probability that two test particles follow the same path during a fraction Q of time t. It depends on the number of effective spikes N. Due to a one level replica symmetry breaking, test particles can either have overlap $Q = 0$ or $Q = 1$; b) The probability density $P(Q)$ of overlaps. It is a sum of two Dirac delta functions. The probability of overlap $Q = 0$ is $1 - \frac{1}{N}$ whereas it is $\frac{1}{N}$ for $Q = 1$. Notice that overlap $Q = 1$ means that two test particles follow the same path, they reach one of the N "spikes". "Spikes" are equally probable (probability $\frac{1}{N}$ of reaching a given "spike").

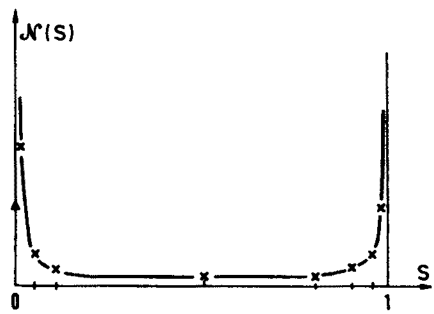

Fig. 5. The weight distribution $\mathcal{N}(S)$. This is the probability density that a "spike" has a weight S, *i.e.* that the region surrounding this high peak has overlap $Q = 1$ with the "spike". It depends on the number of spikes N (here chosen to be $N = 50$). As natural, if the number N is large enough, the weight S is almost surely close to $S = 0$.

As argued in Appendix B, up to a set of Lebesgue measure zero, paths are uncorrelated. Correlation comes from paths ending within the zero Lebesgue measure neighbourhood of a "spike". Carrying out the minimization (38) (see Fig. 5) yields

$$x(Q) = 1 \qquad \text{if} \quad q < q_c\,,$$
$$x(Q) = \frac{q_c}{q} \qquad \text{if} \quad q \geq q_c\,. \tag{41}$$

Introduce the effective number of "spikes" $N = (1 - \frac{q_c}{q})^{-1}$ (see Ref. [20]

and Section 5.2.). This is the number of dominant spikes when the index q is greater than q_c. Interpreting q as an inverse temperature yields that below a critical temperature q_C^{-1} only a finite number of "spikes" play the role of attractors. From (41) the overlap distribution $\mathcal{P}(Q)$ is concentrated in $Q = 0$ and $Q = 1$ [8]:

$$\mathcal{P}(Q) = \left(1 - \frac{1}{N}\right)\delta(Q) + \frac{1}{N}\delta(Q - 1). \tag{42}$$

This is one level replica symmetry breaking as test particles can either remain glued together for a very long time or split up right at the beginning of their evolution.

Above q_c the fractal dimension is zero. Phase space is cut into a point-like set of "spikes" such that each cell of a "spike" has overlap one with its neighbourhood. Between cells belonging to different spikes, the overlap Q is zero. "Spikes" behave as free "quasi particles" as there is no overlap, *i.e.* no correlation between them. Nevertheless the location and the width of the overlap $Q = 1$ neighbourhood of each spike vary from event to event (see Ref. [8]).

5.2. Particle interpretation

The physical quantity which describes a "spike" is its weight, *i.e.* the fraction of particles starting from the same point which fall in the neighbourhood of a "spike" (whose overlap with a given "spike" is one). As shown in Appendix C, the number of "spikes" is infinite. However, choosing q to be greater than q_c allows one to study the flow of test-particles towards an effective finite number N of attractors

$$N = \left(1 - \frac{q_c}{q}\right)^{-1}. \tag{43}$$

Conditionally to the number of attractors N, it is possible to compute the weight distribution $\mathcal{N}(S)$ which measures the probability that the weight of a given "spikes" is between S and $S + dS$

$$\mathcal{N}(S) = \frac{S^{\frac{1}{N}-1}(1-S)^{-\frac{1}{N}}}{\Gamma(\frac{1}{N})\Gamma(1-\frac{1}{N})}, \tag{44}$$

notice that $\langle S\rangle = \frac{1}{N}$. In particular the average probability that a given particle flowing to the given set of N "spikes" goes to a particular spike is $\frac{1}{N}$ (see (42)). The long time behaviour of a steady flow of test particles

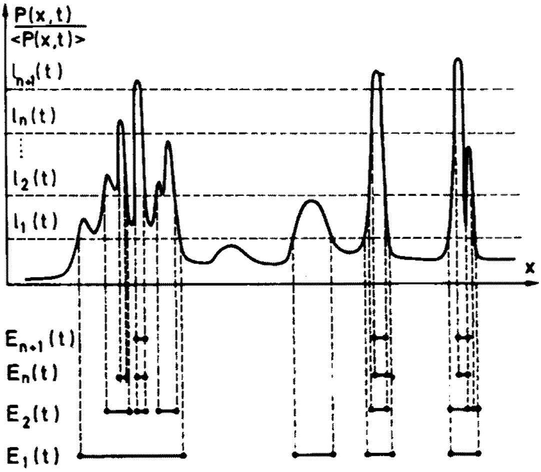

Fig. 6. A single event shape. The distribution $P(y,t)$ is dominated by a few peaks. Each peak is surrounded by other smaller peaks whose overlap with the dominant "spikes" is 1. A way of characterizing the multifractal structure of the density $P(y,t)$ is to introduce divergent levels $\ell_n(t)$[23] $(\ell_{n+1}(t) \gg \ell_n(t))$ in such a way that $E_n(t) = \left\{ x \Big/ \frac{P(x,t)}{\langle P(x,t)\rangle} > \ell_n(t) \right\}$ dominates the evaluation of the partition function $Z(q_n,t)$ for a given value of moment q_n. Furthermore[23], these sets are arranged in a hierarchical way $E_1(t) \supset E_2(t) \cdots \supset E(t)$. The measure of each $E_i(t)$ is small and $\lim_{t\to+\infty} \mu(E_i(t)) = 0$ (μ being the Lebesgue measure). Besides, there exists a limiting set $E(t)$ corresponding to the piece of phase space which dominates the evaluation of the partition function for $q > q_c$, *i.e.* "spikes" are localized on the set $E(t)$ whose measure goes to zero as t gets large.

moving inside the quenched random potential is equivalent to an ensemble of non interacting "quasi particles". It is conspicuous that localization occurs at the end of the evolution. This is a result of the "self organization" of intermittent cascades (Fig. 6 and Ref. [15]).

5.3. Ergodicity breakdown

The existence of "spikes" has an important consequence for the ergodic properties of test particle motions. Test particles almost surely do not have a uniform probability of visiting all phase space, only a few regions corresponding to spikes are probable. Thus, it can be said that the motion of test particles is not ergodic. This is quantitatively proven examining moments of the partition function. From Appendix B, one gets for $q \leq q_c$

$$\frac{1}{\left(\int\limits_{M(t)} \mathrm{d}^D y\right)} \int\limits_{M(t)} \mathrm{d}^D y P^q(y,t) \underset{t\to\infty}{\sim} \langle P^q(y,t)\rangle . \tag{45}$$

This is what is expected from an ergodic system; moments can be obtained integrating a given sample of the system over all phase space. However as soon as q is larger than q_c,

$$\frac{1}{\int\limits_{M(t)} d^D y} \int\limits_{M(t)} d^D y P^q(y,t) \underset{t\to+\infty}{\sim} \langle P^{q_c}(y,t)\rangle . \tag{46}$$

This is precisely a case of ergodicity breakdown. One can interpret it saying that no matter how much the moment index q is increased, the integral $\int\limits_{M(t)} d^D y P^q(y,t)$ is dominated by a few "spikes" which play the rôle of ground states of the system.

6. Conclusion and possible applications

Random cascades on the Cayley tree have been shown to be equivalent to solutions of the random heat equation. The random heat equation, whose solution is given by a Feynman–Kac formula, can be interpreted in terms of cascades through a random medium: it reveals intermittent features as t gets large. Furthermore, multifractal analysis which is performed on the Cayley tree thanks to the replica method is still valid for the random heat equation. It turns out that the long time behaviour is dominated by "spikes" where the probability density $P(y,t)$ is large. These "spikes" are uncorrelated. They represent the regions where a canonical ensemble of test particles get clustered. Moreover ergodicity is broken due to the non uniformity of phase space. Let us now apply the above considerations to different physical situations.

6.1. Deconfinement of a continuum medium

Suppose that a continuum medium is described by a density $P(y,t)$. It evolves from initial conditions which are random homogeneous at $t = 0$, this complies with the fact that small density variations can be present at $t = 0$. In a drastically simplified picture, one can argue that interactions and particle motions can be very crudely approximated saying that particles inside the medium are sensitive to an effective potential $V(y,t)$ depending on space and time. Particle motion generates an effective temperature $T = \beta^{-1}$. The external potential $V(y,t)$ can be of two kinds:

(i) short range interactions yield a gaussian random potential $V(y,t)$ which is the result of an infinite sum of interactions (central limit theorem),

(ii) long-range interactions yield a Levy random potential. In that case the sum of an infinite number of interaction converge towards broad distributions[17] : Levy laws (generalized central limit theorem).

Thence the time evolution of the medium is modelled out by the partition function $Z(q,t)$ ((20), (22)). Depending on the value of β, the continuum medium can undergo a "non thermal" transition:

(i) $\beta < q_c$ — Instead of filling up all space as at $t = 0$, the continuum medium is spread over a fractal region of space whose dimension is $f(\alpha_\beta)$. The continuum medium gets lacunar. One can say that the effective temperature β^{-1} is too high to generates big lumps of medium independent of one another.

(ii) $\beta \geq q_c$ — The continuum medium undergoes a "non-thermal" transition. It gets deconfined as time gets large. Asymptotically localized "quasi-particles" are created. They correspond to the appearance of lumps of medium independent of each other. If the temperature β^{-1} is sufficiently low ($\beta^{-1} < q_c^{-1}$), there are $N = (1 - \frac{q_c}{\beta})^{-1}$ "quasi-particles".

6.2. Confinement of an ensemble of interacting particles

Consider a finite but large number of interacting particles. Following the same path as in Sect. 6.1.) one can model out this ensemble by a random potential $V(y,t)$ and an effective temperature $T = \beta^{-1}$. As above, two cases are possible:

(i) $\beta < q_c$ — Particles flow towards a fractal set whose dimension is $f(\alpha_\beta) > 0$. Therefore no transition is expected as particles are not confined to a point-like asymptotic set.

(ii) $\beta \geq q_c$ — Particles get clustered amongst $N = (1 - \frac{q_c}{\beta})^{-1}$ asymptotically localized "quasi-particles". These "quasi-particles" are independent of each other. They represent the result of the self-organization of the ensemble of interacting particles. Notice that if β is very large (almost zero temperature), interaction creates a single "quasi-particle". This is a gelling transition.

6.3. Intermittency in multiparticle production

The above approach can be utilized to model out intermittency in multiparticle production [2]. It is known that random cascading models describe most of the observed features of intermittent rapidity distributions [22]. One can argue that these cascades can be interpreted as the evolution of non interacting test particles in a random medium. One can assume that there is no fixed effective temperature. A mixture of all possible asymptotic fractal attractors is present. Both cases 6.1. and 6.2. can then be physically interpreted. One can think that case 6.1. represents the creation of hadronic matter out of an initial quark-gluon plasma, whereas case 6.2. could be the formation of jets from an initial partonic distribution. This picture of confinement-deconfinement "non-thermal" transition would yield Levy-type intermittency indices. This is not far from fitting present day experimental data [24]. Needless to say that it would be interesting to carry on in this direction to see whether these are pure analogies or if they could be backed up from fundamental principles, *ie* if intermittency indices could be obtained from Quantum Field Theory.

Let us briefly mention another application which will be developed elsewhere [23]. It is possible to relate the random heat equation to the ϕ^4 Landau–Ginzburg equation (Newell Whitehead equation [21]) describing Rayleigh–Bernard convection. In that case there is a link between Newell–Whitehead equation and log-normal cascades "à la Kolmogorov" [23].

Thus random cascading models seem to be good candidates to understand pattern formation in disordered systems. It would be noteworthy to know other systems which could be mapped onto the random heat equation.

I am very grateful to R. Peschanski for many remarks and suggestions on the manuscript.

Appendix A

Random heat equation intermittency indices

Bounds on the moments $\langle P^q(y,t)\rangle$ which are sufficient to exactly calculate $\langle P^q(y,t)\rangle / \langle P(y,t)\rangle^q$ are obtained. To do so, discretize the random heat equation on the lattice $a\mathbb{Z}^d$ whose lattice spacing is a.

$$\frac{\partial P_a}{\partial \tau_0} = \Delta_d P_a - \tilde{V} P_a \tag{A.1}$$

P_a is the solution of the discretized random heat equation. It is a function of τ_0 and a, Δ_d is the lattice Laplacian

$$\Delta_d f(x) = 2D \sum_{\langle x,y\rangle} [f(y) - f(x)] \tag{A.2}$$

the sum extends to all nearest neighbors of x. The time τ_0 and the random potential $\widetilde{V}$ are related to t and V by

$$\tau_0 = \frac{t}{a^2}, \qquad \widetilde{V} = a^2 V. \tag{A.3}$$

The continuum limit $a \to 0$ has to be taken to retrieve $P(y,t)$ from $P_a(y,\tau_0)$, it amounts to letting a going to zero while t is fixed (A.3).

Notice that the solution of the random heat equation can be written as an expectation value with respect to the Wiener measure $E_y(\cdot)$ such that

$$P_a(y,\tau_0) = E_y\Big(\exp\Big(-\int_0^{\tau_0} \widetilde{V}(y(\tau),\tau)d\tau\Big) P_a(y(\tau_0),0)\Big), \tag{A.4}$$

where $P_a(\cdot,0)$ is the initial configuration and E_y is normalized as:

$$E_y(1) = 1. \tag{A.5}$$

As initial conditions are homogeneous, one can single out the configuration which remains the same for all times between 0 and τ_0, it yields:

$$P_a(y,\tau_0) \geq P_a(y,0)\exp\Big(-\int_0^{\tau_0} \widetilde{V}(y,\tau)\mathrm{d}\tau\Big)\mathrm{e}^{-\tau_0}, \tag{A.6}$$

where the exponential term $\mathrm{e}^{-\tau_0}$ represents the probability of staying at the origin. Two cases have to be distinguished:

Autonomous random potentials

If $V(y,t) = V(y)$ independent of t then one gets

$$P_a(y,\tau_0) \geq P_a(y,0)\exp\big[-\tau_0\widetilde{V}(y) - \tau_0\big]. \tag{A.7}$$

Taking moments on both sides (q>1)

$$\langle P_a^q(y,\tau_0)\rangle \geq \langle P_a^q(y,0)\rangle \exp(-q\tau_0)\exp\big(+K(qa^2\tau_0)\big). \tag{A.8}$$

Time dependent random potentials

Write

$$P_a(y,\tau_0) \geq P_a(y,0)\exp\left\{-\widetilde{V}(y,0)\tau_0 - \int_0^{\tau_0}\left[\widetilde{V}(y,\tau) - \widetilde{V}(y,0)\right]d\tau - \tau_0\right\}. \tag{A.9}$$

Suppose that increments of the random process $\widetilde{V}(y,\cdot)$ are independent then

$$\langle P^q(y,\tau_0)\rangle \geq \langle P^q(y,0)\rangle \exp[K(qa^2\tau_0)] \times\left\langle \exp\left\{-q\int_0^{\tau_0}[\widetilde{V}(y,\tau) - \widetilde{V}(y,0)]\right\}\right\rangle \exp(-q\tau_0). \tag{A.10}$$

Splitting up the expectation value

$$\langle P_a^q(y,\tau_0)\rangle \geq \langle P_a^q(y,0)\rangle \exp(K(qa^2\tau_0) - q\tau_0)\Bigg[\exp(-qA\tau_0) + \left\langle \exp\left\{-q\int_0^{\tau_0} d\tau[\widetilde{V}(y,\tau) - \widetilde{V}(y,0)]\right\}\mathbf{I}_{\left(\widetilde{V}(y,\tau)-\widetilde{V}(y,0)\right)>A}\right\rangle\Bigg], \tag{A.11}$$

where A is chosen such that the probability $\mathrm{Pro}(\{(\widetilde{V}(y,\tau) - \widetilde{V}(y,0)) \leq A\}$, for all $\tau \leq t)$ is arbitrarily close to one; and finally

$$\langle P_a^q(y,\tau_0)\rangle \geq \langle P_a^q(y,0)\rangle \exp\left[K(qa^2\tau_0) - qA\tau_0 - q\tau_0\right]. \tag{A.12}$$

Observe that in both cases, one can write

$$\langle P_a^q(y,\tau_0)\rangle \geq \langle P_a^q(y,0)\rangle \exp\left[K(qa^2\tau_0) - q\chi\tau_0\right] \tag{A.13}$$

for a given constant χ.

Furthermore, one has

$$\langle P_a^q(y,\tau_0)\rangle = \langle P_a^q(0,\tau_0)\rangle = \left\langle\left(E_y\left(\exp\left[-\int_0^{\tau_0}\widetilde{V}(y(\tau),\tau)d\tau\right]P_a(y(\tau_0),0)\right)\right)^q\right\rangle. \tag{A.14}$$

Using Hölder's inequality, it yields

$$\langle P_a^q(y,\tau_0)\rangle \leq \left\langle E_y\left(\exp\left[-q\int_0^{\tau_0}\widetilde{V}(y(\tau),\tau)d\tau\right]P_a^q(y(\tau_0),0)\right)\right\rangle \tag{A.15}$$

then

$$\langle P_a^q(y,\tau_0)\rangle \le \langle P_a^q(y,0)\rangle E_0\left(\left\langle \exp\left[-q\int_0^{\tau_0}\widetilde{V}(y(\tau),\tau)\mathrm{d}\tau\right]\right\rangle\right), \tag{A.16}$$

because $P_a(y(\tau_0),0)$ and $\exp\left[-q\int_0^{\tau_0}\widetilde{V}(y(\tau),\tau)\mathrm{d}\tau\right]$ are independent. Applying Jensen's inequality

$$\exp\left(-\frac{1}{\tau_0}\int_0^{\tau_0} q\tau_0\widetilde{V}(y(\tau),\tau)\mathrm{d}\tau\right) \le \frac{1}{\tau_0}\int_0^{\tau_0}\exp\left(-q\tau_0\widetilde{V}(y(\tau),\tau)\right)d\tau \tag{A.17}$$

yields

$$\langle P_a^q(y,\tau_0)\rangle \le \langle P_a^q(y,0)\rangle \exp\left[K(q\tau_0 a^2)\right]. \tag{A.18}$$

Therefore, from (A.13) and (A.18)

$$\exp\left[K(qt) - q\chi\frac{t}{a^2}\right] \le \frac{\langle P_a^q(y,t)\rangle}{\langle P_a^q(y,0)\rangle} \le \mathrm{e}^{K(qt)}. \tag{A.19}$$

Impose on $K(\cdot)$ the following asymptotic behaviour

$$K(t) \underset{t\to+\infty}{\sim} B(t)t^{\eta}, \tag{A.20}$$

where $\eta > 1$ and $\lim_{t\to+\infty} = B > 0$. In order to reach the asymptotic regime $t \to +\infty$, and then perform the continuum limit $a \to 0$, choose the following scaling behaviour for t

$$t \underset{a\to 0}{\gg} a^2\left(\frac{a}{L}\right)^{\frac{-2\eta}{\eta-1}}, \tag{A.21}$$

where L is a fixed length. This ensures that as $t \to +\infty$

$$K(qt) \underset{t\to+\infty}{\gg} q\chi\frac{t}{a^2}. \tag{A.22}$$

Then, taking $a \to 0$ gives for very large t

$$\langle P^q(y,t)\rangle \underset{t\to+\infty}{\propto} \langle P^q(y,0)\rangle \mathrm{e}^{K(qt)}. \tag{A.23}$$

From (A.23), one can conclude that

$$\frac{\langle P^q(y,t)\rangle}{\langle P(y,t)\rangle^q} \underset{t\to+\infty}{\propto} \frac{\langle P^q(y,0)\rangle}{\langle P(y,0)\rangle^q}\mathrm{e}^{K(qt)-qK(t)}. \tag{A.24}$$

This is the relation which is used to compute intermittency indices of the regularized random heat equation.

Appendix B

The replica approach to the random heat equation

It will be shown that the naïve way of calculating multifractal spectra utilized in Section 3 is indeed valid if one takes care of the freezing of the spectrum $f(\alpha)$ above a certain threshold q_c. In order to prove that

$$\langle Z^p(q,t)\rangle \underset{t\to+\infty}{=} \langle Z(q,t)\rangle^p \tag{B.1}$$

if q is sufficiently small, the replica method is used.

The random heat equation is solved once the propagator $\mathcal{K}(y,t,\,y_0,0)$ is known

$$\mathcal{K}(y,t,y_0,0) = \int_{y(0)=y_0}^{y(t)=y} [dy]\exp\left[-\frac{1}{4}\int_0^t \left(\frac{dy}{d\tau}\right)^2 d\tau - \int_0^t V(\tau,y(\tau))\mathrm{d}\tau\right], \tag{B.2}$$

where the sum is over all paths from y_0 at $t=0$ to y at t. Then the solution reads

$$P(y,t) = \frac{1}{Z_0}\int d^D y_0 \mathcal{K}(y,t,y_0,0)P(y_0,0), \tag{B.3}$$

where Z_0 is a normalization factor. Densities $P(y,t)$ and $P(y',t)$ are uncorrelated if the points are different $y\neq y'$. As explained in Section 5, dominant patterns of the long time behaviour of the random heat equation are located on sets whose measure goes to zero (see Fig. 6) [15, 23]. Within these sets correlation is maximum. In order to show (B.1), one has to regularize (B.3) on the lattice.

Then the p^{th} moment of the partition function is given by

$$\langle Z^p(q)\rangle \underset{\substack{a\to 0\\ t\to+\infty\\ \Lambda\to+\infty}}{\propto} a^{Dp(1-q)}\delta^{pq}(a^2\tau_0)\mathrm{e}^{-pqK(a^2\tau_0)}$$

$$\times \sum_{\alpha_1+\cdots\alpha_\Lambda=p}\frac{p!}{\alpha_1!\cdots\alpha_\Lambda!}\left\langle\prod_{i=1}^{\Lambda} P_a^{q\alpha_i}(i,\tau_0)\right\rangle. \tag{B.4}$$

The independence of the densities $P(y,t)$ in the thermodynamic limit entails that

$$\lim_{a\to 0}\lim_{\Lambda\to+\infty}\lim_{t\to+\infty}\left\langle\prod_{i=1}^{\Lambda}P_a^{\alpha_i q}(i,\tau_0)\right\rangle=\prod_i\langle P^{\alpha_i q}(i,t)\rangle\,. \tag{B.5}$$

Therefore $\langle Z^p(q)\rangle$ is a sum of exponential terms whose leading behaviour is given by a dominant exponential. Choosing t large, Λ large and a sufficiently small one can write

$$\langle Z^p(q)\rangle \underset{\substack{t\to+\infty\\ \Lambda\to+\infty\\ \alpha\to 0}}{\sim} a^{Dp(1-q)}\,\delta^{pq}(t)\,\mathrm{e}^{-pqK(t)}$$

$$\times\sum_{\alpha_1+\cdots\alpha_\Lambda=p}\frac{p!}{\alpha_1!\cdots\alpha_\Lambda!}\prod_{i=1}^{\Lambda}\langle P^{\alpha}{}_i{}^{q}(i,t)\rangle\,. \tag{B.6}$$

This can be written in terms of the free energy function ε_q given by (14) as:

$$\langle Z^p(q)\rangle \underset{\substack{t\to+\infty\\ \Lambda\to+\infty\\ a\to 0}}{\sim} a^{Dp(1-q)}$$

$$\times\prod_{i=1}^{\Lambda}\delta(t)\sum_{\alpha_1+\cdots+\alpha_\Lambda=p}\frac{p!}{\alpha_1!\cdots\alpha_\Lambda!}\delta(t)^{-\sum_{i=1}^{\Lambda}q\alpha_i\varepsilon_{q\alpha_i}}\,. \tag{B.7}$$

Notice that:

$$\mathrm{Vol}(G)=\frac{\alpha_1!\cdots\alpha_\Lambda!}{p!}=\frac{\Gamma(\alpha_1+1)\cdots\Gamma(\alpha_\Lambda+1)}{\Gamma(p+1)} \tag{B.8}$$

is the volume of the symmetry group amongst replicas clustered within Λ blocks of α_i replicas ($i\in\{1,\ldots,\Lambda\}$). The symmetry group is $G=S_{\alpha_1}\times\cdots\times S_{\alpha_\Lambda}$. One shuffles the partition $\{\alpha_i\}_{i=1\ldots\Lambda}$ of P in such a way that

$$\alpha_1\geq\alpha_2\cdots\geq\alpha_\Lambda\,. \tag{B.9}$$

Define the overlap function Q_j by

$$Q_j=\sum_{i=1}^{j}\frac{\alpha_i}{p}, \tag{B.10}$$

where $Q_0 = 0$ and $Q_\Lambda = 1$.
Denote by

$$x(Q_j) \equiv \alpha_j \tag{B.11}$$

the value of α_j arranged according to the ordering in (B.9). This allows one to write moments of the partition function as:

$$\langle Z^p(q)\rangle \underset{\substack{t\to+\infty\\ \Lambda\to+\infty\\ a\to 0}}{\sim} a^{Dp(1-q)}$$

$$\times \prod_{i=1}^{\Lambda} \delta(t) \sum_{\alpha_1+\cdots+\alpha_\Lambda=p} \frac{1}{\mathrm{Vol}(G)} \delta(t)^{-pq\sum_{i=0}^{\Lambda-1}(Q_{i+1}-Q_i)\epsilon_{qx(Q_{i+1})}} . \tag{B.12}$$

When p is not an integer, this expression holds true if one takes into account the expression of Vol(G) as a function of α_i and p is given in (B.9). Moreover the above construction of the overlap Q_j and the partition element $x(Q_j)$ has to be generalized. One introduces a hierarchy amongst the p replicas of the system. To do so, denote by Q the overlap between two replicas, *i.e.*the fraction of the replica hierarchy shared by two test particles emanating from the same initial point (see Fig. 7). At any overlap Q_j, one puts p/x_j groups of x_j replicas. The number x_j is the proportion of replicas which are clustered, *i.e.*their overlap is larger than Q_j. Then going from one step Q_j to the next one Q_{j+1}, each group of replicas is divided into x_j/x_{j+1} branches. When p is less than one, this entails the following inequalities

$$\begin{aligned} 0 = Q_0 < \cdots < Q_j < Q_{j+1} \cdots < Q_\Lambda = 1\,, \\ p = x_0 < \cdots < x_j < x_{j+1} \cdots < x_\Lambda = 1\,. \end{aligned} \tag{B.13}$$

The order of the second inequality is reversed compared to the above explanation to comply with the limit $p \to 0$. In order to obtain the scaling exponent $\tau(q)$, one takes the limit $\Lambda \to +\infty$ (the thermodynamic limit for the replica tree). This amounts to considering an infinite tree representing the breaking pattern of p replicas. This tree can be viewed as a copy of the physical time evolution of test particles. Equation (B.12) is still valid when considering the above hierarchy. It is now possible to take the continuum limit $\Lambda \to +\infty$ and then $a \to 0$. In this limit x_j becomes a function $x(Q)$ where Q varies from 0 to 1. The sum over all partitions of p is replaced by an integral over all monotonic increasing function $x(Q)$ whose range is between 0 and 1. Therefore

$$\langle Z^p(q)\rangle \underset{t\to+\infty}{\sim} \int [dx]\delta(t)^{-pq\int_0^1 dQ\, \epsilon_{qx(Q)}} \tag{B.14}$$

and the normalization is such that $\int [dx] = 1$. As t gets large, moments of the partition function are dominated by the saddle point. Due to the limit $p \to 0$, one replaces the maximization by a minimization, one gets

$$\langle Z^p(q) \rangle \underset{\substack{t \to +\infty \\ p \to 0}}{\propto} \delta(t)^{-pq \underset{\{x(Q)\}}{\mathrm{Min}} \left[\int_0^1 dQ \varepsilon_{qx(Q)} \right]}, \tag{B.15}$$

where $x(Q)$ is now a monotonic increasing function whose range lies between 0 and 1. It corresponds to the total number of replicas at a given overlap Q. Performing the minimization is rendered straightforward noticing that $x(Q)$ has to be a constant function. Depending on whether the index q is less or greater than q_c (the critical index where ε_q is minimum) one gets different values for the scaling behaviour of $\langle Z^p(q) \rangle$. Thus there are two regimes. When $q \leq q_c$ the p replicas are equivalent (symmetry group $\mathcal{S}_p$) whereas when $q > q_c$ replica symmetry is broken (symmetry group $\mathbf{I} \times \mathbf{I} \cdots \times \mathbf{I}$ p times) [8, 9, 19]. This entails that (see Section 5)

$$\begin{aligned} q \leq q_c \quad & \left\langle Z^p(q) \right\rangle \underset{\substack{t \to +\infty \\ p \to 0}}{\propto} \delta(t)^{-pq\varepsilon_q} \\ q > q_c \quad & \left\langle Z^p(q) \right\rangle \underset{\substack{t \to +\infty \\ p \to 0}}{\propto} \delta(t)^{-pq\varepsilon_{q_c}} \end{aligned} \tag{B.16}$$

As explained in Section 5, this is a one level replica symmetry breaking. From (B.16) the function $\tau(q)$ is given by

$$\begin{aligned} q \leq q_c \quad & \tau(q) = -Dq\varepsilon_q, \\ q > q_c \quad & \tau(q) = -Dq\varepsilon_{q_c}. \end{aligned} \tag{B.17}$$

From thermodynamics, the free energy is defined by $-\tau(q)/q$. This shows that ε_q is the free energy, it remains frozen above q_c. The multifractal spectrum plays the rôle of a local entropy (see Eq. (13)). Thence

$$\begin{aligned} q < q_c \quad & f(\alpha) = -Dq^2 \frac{d\varepsilon_q}{dq}, \\ q \geq q_c \quad & f(\alpha) = 0. \end{aligned} \tag{B.18}$$

Therefore, the naïve result resulting from the calculation of $\langle Z(q,t) \rangle$ for t large is retrieved for $q < q_c$. Above q_c the multifractal spectrum remains

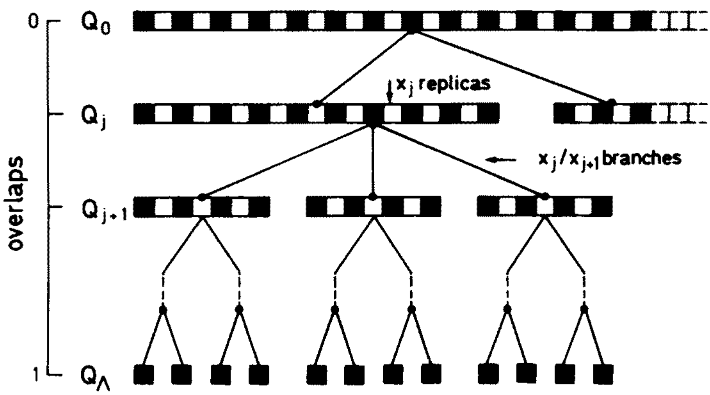

Fig. 7. The tree structure of replica symmetry breaking. The p different replicas are organized in a tree structure whose vertical coordinate is the overlap Q_j, the p replicas are put in p/x_j groups, each containing x_j replicas. At the next value Q_{j+1}, each one is separated in x_j/x_{j+1} subgroups, each with x_{j+1} replicas, and so on and so forth... In this tree structure, the root corresponds to the overlap $Q_0 = 0$ (symmetry group $\mathcal{S}_p$) containing all replicas ($x_0 = p$); the last step gathers the cluster of individual replicas ($x_\Lambda = 1$) with individual overlap $Q_\Lambda = 1$. The replica methods consists in an optimization method to get at the scaling exponent $\tau(q)$ in the space of all pairs (Q_j, x_j) between these two limits.

equal to zero. The consequence of replica symmetry on ergodicity breaking are spelt out in Section 5.

Appendix C

Weights of asymptotically localized "quasi particles"

Define S_i as the weight of the $i - th$ "spike" which plays the rôle of an attractor for particles evolving through the random medium when q is greater than q_c.

$$S_i = \lim_{N \to +\infty} S_i^{(\mathcal{M})}, \tag{C.1}$$

where $S_i^{(\mathcal{M})}$ measures the fraction of the $\mathcal{M}$ initial particles flowing to the i^{th} "spike".

$$S_i^{(\mathcal{M})} = \frac{1}{\mathcal{M}} \sum_{\alpha \to i} \mathbf{I}_\alpha \tag{C.2}$$

The symbol $\alpha \to i$ means that the α^{th} particle goes to the i^{th} "spike".

Notice that for $\mathcal{M}$ sufficiently large:

$$\sum_i S_i^{(\mathcal{M})} = \sum_i \frac{1}{\mathcal{M}} \sum_{\alpha \to i} \mathbf{I}_\alpha = 1 \tag{C.3}$$

which implies that weights S_i are normalized according to

$$\sum_i S_i = 1 \,. \tag{C.4}$$

In order to compute fluctuations of the weights S_i, one calculates moments $\left\langle S_i^{(\mathcal{M})q} \right\rangle$ for $\mathcal{M}$ large. At the end of the calculation, the limit $\mathcal{M} \to +\infty$ is taken. Then

$$\left\langle S_i^{(\mathcal{M})q} \right\rangle = \left\langle \frac{1}{\mathcal{M}^q} \sum_{\substack{\alpha \to i \\ \vdots \\ \alpha_q \to i}} \mathbf{I}_{\alpha_1} \cdots \mathbf{I}_{\alpha_q} \right\rangle , \tag{C.5}$$

where the average is taken over fluctuations of the random medium. Using (C.1) one can insert $\sum_i \frac{1}{\mathcal{M}} \sum_{\alpha_{q+1} \to i} \mathbf{I}_{\alpha_{q+1}} = 1$ and get

$$\left\langle S_i^{(\mathcal{M})q} \right\rangle = \left\langle \sum_i \frac{1}{\mathcal{M}^{q+1}} \sum_{\substack{\alpha_1 \to i \\ \vdots \\ \alpha_{q+1} \to i}} \mathbf{I}_{\alpha_1} \cdots \mathbf{I}_{\alpha_{q+1}} \right\rangle \tag{C.6}$$

This is nothing but the total probability that $(q+1)$ particles starting at the same point get clustered amongst the same dominant "spike". One can express this condition saying that overlaps between these $(q+1)$ particles have to be of order 1:

$$\left\langle S_i^{(\mathcal{M})q} \right\rangle = \lim_{\varepsilon \to 0} \left\langle \sum_i \frac{1}{\mathcal{M}^{q+1}} \sum_{\substack{\alpha_1 \to i \\ \vdots \\ \alpha_{q+1} \to i}} \mathbf{I}_{\alpha_i} \cdots \mathbf{I}_{\alpha_{q+1}} \theta(Q^{\alpha_1 \alpha_2} - 1 \right.$$
$$\left. + \varepsilon) \cdots \theta(Q^{\alpha_1 \alpha_{q+1}} - 1 + \varepsilon) \right\rangle , \tag{C.7}$$

where θ is the step function and $Q^{\alpha\beta}$ is the overlap between two particles α and β. This is most conveniently computed going to replica space as this is the probability of choosing one cluster formed by m replicas and then q replicas amongst these m replicas. Therefore

$$\left\langle S_i^q \right\rangle = \lim_{p\to 0}\lim_{\varepsilon\to 0} \frac{p(m-1)\cdots(m-q)}{p(p-1)\cdots(p-q)}\,, \tag{C.8}$$

where one lets the number of replicas p going to zero in order to obtain thermodynamic quantities. Taking into account that

$$\lim_{p\to 0} m = x(1-\varepsilon) \tag{C.9}$$

and eventually

$$\lim_{\varepsilon\to 0} x(1-\varepsilon) = 1 - \frac{1}{N}\,, \tag{C.10}$$

where N is the effective number of "spikes". Notice that this is different from $x(1) = 1$. This entails that one chooses N effective "spikes" and only consider particles flowing to these N attractors. Then from (C.8) and (C.10) one gets

$$\langle S_i^q \rangle = \frac{\Gamma(q+\frac{1}{N})}{\Gamma(\frac{1}{N})\Gamma(q+1)}\,. \tag{C.11}$$

This can be easily inverted to yield the density $\mathcal{N}(S)$. The density of the weight S is given by (see Fig. 5).

$$\mathcal{N}(S) = \frac{S^{\frac{1}{N}-1}(1-S)^{-\frac{1}{N}}}{\Gamma\left(\frac{1}{N}\right)\Gamma\left(1-\frac{1}{N}\right)}\,. \tag{C.12}$$

This is the number of weights between S and $S + dS$. The normalization is chosen such that $N\langle S\rangle = 1$. This proves that the average number of "spikes" is infinite as

$$\left\langle \sum_i 1 \right\rangle = \langle S_i^{-1} \rangle = +\infty\,. \tag{C.13}$$

However, due to the condition $N\langle S\rangle = 1$, the density $\mathcal{N}(S)$ can be interpreted as the conditional density having chosen a given set of N "spikes". This corresponds to studying the flow of particles at a given fixed "temperature" lower than q_c^{-1}. Taking only into account particles flowing to this given set of N "spikes", statistical properties of weights (S_i) are given by (54).

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