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RELATIVISTIC THEORIES OF PARTICLES AND FIELDS WITH FRACTIONAL SPIN AND STATISTICS *

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We discuss the general structure of relativistic theories in 2+1 dimensions whose physical states carry fractional spin and statistics at both the first–quantized and second–quantized level. We show that the Poincaré representations carried by the physical states of the theory are modified by coupling the particle-number current to a topological term. We discuss the spin-statistics theorem and the dependence of the total angular momentum on the number of particles and we show that due to short-distance divergencies they are different in the first– and second–quantized theories.

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1. Introduction

The possibility of arbitrary spin and statistics in the quantum mechanics of planar systems opens the way to a variety of new theoretical issues which have been intensely investigated in recent times (see Ref.[1] for a review). Largely because of the possible relevance to realistic systems which occur in condensed matter physics, most of the theoretical effort has gone into the elucidation of the *nonrelativistic* quantum mechanics (and, to a lesser extent, field theory [2]).

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However, experience with the spin $^1/_2$case, as well as general arguments, suggest that it is only in the relativistic theory — in particular in field theory — that most of the physical effects of spin manifest themselves. For example, whereas in nonrelativistic quantum mechanics the wave function is just a tensor product of its spin and space parts, in the relativistic case the spin and spatial degrees of freedom are coupled dynamically by the requirement that the wave function be an eigenstate of the Pauli-Lubanski operator; also, the spin-statistics theorem [3] can be proven only in the setting of second quantization.

On the other hand, most of the conventional wisdom on fractional spin and statistics fails in the relativistic case. For example, the possibility of fractional spin is usually viewed as a consequence of the fact that the rotation group for planar systems is the abelian group O(2), therefore there are no quantization conditions due to commutation relations on the spectrum of its single generator J, the angular momentum operator. Also, the path-integral for non-relativistic particles with fractional statistics is constructed [4] by exploiting the fact that paths which belong to different homotopy classes may be assigned weights which provide a one-dimensional unitary representation of the fundamental group of the configuration space $\pi_1(C)$. For indistinguishable particles in the plane $\pi_1(C)$ is the braid group, which admits an infinity of such representations. But if the configuration space is the set of points in $2+1$ dimensional Minkowski space then $\pi_1(C)$ is the permutation group, which has only two such representations, the trivial and the alternating, corresponding to bosons and fermions.

Many other problems of this kind appear if the relativistic generalization of the usual approach is pursued in detail. We shall sketch here how these problems are solved both in quantum mechanics [5] and field theory [6], concentrating on the effects which are not seen in a nonrelativistic treatment. We shall see that, in particular, a new spin-statistics relation appears, the second quantization of the theory does not commute with its point particle limit, and more precisely that the mechanism for quantized theories, is due to a quantum anomaly in the latter case.

1. Group theory and the cocycle formalism

The possibility of fractional spin and statistics in a relativistic theory may be seen from a purely group theoretical viewpoint. Just as nonrelativistic state vectors (wave functions in quantum mechanics, and functionals in field theory) carry a generally reducible representation of the rotation group (O(2), in the plane), relativistic ones carry a representation of the Lorentz group, which in $2+1$ dimensions is SO(2,1). Because the phase of the state vectors is unobservable, the representation may be generally projective, and in particular multivalued, provided the multivaluedness is contained in a phase. Now, the group manifold of SO(2,1) is a one-sheeted hyperboloid, thus the group is infinitely connected, $\pi_1(SO(2,1)) = \mathbb{Z}$, and admits arbitrarily multivalued representations; furthermore, multivalued representations of SO(2,1) correspond to multivalued representations of its rotation subgroup SO(2). Since rotations are generated by the angular momentum operator J, a state vector $\psi_s(q)$ which is multivalued upon rotations, *i.e.*, which upon rotation of 2π acquires a phase

$$e^{i2\pi J}\psi_s(q) = e^{i2\pi s}\psi_s(q) \tag{1}$$

carries fractional angular momentum, equal to $s \bmod(\mathbb{Z})$.

The standard procedure to construct the quantum mechanics and field theory of states that carry a representation of the Lorentz group associated to a given value of spin is to seek for a wave function which carries an irreducible representation of the Poincaré group up to a phase, and more specifically a Poincaré irrep induced by a multivalued Lorentz representation. Poincaré irreps (in 2+1 dimensions) are characterized by the respective eigenvalues m^2 and ms of the two Casimir operators $P^\mu P_\mu$ and $\epsilon_{\mu\nu\rho}M^{\mu\nu}P^\rho$, where P^μ are the momentum operators, which generate translations, and $M^{\mu\nu}$ are the Lorentz generators, related to the angular momentum J and boost B^a, $a = 1,2$ operators by $1/2\left(M^{(12)} - M^{(21)}\right) = R$ and $1/2\left(M^{(0a)} - M^{(a0)}\right) = B^a$. The eigenvalue conditions are interpreted as wave equations, while the values of m and s are interpreted as mass and spin of the state, respectively. In the second quantized theory the wave function is promoted to a local field, which upon adjoint action of the group transforms according to an irrep of the universal covering of the Lorentz group associated to the given value of spin. The field equations then select the one particle irreps of the first-quantized theory.

There is a technical complication in pursuing this approach in the case of generic spin, because finite dimensional representations of SO(2,1) can at most be double-valued, therefore, a representation associated to spin which is neither integer nor half-integer, and is therefore more than double valued (compare Eq.(1)) requires infinite-component wave functions and fields [7].

However, a different approach to the quantization of spinning particles is also available, recently proposed and developed by Polyakov [8] in the spin $1/2$ case. In this approach the 2+1 dimensional propagator is derived from a path integral that contains an extra weight which provides quantization of the spin degrees of freedom. From this viewpoint, wave functions which are multivalued upon Lorentz transformation, as required for noninteger spin, are obtained because the extra weight supplements the state vectors of the theory with a mulitivalued locally trivial one-cocycle of the group. This approach turns out to be viable in the case of generic spin and statistics as well, both at the first and second-quantized level.

Quite in general, a multivalued wave function ψ may be written as

$$\psi_0(q) = e^{is\alpha_0(q)}\psi(q)\,, \tag{2}$$

where ψ is a single-valued wave function, while $\alpha_0(q)$ is function such that if $g_0^n \in \widetilde{SO(2,1)}$ is in the n-th Riemann sheet of the group manifold of $\widetilde{SO(2,1)}$, but it projects down to the identity of SO(2,1), then

$$\alpha_0(q^{g_0^n}) - \alpha_0(q) = n\,, \tag{3}$$

where q^g denotes the Lorentz transform of point q in configuration space by the element g of the Lorentz group. The transformation properties of α_0 imply that ψ_0 transforms with a cocycle $\omega_1(q;g)$, according to

$$U(g)\psi_0(q) = e^{i\omega_1(q;g)}\psi_0(q^g)\,, \tag{4}$$

where $U(g)$ is the action of the Lorentz group element g on the Hilbert space spanned by wave functions, and the cocycle is given by

$$\omega_1(q;g) = s\,(\alpha_0(q^g) - \alpha_0(q)) = s\Delta^g\alpha_0\,. \tag{5}$$

Eq.(5) expresses the fact that the cocycle is trivial; nevertheless, due to the multivaluedness (3) of α_0 the triviality is only local, *i.e.*, it is not possible to eliminate the cocycle by a local phase redefinition of the wave function.

An explicit expression of $\omega_1(q;g)$ can be given in terms of the winding number density over the group SO(2,1). The winding number density is a function $w[g(t)]$ such that if $g(t)$ is a one-parameter smooth family of elements of the group, *i.e.*, a path over the group manifold parameterized by t, then the line integral of w along a closed path P on SO(2,1) is equal to the homotopy class of the path, *i.e.*, if P is in the p-th homotopy class, then

$$\oint_P dt\, w(t) = p\,. \tag{6}$$

The cocycle is constructed by choosing a reference point q_0 in configuration space, and it is given by

$$\begin{aligned} \omega_1(q;g) &= s\int_{t_0}^{t_1} w(t)\,,\\ q_0^{g(t_0)} &\equiv \Lambda(t_0)q_0 = q\,,\\ q_0^{g(t_1)} &\equiv \Lambda(t_1)q_0 = q^g\,. \end{aligned} \tag{7}$$

Now, a cocycle may be induced on the state vectors of the theory by adding a topological term to the action which appears as a weight in the sum over paths that provides the propagator of the theory. Indeed, this is the standard path-integral approach to the nonrelativistic quantum mechanics with fractional spin [4]. Given a theory with Lagrangian $L(t)$ and canonical statistics, add a total derivative (topological) term to the Lagrangian: $L \to L - \frac{d\Omega}{dt}$. The propagator of the theory is $\langle\psi_f|\psi_i\rangle = \langle\psi_f|q',t'\rangle K(q',t';q,t)\langle q,t|\psi_i\rangle$, where

$$K(q',t';q,t) \equiv \langle q',t'|q,t\rangle = \int\limits_{\substack{q(t)=q\\ q(t')=q'}} \mathcal{D}q(t_0)\ \exp\left(i\int_t^{t'} dt_0\left(L - \frac{d\Omega}{dt_0}\right)\right). \quad (8)$$

The topological term depends only on the boundary conditions and can be taken out of the path-integral:

$$K(q',t';q,t) = \sum_n e^{-i\Omega^n(q')}K_0(q',t';q,t)e^{i\Omega(q)}\,, \quad (9)$$

where

$$K_0(q',t';q,t) = \int \mathcal{D}q(t_0)\exp\left(i\int_t^{t'} dt_0\, L\right), \quad (10)$$

and the sum runs over homotopy classes of paths, since the configuration space to which q belongs in general is multiply connected, and in such case the value of the surface term $\Omega(q')$ may depend both on the boundary condition at q', and on the homotopy class of the path from q to q'. Then, the phases $e^{i\Omega}$ may be absorbed in the wave function $\langle q,t|\psi\rangle = \psi(q,t)$:

$$\psi_0(q,t) = e^{i\Omega(q)}\psi(q,t). \quad (11)$$

The path-dependence (the sum over n in Eq.(8)) may be reproduced by simply fixing the choice of branch of the generally multivalued function Ω in Eq.(11) by

$$\Omega(q) = \int_{q_0}^{q} dq'\frac{d}{dq'}\Omega(q'), \quad (12)$$

where the integration runs along a path which joins a fiducial reference point q_0 to the point q at which Ω is evaluated.

The wave functions ψ_0 are propagated by K_0 (10); for particular choices of the functions Ω the phase $e^{i\Omega}$ may have nontrivial transformation properties upon rotations, so that ψ_0 carries fractional spin even though ψ has canonical spin. In nonrelativistic quantum mechanics the points q are given for a system of n particles by n two-vectors $(\vec{x}_1, \ldots, \vec{x}_n)$, and a choice of Ω which provides fractional spin and statistics is

$$\Omega(\vec{x}_1, \ldots, \vec{x}_n) = 2s \sum_{i=2}^{n} \sum_{j=1}^{i-1} \Theta(\vec{x}_i - \vec{x}_j), \tag{13}$$

where $\Theta(\vec{x})$ is the *multivalued* polar angle of the vector $\vec{x}$ (with respect to an arbitrary reference axis). Obviously upon rotation by θ Ω (13) varies by

$$\Delta^{\theta} \Omega = sn(n-1)\theta \tag{14}$$

thereby endowing the wave function (11) with angular momentum

$$j = sn(n-1) + \ell, \quad \ell \in \mathbb{Z} \tag{15}$$

according to Eq.(1).

Notice that there are two complementary views of this phenomenon: either one deals with single-valued wave functions and a dynamics including a topological action, or one deals with multivalued wave functions with an ordinary Lagrangian, *i.e.*, not modified by surface terms. In the latter case the angular momentum operator J is the canonical one and has the unusual spectrum (15) due to the boundary conditions, while in the former case the operator itself is shifted with respect to the canonical one.

This above procedure may be applied also to relativistic quantum mechanics, provided a suitable relativistic generalization of Ω (13) is found. The generalization to field theory of this approach is in principle also feasible [9,6], by means of the Schrödinger functional formulation of field theory. One singles out time t and quantizes the fields $\phi(\vec{x})$ canonically at fixed time; the state vectors are functionals of the field configurations: $\langle q, t|\Psi\rangle = \langle \phi(\vec{x}), t|\Psi\rangle = \Psi[\phi(x); t]$. The propagator is

$$K(\phi'(\vec{x}), t'; \phi(\vec{x}), t) = \int \mathcal{D}\phi(\vec{x}, t_0) \exp\left(i \int_t^{t'} dt_0 \int d\vec{x} \mathcal{L}[\phi(\vec{x}, t_0)] \right), \tag{16}$$

where the boundary conditions are the field configurations $\phi(\vec{x})$ at initial time t and $\phi'(\vec{x})$ at final time t'. This procedure is not manifestly relativistically invariant at intermediate stages, although obviously physical amplitudes (S-matrix elements) are. Rather, the state functionals transform covariantly: upon a Lorentz boost that takes the vector $\hat{t}$ into the time-like vector $\hat{n}$ the physical states are transformed into functionals of the fields quantized on the plane orthogonal to $\hat{n}$, at fixed values of the coordinate along $\hat{n}$. In general, one may choose to quantize the system canonically on a space-like plane Σ and take the coordinate orthogonal to Σ to parametrize its evolution. Generic spin is obtained by adding to a bosonic Lagrangian $\mathcal{L}_0$ a topological Lagrangian $\mathcal{L}_t$, which is the total divergence of a three-vector density $\mathcal{L}_t = \partial_\mu \Omega^\mu(x)$. If we demand that fields fall off at infinity, this leads to nonvanishing contributions at initial and final times only, since there the field configuration is nontrivial because of the boundary conditions:

$$\int d\vec{x} dt_0 \, \partial_\mu \Omega^\mu[\phi(\vec{x}, t_0)] = H(t') - H(t)$$
$$H(t) = \int d\vec{x} \, \Omega_0(\vec{x}, t) \,. \tag{17}$$

Again, the state functionals may be redefined according to Eq. (11), with $\Omega(t) \equiv H(t)$; the argument then is a rerun of the above. Notice however, that explicit construction of the topological Lagrangian poses several technical complications, which we shall discuss in Section 4.

2. Relativistic quantum mechanics

The topological Lagrangian $L_t = \frac{d\Omega}{dt}$ associated to the cocycle Ω given by Eq.(13) can be written in a covariant form by defining a particle number current

$$j^\mu(x^\alpha) = \sum_{i=1}^{n} \int ds \, \delta^{(3)}(x - x_i) \frac{dx^\mu}{ds}. \tag{18}$$

Then, we define the (bilocal) action

$$I_t[j] \equiv s\pi \int d^3x \, d^3y \, j^\mu(x) K_{\mu\nu}(x,y) j^\nu(y) \tag{19}$$

(up to an irrelevant infinite normalization) where the bilocal kernel

$$K_{\mu\nu}(x,y) = -\frac{1}{2\pi} \epsilon_{\mu\rho\nu} \frac{(x-y)^\rho}{|x-y|^3} \tag{20}$$

is the inverse of the operator $\epsilon^{\mu\nu\rho}\partial_\nu$ when acting on the current j^ν, *i.e.*, it satisfies

$$\epsilon_{\mu\nu\rho} \partial^\nu K^{\rho\sigma}(x,y) = \delta_\mu{}^\sigma \delta^{(3)}(x-y) \tag{21}$$

up to the addition of terms proportional to ∂_σ which are irrelevant because j^σ is conserved.

Substituting the point-particle form of the current (18) in the action (19) one obtains

$$I_t = \sum_{i=1}^{n} I_t^1[x_i] + 2\sum_{i=2}^{n}\sum_{j=1}^{i-1} I_t^2[x_i - x_j], \tag{22}$$

$$I_t^1[x_i] = -\frac{s}{2}\int dx_i^\mu \, dx_i^\nu \, \epsilon_{\mu\rho\nu} \frac{(x(s)-x(t))^\rho}{|x(s)-x(t)|^3}, \tag{23}$$

$$I_t^2[x_i - x_j] = -\frac{s}{2}\int dx_i^\mu \, dx_j^\nu \, \epsilon_{\mu\rho\nu} \frac{(x_i - x_j)^\rho}{|x_i - x_j|^3}, \tag{24}$$

where the integrations run over the space-time paths traversed by the particles' trajectories, and the reparameterization independence of I_t has been made explicit.

If we neglect the particle self-interactions (23) then it can be shown [1] that I_t coincides with the nonrelativistic topological action associated to Ω (13). Whereas it is easy to give a relativistic meaning to this quantity as the sum of the linking numbers of the various one-particle paths in space time, and while in the nonrelativistic limit the self-interaction terms can be set to zero by a choice of regularization [10], it is clear that in a relativistic theory this cannot be possible because paths contributing to a relativistic path integral can go both forward and backwards in time, thereby making the distinction between self-interaction and two-body interaction a Lorentz noninvariant concept. This is just what we would expect since in the one-particle case only the self-interaction exists, and we expect spin to modify the single-particle path integral in the relativistic theory only. Most of the new results in the relativistic quantum mechanics thus come from the treatment of the self-interaction terms (23).

Although the kernel $K_{\mu\nu}$ (20) is singular when $x \to y$ the integrand in Eq.(23) is everywhere regular and the integral (22) can be computed without need of regularization (contrary to occasional statements in the literature). The result is best expressed in terms of the istantaneous unit tangent to the path

$$e^\mu(t) = \frac{\dot{x}^\mu}{|\dot{x}|}. \tag{25}$$

which spans a two-sheeted hyperboloid and may be parametrized as

$$e^\mu_{(0)}(t) = \begin{pmatrix} \cosh\theta \\ \sinh\theta\sin\phi \\ \sinh\theta\cos\phi \end{pmatrix}. \tag{26}$$

Then it can be shown [5] that

$$I_t^1 = s\int dt\, \dot{\phi}\, \cosh\theta. \tag{27}$$

This demonstrates immediately that the self interaction I_t^1 is not quite "topological", but rather, it depends on the metric properties of the path. The quantity (27) (when rotated to Euclidean space) is actually known as a knot invariant, called the writhing number: it is equal to a topological invariant of the curve, its self-linking number, minus its total geometric torsion (which is of course a metric quantity). The self-linking number is defined as the number of intersection of the curve with the envelope of its tangents; it is a measure of the number of coils which the curve forms. Whereas the torsion varies continuously upon small deformations of the curve, the self-linking varies discontinuously. The discontinuity may be seen explicitly in the expression (27) which is ill defined when $\theta = 0$, *i.e.*, when $e = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. The presence of a singular point is crucial in order for the phases induced by the action (27) to be multivalued: because e spans a simply connected space it is necessary to excise a point from it in order to construct a multivalued function on this space. In the spin $\frac{1}{2}$ case the action (27) reduces to that evaluated by Polyakov [8], thus reproducing the Dirac propagator when used in the path integral (8).

Because the action (27) is not entirely topological it cannot be eliminated by a local phase redefinition of the wave function. However, the effects of the topological interaction can still be reabsorbed in a phase redefinition of the form (11) provided the wave function is localized on a path, rather than in a point. This is possible because Eq.(27) evaluated for a closed path can be rewritten as the holonomy of a certain potential $\tilde{A}[e]$ (actually the Dirac monopole potential) which, in turn, can be cast as a surface integral on the surface S bound by the path:

$$I_t^1 = s \oint dt\, \frac{de}{dt} \cdot \tilde{A}[e] = s \int_S dS^\mu \epsilon_\mu{}^{\nu\rho} \partial_\nu \tilde{A}'_\rho[e]. \tag{28}$$

The generally open paths which appear in Eq.(27) can always be closed without modifying the value of the integral (*i.e.*, they may be closed by a curve that carries vanishing writhing). Then, we may define a wave function

$$\psi_0(x) = e^{-is\Theta_{P_0}(x)}\psi(x)\,, \tag{29}$$

$$\Theta_{P_0}(x) = \int_{x_0\,P_0}^{x} dx'\, \frac{de}{dx'} \cdot \tilde{A}[e]\,, \tag{30}$$

where P_0 is a path that joins a reference point x_0 to x in such a way that the closed path $P_0 \cup [x, x']$ has the same writhing as the open path $[x, x']$ for all x, x'. It can be shown [5] that ψ_0 (29) is propagated by the kernel K_0 (10) without topological term, because the phase (30) transported along the path traversed by x in the path integration yields a contribution to the propagator on the surface swept by P_0 which exactly reproduces the phase (27). The fact that a relativistic wave function with fractional spin must be localized on a path has been shown in Ref.[11] on axiomatic grounds; Eqs(27)–(30) give an explicit expression of the nonlocal part of the wave function.

Finally, it can be verified explicitly that ψ_0 (29) transforms with a Lorentz cocycle (7), which reproduces the correct Poincaré irreps for momentum eigenstates. It thus appears that in a relativistic theory, too, a dual description of fractional spin is possible, either with an extra term in the action and usual wave functions, or with the ordinary action and unusual wave functions. Notice that in the relativistic case it is not enough to shift the angular momentum operator (or its spectrum): in order to preserve the Poincaré algebra all the generators must be modified. Indeed, the above construction is made possible by the existence of a redefinition of the Poincaré generators which preserves the algebra while shifting the angular momentum spectrum [7].

When $n > 1$, the self-interaction and two-body interaction are both present, and can both be absorbed in the wave function by setting

$$\psi_0(\vec{x}_1,\ldots,\vec{x}_n;t) = \exp\left(is\sum_{i=1}^{n}\sum_{j=1}^{n}\Theta_{ij}(t)\right)$$

$$\times\int\frac{m}{E_1}d^2k_1\ldots\frac{m}{E_n}d^2k_n\,\exp\left(-is\sum_{i=1}^{n}\Theta_{P_0}(k_i)\right)\langle k_1,\ldots,k_n|\psi(\vec{x}_1,\ldots,\vec{x}_n;t)\rangle, \tag{31}$$

where Θ_{P_0} is as in Eq.(29). Whereas the phases Θ_{P_0} contribute to the spin, but not to the statistics (they are invariant upon the interchange of two particles), the phases Θ_{ij} contribute to the statistics (but not to the spin) because upon interchange of two particles they vary by π. If we define the statistics σ by

$$\psi_0(\ldots,x_i,\ldots,x_j,\ldots) = e^{2i\sigma\pi}\psi_0(\ldots,x_j,\ldots,x_i,\ldots), \tag{32}$$

(where the interchange is performed by an anticlockwise rotation), then the two-body interaction (24) endows ψ_0 with statistics $\sigma = s$. Since the interchange can be realized by the action of the orbital angular momentum operator this also yields a contribution to the spectrum of the relative orbital angular momentum $L_{x_ix_j}$ of particles i and j, which becomes

$$l_{x_ix_j} = \ell + 2\sigma; \quad \ell\in\mathbb{Z}\,,\ \sigma = s. \tag{33}$$

For an n particle system there are $\frac{1}{2}n(n-1)$ contributions of the form (33), and n contributions to the spin from the n self-interaction terms, thus the orbital angular momentum L, spin S and total angular momentum J carried by the wave function ψ_0 (31) are given by

$$\begin{aligned} &L = n(n-1)s + \ell; \quad \ell \in \mathbb{Z} \;, \\ &S = ns \,, \\ &J = L + S = \ell + n^2 s \,, \end{aligned} \tag{34}$$

in agreement with the result of Ref.[11], based on the algebraic approach.

This shows that in the present theory there is a fixed spin-statistics relation $\sigma = s$. Notice that this is not a spin-statistics theorem in the usual sense of the word: by construction in this theory $\sigma = s$, but we have not proven that a different theory with $\sigma \neq s$ would necessarily be nonlocal or otherwise ill-defined. Also, the spin-statistics relation, which connects the coefficients of the one particle and two particle terms (23),(24) should not be confused with the relation between σ defined in Eq.(32) and the orbital angular momentum spectrum (33), which is a kinematical identity and is always true.

4. Relativistic field theory

Although formally the generalization of the above treatment to field theory seems easy (after all the topological action (19) can be viewed as a field theoretic object provided the currents j^μ are written in terms of some covariant fields), a closer look at it reveals several problems. First of all, in order to compute the effects of the topological interaction (19) we have used crucially its explicit expression (22)–(24) in terms of one-particle trajectories. As a matter of fact, if the current j^μ is a smooth density, rather than being a sum of delta functions, it is not clear how we can tell the particle self-interaction from the two-particle interaction, whereas this separation is required if the topological interaction is to have effect both on the spin (which is related to the self-interaction) and on the statistics (which comes from the particle-particle interaction). Moreover, if the currents are smooth densities, the action (19) (as a classical object) is inevitably Lorentz invariant, and surface terms produced by it are Lorentz covariant: indeed, in the quantum mechanical setting of the previous section all noncovariant transformation properties could be traced back to the singular nature of the point particle current (18). Finally, in a quantum field theory the propagator (16) is going to be the same operator regardless of the number of particles contained in the in and out states of the theory. The phase acquired by the states upon rotation, instead, must depend on the number of particles (for instance, it must be zero for the vacuum, and nonzero for the one-particle states). The solution to all these problems is related to the fact that the lack of rotational invariance of the surface terms $H(t)$ (17) arises as a consequence of an anomaly.

Let us therefore assume that the action (19) is added to that of some bosonic field theory which has a global U(1) symmetry, and thus admits a conserved U(1) current j^μ, which coincides with that which enters the action (19). It turns out that the surface terms $H(t)$ (17) can actually be computed explicitly [6] for an arbitrary field configuration, up to terms which have trivial (*i.e.*, covariant) Lorentz transformation properties. The key remark is that the kernel $K^{\mu\nu}$ (20) has the form of the magnetic field of a Dirac monopole; thus, it can be written as the curl of a gauge potential with a Dirac string of singularities:

$$\frac{x^\alpha}{|x|^3} = \epsilon^{\alpha\mu\lambda}\partial_\mu \tilde{A}_\lambda. \tag{35}$$

An explicit expression of $\tilde{A}$ is

$$\tilde{A}_t = 0,\ \tilde{A}_a = -\frac{\epsilon_{ab}x^b}{r(t-r)}, \tag{36}$$

where $a = 1,2$, and $r^2 = t^2 - x_1^2 - x_2^2$. The current-current interaction (19) is thus of the form

$$I_t = -\frac{s}{2}\int d^3x\, d^3y\, j^\mu(x)\Big[\partial_\mu \tilde{A}_\nu(x-y) - \partial_\nu \tilde{A}_\mu(x-y)\Big] j^\nu(y), \tag{37}$$

or

$$I_t = -\frac{s}{2}\int d^3x\, j^\mu(x)\partial_\mu\Phi(x),$$
$$\Phi(x) = \int d^3y\left(\tilde{A}_\rho(x-y)j^\rho(y) + \tilde{A}_\rho(y-x)j^\rho(y)\right), \tag{38}$$

which, in turn, may be cast as a divergence:

$$I_t = -\frac{s}{2}\int d^3x\, \partial_\mu \Omega^\mu, \tag{39}$$

where $\Omega^\mu = \Phi j^\mu$.

Although Eq.(39) is still nonlocal in time, because $\Omega^0(t)$ is defined in terms of Φ (38) as an integral over all times, the divergent nature of the monopole potential when $x \to y$ allows an explicit local determination [6] of the surface terms (17):

$$I_t = -2s\left[H(t') - H(t)\right] + I_{\mathrm{cov}}, \tag{40}$$

$$H(t) = \frac{1}{2}\int d^2x\, d^2y\, j^0(\vec{x};t)\Theta(\vec{x}-\vec{y})j^0(\vec{y};t), \tag{41}$$

where $I_{\rm cov}$ denotes a contribution which cannot be simply cast as a surface term, but is covariant upon Lorentz transformation. The function $H(t)$ may seem at first quite ill-defined, since the function $\Theta(\vec{x})$ is ill-defined when $|\vec{x}| \to 0$. Indeed, at the classical level I_t (40) is Lorentz invariant, implying that $H(t)$ (41) is a rotationally invariant and Lorentz covariant quantity; on the other hand if we use a point-particle expression (18) for the charge densities in Eq.(41) then $H(t)$ reduces to a sum of manifestly rotationally noninvariant terms of the form (13). This contradiction entails several paradoxes if $H(t)$ is treated as a classical object [2,12]. However, in quantum field theory the propagation kernel (16) is an operator, a functional of the field operators on which the currents j^μ depend. The phases $e^{2siH(t)}$ (17) induced on the state functionals should therefore be viewed as operator-valued quantities. The fact that the bilocal kernel in $H(t)$ (41) is ill-defined at $x = y$ is irrelevant because the product of the two charge densities diverges when their arguments coincide as $j^0(x)j^0(y) \underset{x\to y}{\sim} \frac{1}{|x-y|^4}$. This point is thereby effectively excluded from the integration domain in Eq.(40).

This can be verified by checking that $H(t)$ has a well defined expectation value when acting on physical states. For this, we need to define particle creation and annihilation operators $\phi^\dagger(x)$ and $\phi(x)$, respectively, which satisfy (by definition)

$$[j^0(\vec{y},t), \phi^\dagger(\vec{x},t)] = \delta^{(2)}(\vec{x}-\vec{y})\phi^\dagger\,,$$
$$[j^0(\vec{y},t), \phi(\vec{x},t)] = -\delta^{(2)}(\vec{x}-\vec{y})\phi\,. \tag{42}$$

Then, if the states of the theory are constructed from a vacuum which is annihilated by the charge operator, it is easy to show that

$$\exp(2isH)\phi^\dagger(\vec{x};t)|0\rangle = \exp\big(2isS(\vec{x})\big)\phi^\dagger(\vec{x};t)|0\rangle\,, \tag{43}$$

$$\exp(2isH)\phi^\dagger(\vec{x};t)\phi^\dagger(\vec{y};t)\widetilde{|0\rangle}$$
$$= \exp\big(2is\,[S(\vec{x})+S(\vec{y})]\big)\exp\big(-2is\Theta(\vec{x}-\vec{y})\big)\phi^\dagger(\vec{x};t)\phi^\dagger(\vec{y};t)\widetilde{|0\rangle}. \tag{44}$$

Here $S(\vec{x})$ and $\Theta(\vec{x})$ are the single and double commutator of $H(t)$ with the creation operator, respectively:

$$[H(t), \phi^\dagger(\vec{z},t)] = S(\vec{z},t)\phi^\dagger(\vec{z},t)\,, \tag{45}$$

$$\Big[S(\vec{x}), \phi^\dagger(\vec{y})\Big] = \Theta\,(\vec{x}-\vec{y})\,\phi^\dagger(\vec{y}), \tag{46}$$

where

$$S(\vec{x},t) = \int d^2y\,\Theta(\vec{x}-\vec{y})j^0(\vec{y},t), \tag{47}$$

$\Theta(\vec{x})$ is as in Eq.(13), and

$$\widetilde{|0\rangle} = e^{2isH}|0\rangle. \tag{48}$$

The redefined vacuum $\widetilde{|0\rangle}$ (48) is generally different from $|0\rangle$ because although $Q|0\rangle = 0$, in general $j^0(\vec{x})|0\rangle \neq 0$. However, the redefinition does not affect the Poincaré invariance of the vacuum, and amounts to normal ordering. The action on a generic n-particle state is

$$\exp\left(2isH\right)|\Psi_n\rangle = \prod_{i=1}^{n}\left(\exp\left(2isS(\vec{x}_i)\right)\phi^\dagger(\vec{x}_i)\right)|0\rangle \tag{49}$$

$$= \exp\left(-2is\sum_{j=1}^{n}\sum_{i=1}^{j-1}\Theta(\vec{x}_i - \vec{x}_j)\right) \times\left[\exp\left(2is\sum_{i=1}^{n}S(\vec{x}_i)\right)\prod_{i=1}^{n}\phi^\dagger(\vec{x}_i)|0\rangle\right]. \tag{50}$$

The phases induced on antiparticles, created by ϕ are the same but with the opposite sign, as required by angular momentum conservation. The phases Θ are obviously rotationally noninvariant, and so are the phases $S(\vec{x})$:

$$R^\beta \exp\left(2isS[j^0(\vec{x})]\right)R^{\beta^{-1}} = \exp\left(2is\beta Q\right)\exp\left(2isS[j^0(R^\beta\cdot\vec{x})]\right), \tag{51}$$

where Q is the charge operator and $S[j^0(R^\beta\cdot\vec{x})]$ denotes the covariant Lorentz transform of $S[j^0(\vec{x})]$, obtained by transforming the argument of the field operators on which S depends. It may further be shown [6] that upon generic Lorentz transformation the states (50) transform with the appropriate Lorentz cocycle, weighted by the sum of the coefficients of the phases in Eq.(48).

Eqs (42)–(50) demonstrate that thanks to short distance divergencies in the product of currents which appears in the definition of the bilocal action (37) the classically invariant action I_t (37) induces an operator-valued phase on the state functional of the field theory whose matrix elements are generally Lorentz nonivariant. This shows that the noninvariance which lifts the Lorentz representation provided by the physical states of the theory to one of the universal cover of the Lorentz group appears as a quantum anomaly in this theory. Moreover, due to the short distance divergencies which are at the base of the commutators (45)–(46), the phase induced on physical states depends on the particle content of the states, as it ought to, and naturally splits in a spin and a statistics phase, corresponding respectively to the term in square brackets and that outside brackets in Eq.(50).

An independent check that indeed short distance divergencies in the product of the operator phase induced by the topological action (37) and the field creation operators are responsible for the rotational noninvariance of physical states may be obtained by computing directly the leading divergence in the operator product expansion of e^{iI_t} and a field creation operator [13]. Let us look in particular at the lowest order in the topological coupling s:

$$I_t\phi^\dagger(z) = \Big[-\frac{s}{2}\int d^3x\, d^3y\, j^\mu(x) \times\Big(\partial_\mu \tilde{A}_\nu(x-y) - \partial_\nu \tilde{A}_\mu(x-y)\Big) j^\nu(y)\Big]\phi^\dagger(z)\,. \tag{52}$$

The leading divergence comes from the operator product expansion

$$j_\mu(x)\phi(y) \underset{x\to y}{\sim} C\frac{(x-y)_\mu}{|x-y|^3}\phi\left(\frac{x+y}{2}\right) + O(|x-y|^{-1}). \tag{53}$$

The coefficient C can be calculated perturbatively, say in the ϕ^4 theory; if the current has the Klein-Gordon form $j_\mu =: i\phi^\dagger\partial_\mu\phi :$ then $C = -\frac{1}{4\pi}$.

It follows that the leading divergence in the operator product (52) is

$$I_t\phi^\dagger(z) = -\frac{s}{4\pi}\int d^3x\, d^3y\, j^\mu(x)\Big[\partial_\mu \tilde{A}_\nu(x-y) - \partial_\nu \tilde{A}_\mu(x-y)\Big] \times \frac{(y-z)^\nu}{|y-z|^3}\phi^\dagger(\frac{y+z}{2}) + \text{ less divergent terms}\,. \tag{54}$$

A lengthy but relatively straightforward computation leads to

$$I_t\phi^\dagger(z) = 2sS(\vec{z})\phi^\dagger(z) + \text{covariant terms} \tag{55}$$

in agreement with Eq. (43). The check may be pursued at higher orders and for generic n-particle states. This shows that the rotationally noninvariant phases in Eqs (43)–(44) are generated by the leading divergence in the operator product expansion of the exponential of the action (38), viewed as a composite operator, and the field creation and annihilation operators, without having to manipulate classically ill-defined expression as that of Eq. (41).

The spin, statistics and angular momentum of n-particle states can now be read off Eq.(50). It should be noticed, however, that the statistics of states should be a free parameter in field theory: namely, we are free to choose the symmetry of physical states by symmetrizing the states on which the operator phase e^{2isH} acts, *i.e.*, by freely adjusting the coefficient of the statistics phases $\Theta(\vec{x}-\vec{y})$ in Eq.(50). Then, we have

$$\begin{aligned} L &= n(n-1)\sigma + \ell; \quad \ell \in \mathbb{Z} \,, \\ S &= 2n^2 s \,, \\ J &= L + S \,, \end{aligned} \tag{56}$$

to be contrasted with the point-particle results, Eq. (34).

Eq. (56) shows that if we require the field theory to be local and well-defined in the thermodynamic limit then a particular spin-statistics relation is singled out [6]. Indeed, noninteracting in and out states can exist only if the total angular momentum (which is an additive quantum number, because the rotation group is abelian) is linear in the number of particles. Otherwise, either noninteracting states do not exist, in which case the thermodynamic limit is ill-defined, or causality is violated. This requirement is satisfied if

$$2s = -\sigma \tag{57}$$

which implies

$$J = \ell + 2ns. \quad \ell \in \mathbb{Z} \,. \tag{58}$$

For this relation to be satisfied a nontrivial symmetry has to be imposed on physical states, *i.e.*, the statistics must differ from that automatically generated by the operator phase, and displayed in Eq. (49). This prevents the identification of the operators in brackets in Eq.(50) as the creation operators for particles with fractional spin, and shows that the results found in a semiclassical approach [14] do not carry over to the full second-quantized theory. Notice that this is genuine spin-statistics theorem; it has the opposite sign as that which one might have been naively guessed, and which is displayed by the point particle theory discussed in Section 3 and derived by some authors [11] in point particle and soliton theories. However, if spin is integer or half-integer Eq. (57) reduces to the usual relation and there is no difference between the field theory and the point particle case.

5. Discussion

We have seen that the cocycle approach to path integrals and wave functions with fractional spin and statistics, which is standard in the non-relativistic case, can be generalized both to relativistic quantum mechanics and field theory. In relativistic quantum mechanics the main new results are dynamical effects of spin even in the one-particle case through a generalization of Polyakov's spin action, paths weighted by a knot invariant of the path, wave functions localized on paths which transform with Poincaré cocycles, an explicit realization of the possibility of an internal redefinition of the Poincaré algebra which preserves the algebra while shifting the angular momentum spectrum, and finally a generalized spin-statistics relation (although not a spin-statistics theorem). In field theory we have found an operator valued spin-statistics phase, the generation of rotational noninvariance via an anomaly due to short distance divergences in the operator product expansion, state functionals which transform with Poincaré cocycles and obtain spin and statistics phases which scale with the number of particles thanks to the action of the operator phase, and the possibility of proving a novel spin-statistics theorem.

Comparison of the point particle and field theoretical angular momentum spectra, Eqs (34) and (56) shows that: *(i)* the dependence of the spin and statistics on the coefficient of the topological action is by a factor of 2 larger in the field theory; *(ii)* the statistics is a free parameter in the field theory while it is fixed *a priori* in the particle theory; *(iii)* the dependence of the statistics on the number of particles is the same while that of the spin is not. This means that the second quantization of the theory does not commute with the point particle limit, and can be traced to the different way the repulsive core which gives rise to fractional spin and statistics is treated. Namely, in both case the repulsive interaction which gives rise to a multiply connected configuration space (required for fractional spin) and the exclusion principle (required for fractional statistics) is due to the divergence of the bilocal kernel Eq. (20) as its arguments x, y coincide. This divergence however is regulated differently. In particle mechanics it is regulated geometrically, by evaluating the kernel over particle trajectories, which leads to the regular integrand of Eq. (19). In field theory it is regulated by the current-current repulsion due to their short-distance divergencies.

This, in particular, explains the different scaling with the number of particles of the spin, which is generated by each particle's self-interaction. The peculiar (quadratic) dependence of S on n (56) in field theory is responsible for the unexpected possibility of defining local in and out states even in theories with fractional spin, which are conventionally viewed as being intrinsically interacting.

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REFERENCES

[1] S. Forte, *Quantum Mechanics and Field Theory with Fractional Spin and Statistics, Rev. Mod. Phys.*, in press (1992).
[2] R. Jackiw, S.Y. Pi, *Phys. Rev.* **D42**, 2500 (1990).
[3] R.F. Streater, A.S. Wightman, *PCT, Spin and Statistics, and All That*, Benjamin, New York 1964.
[4] Y.-S. Wu, *Phys. Rev. Lett.* **52**, 2103; **53**, 111 (1984).
[5] S. Forte, *Int. J. Mod. Phys.*, in press (1991).
[6] S. Forte, T. Jolicœur, *Nucl. Phys.* **B350**, 589 (1991).
[7] R. Jackiw, V.P. Nair, *Phys Rev.* **D43**, 1933 (1991).
[8] A.M. Polyakov, *Mod. Phys. Lett.* **A 3**, 325 (1988); lectures in *Fields, Strings and Critical Phenomena*, Les Houches proc., ed. by E. Brézin and J. Zinn-Justin, North Holland, Amsterdam, 1989.
[9] Y.-S. Wu, A. Zee, *Phys. Lett.* **147B**, 325 (1984).
[10] R. Jackiw, *Nucl. Phys. B* (Proc. Suppl.) **18A**, 107 (1991).
[11] J. Fröhlich, P.A. Marchetti, *Comm. Math. Phys.* **121**, 177 (1989).
[12] C.R. Hagen, *Phys. Rev. Lett.* **63**, 1025 (1989).
[13] S. Forte, T. Jolicœur, unpublished (1990).
[14] G.W. Semenoff, *Phys. Rev. Lett.* **61**, 517 (1988); **63**, 1026 (1989); G.W. Semenoff, P. Sodano, *Nucl. Phys.* **B328**, 753 (1989).