ALGEBRAICALLY DEFORMED HARMONIC OSCILLATOR AND ITS PROSPECTS*

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An analogy is pointed out to exist between the algebraically deformed harmonic oscillator introduced by the author some years ago and the algebraically deformed rotators described by the examples of deformed or "quantum" Lie algebras originated in the last years from the "classical" algebra of SU(2). An experimental question is asked, if in fact physical harmonic oscillators are not (slightly) algebraically deformed. A loose relation to the Pöschl–Teller potential is discussed.

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Deformed Lie algebras called also "quantum" algebras or "quantized" enveloping algebras [1,2,3] arouse recently much interest, although their applications in physics are as yet rather technical. Perhaps, the adjective "quantum" used by some not only distinguishes the new algebraic structures from the "classical" Lie algebras, but also reflects certain expectations that they may imply an important step in quantum physics.

In the prototype case of the "classical" algebra of SU(2), where

$$[J_+, J_-] = J_0\,, \quad [J_0, J_+] = J_+\,, \quad [J_-, J_0] = J_- \tag{1}$$

with $J_\pm \equiv (J_x \pm iJ_y)/\sqrt{2}$ and $J_0 \equiv J_z$, the best known are three deformed Lie algebras:

$$[J_+, J_-] = \frac{q^{2J_0} - q^{-2J_0}}{q^2 - q^{-2}}\,, \quad [J_0, J_+] = J_+\,, \quad [J_-, J_0] = J_- \tag{2}$$

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of Drinfeld and Jimbo [1],

$$s^{-1}J_+J_- - sJ_-J_+ = J_0\,, \quad s^2J_0J_+ - s^{-2}J_+J_0 = J_+\,,$$
$$s^2J_-J_0 - s^{-2}J_0J_- = J_- \tag{3}$$

of Woronowicz [2], and

$$s^{-1}J_+J_- - sJ_-J_+ = J_0\,, \quad s^{1/2}J_0J_+ - s^{-1/2}J_+J_0 = J_+\,,$$
$$s^{1/2}J_-J_0 - s^{-1/2}J_0J_- = J_- \tag{4}$$

of Woronowicz and Witten [2]. These deformed Lie algebras can be considered as different "quantum" maps of the underlying "classical" Lie algebra [4,5]. Speaking in a more physical way, they describe different algebraically deformed rotators.

It happens that as early as 1983 the analogical idea of an algebraically deformed harmonic oscillator was discussed. In fact, a deformed commutation relation for one-dimensional harmonic oscillator was introduced [6]:

$$[a,\, a^+] = 1 + (\lambda^2 - 1)a^+a\,, \tag{5}$$

or

$$a\,a^+ - \lambda^2 a^+ a = 1\,, \tag{6}$$

where $\lambda \neq 0$ denote a real constant. (This was done in the context of some speculations about the origin of lepton and quark generations which are irrelevant for the present paper). It was also shown that if $\lambda^2 \neq 1$ the exact spectrum of the deformed excitation-number operator

$$N = a^+a \tag{7}$$

is exponential (and bounded from below):

$$N_n = \frac{\lambda^{2n} - 1}{\lambda^2 - 1} = \begin{cases} 0 & \text{for } n = 0 \\ 1 + \lambda^2 + \ldots + \lambda^{2n-2} & \text{for } n \geq 1 \end{cases}$$
$$(n = 0, 1, 2, \ldots)\,. \tag{8}$$

For $\lambda^2 > 1$ or $\lambda^2 < 1$ the spectrum N_n is rising to $+\infty$ or to the finite limit $1/(1 - \lambda^2)$, respectively, when n is increasing. If $\lambda^2 \to 1$, then $N_n \to n$, reproducing the spectrum for a "classical" harmonic oscillator. The deformed annihilation and creation operators, a and a^+, satisfy the relations

$$a|n> = \sqrt{N_n}|n-1>\,, \quad a^+|n> = \sqrt{N_{n+1}}|n+1>\,, \tag{9}$$

where $a|0>=0$ and $N_{n+1}=\lambda^2 N_n+1$, while

$$N|n>=N_n|n>\,,\quad <n|n>=1\,,\quad (n=0,1,2,\ldots)\,. \tag{10}$$

This is due to the deformed commutation relations between N and a or a^+,

$$aN-\lambda^2 Na=a\,,\qquad Na^+-\lambda^2 a^+N=a^+\,, \tag{11}$$

following from the basic Eq. (6).

The Hamiltonian of the algebraically deformed harmonic oscillator is

$$H=\tfrac{1}{2}(p^2+\omega^2 q^2)\,, \tag{12}$$

where

$$q=\frac{1}{\sqrt{\omega}}\,\frac{a+a^+}{\sqrt{2}}\,,\qquad p=\sqrt{\omega}\,\frac{a-a^+}{i\sqrt{2}} \tag{13}$$

are deformed canonical variables satisfying through Eq. (5) the Heisenberg deformed commutation relation

$$[p,q]=-i\left(1+\frac{2}{\omega}\,\frac{\lambda^2-1}{\lambda^2+1}H-\frac{\lambda^2-1}{\lambda^2+1}\right)\,. \tag{14}$$

Then, from Eqs (12), (13), (6) and (7)

$$H=\tfrac{1}{2}(a^+a+aa^+)\omega=\left(\frac{\lambda^2+1}{2}N+\tfrac{1}{2}\right)\omega\,. \tag{15}$$

Hence, the energy spectrum of our deformed harmonic oscillator is

$$E_n=\left(\frac{\lambda^2+1}{2}N_n+\tfrac{1}{2}\right)\omega \tag{16}$$

with N_n as given in Eq. (8). For $\lambda^2>1$ or $\lambda^2<1$ the spectrum E_n rises to $+\infty$ or to the finite limit $(1+\lambda^2)\omega/2(1-\lambda^2)$, respectively, when n increases. Thus, in the case of $\lambda^2<1$ a new phenomenon of saturation appears for the spectra N_n and E_n of our deformed harmonic oscillator. If $\lambda^2\to 1$, then $E_n\to(n+\frac{1}{2})\omega$, giving the spectrum of a "classical" harmonic oscillator.

Unless the algebraically deformed harmonic oscillator and/or the algebraically deformed rotator are purely mathematical exercises only, their existence in Nature should manifest itself in some experiments. *A priori*, these manifestations might either be restricted to a few yet unexplained phenomena or, on the contrary, be universal. Obviously, in the second case the algebraic deformations should be extremely small and governed by universally fixed parameters. In the present note we would like to ask a long-distance experimental question, if in fact physical harmonic oscillators are not (slightly) algebraically deformed in the sense of the deformed commutation relation (5) with $\lambda^2 \simeq 1$ but $\neq 1$.

Good examples of physical harmonic oscillators are particular modes of Bose physical fields, first of all, of the electromagnetic field. For instance, the radiation field of monochromatic laser represents such an oscillator. Thus, in the case of $\lambda^2 < 1$ the phenomenon of saturation might appear for the energy of the laser's radiation field. Of course, the saturation limit should be very high, increasing infinitely with $\lambda^2 \to 1$.

It would be interesting to investigate the possibility of analogical saturation phenomenon for an adequate example of algebraically deformed rotator, when its "classical" angular momentum j increases. A physical candidate-system might be here a nucleus with very high spin. As is well known, the physics of nuclei with very high spins is presently a dynamically developing domain.

In the case of the Heisenberg-deformed commutation relation (14) we can find explicitly its exact position-type representation. To this end, let us use the *ansatz*

$$p = p_c = -i\frac{\partial}{\partial q_c}, \quad q = f(q_c, p_c), \tag{18}$$

where q_c and p_c are the "classical" canonical variables:

$$[p_c, q_c] = -i. \tag{19}$$

Within the function $f(q_c, p_c)$ they are presumed to be ordered in such a way that all q_c stand on the left of all p_c. Then, Eqs (14) and (12) imply the relation

$$\frac{\partial f(q_c, p_c)}{\partial q_c} \equiv i[p, q] = 1 - \tfrac{\ell}{2}\omega + \tfrac{\ell}{2}p_c^2 + \tfrac{\ell}{2}\omega^2 f^2(q_c, p_c), \tag{20}$$

where $\ell = 2(\lambda^2 - 1)/\omega(\lambda^2 + 1)$. Of course, $\partial/\partial q_c$ commutes here with p_c. If the operator $f(0, p_c)$ is given, the differential equation (20) should determine the operator $f(q_c, p_c)$. Let us assume that $q = 0$ when $q_c = 0$. Then $f(0, p_c) = 0$ and we obtain from Eq. (20)

$$q \equiv f(q_c, p_c) =: \frac{C(p_c)}{\omega\sqrt{\ell/2}} \tan\left(\omega\sqrt{\tfrac{\ell}{2}}\, C(p_c) q_c\right):, \tag{21}$$

where :( ): denotes our ordering of q_c and p_c, while the operator

$$C(p_c) = \sqrt{1 - \tfrac{\ell}{2}\omega + \tfrac{\ell}{2}p_c{}^2} \tag{22}$$

may be defined by its spectral representation or its formal series. If $\ell < 0$ (*i.e.* $\lambda^2 < 1$) then $\sqrt{\ell} = i\sqrt{|\ell|}$ and in Eq. (21) there appears the hyperbolic tangent of $\omega\sqrt{|\ell|/2}\, C(p_c)\, q_c$. Observe from Eqs (21) and (22) that $q \to q_c$ for $\ell \to 0$ (*i.e.* $\lambda^2 \to 1$), as it should be. Let us mention that an analogical, momentum-type representation also exists for the commutation relation (14). Then $p = g(q_c, p_c)$ and $q = q_c = i\partial/\partial p_c$.

The deformed position operator q in its position-type representation (21),when acting on the plane wave $< q_c|p_c >= \exp(ip_cq_c)/\sqrt{2\pi}$, gives

$$q\tfrac{1}{\sqrt{2\pi}}\exp(ip_cq_c) = \frac{C(p_c)}{\omega\sqrt{\ell/2}}\tan\left(\omega\sqrt{\tfrac{\ell}{2}}C(p_c)q_c\right)\,\tfrac{1}{\sqrt{2\pi}}\exp(ip_cq_c)\,. \tag{23}$$

Here, q_c and p_c are eigenvalues of the operators q_c and p_c. For $\ell \to 0$ (*i.e.* $\lambda^2 \to 1$) Eq. (23) becomes trivial since $q \to q_c$. If $\ell > 0$ (*i.e.* $\lambda^2 > 1$) and $C(p_c)$ is real and $\neq 0$ (it is so for all p_c when $0 < \omega\ell/2 < 1$), the effect of q is periodic in q_c with the period $\pi/(\omega\sqrt{\ell/2}\, C(p_c))$ and singular at the equidistant poles

$$q_c^{(\nu)}(p_c) = \frac{2\nu+1}{\omega\sqrt{\ell/2}C(p_c)}\,\frac{\pi}{2} \qquad (\nu = 0, \pm1, \pm2, \ldots)\,. \tag{24}$$

Thus, in this case, a one-dimensional lattice structure is implied by the form of q. For $\ell \to +0$ (*i.e.* $\lambda^2 \to 1+0$ all poles $q_c^{(\nu)}$ are removed to $\pm\infty$. If $\ell < 0$ (*i.e.* $\lambda^2 < 1$) and still $C(p_c)$ is real (it is so for $p_c^2 \leq \omega + 2/|\ell|$ when $|\ell < 0$), there is no lattice structure, but it appears again if $\ell < 0$ and $C(p_c)$ becomes imaginary (for $p_c^2 > \omega + 2/|\ell|$ when $\ell < 0$). Note that, strictly speaking, the position operators are $q/\sqrt{m}$ and $q_c/\sqrt{m}$, where m is the mass of the harmonic oscillator. Similarly, $\sqrt{m}\,p$ and $\sqrt{m}\,p_c$ are momentum operators.

Due to Eqs (18) and (21) the Hamiltonian (12) of our deformed harmonic oscillator can be considered as the Hamiltonian of "classical" particle moving in a complicated momentum-dependent potential:

$$H = \tfrac{1}{2}p_c^2 + \tfrac{\omega^2}{2}\left\{:\frac{C(p_c)}{\omega\sqrt{\ell/2}}\tan\left(\omega\sqrt{\tfrac{\ell}{2}}\,C(p_c)q_c\right):\right\}^2\,. \tag{25}$$

Fortunately, we know exactly the spectrum of H.

If the momentum dependence of the potential in Eq. (25) is formally neglected by putting p_c equal to zero, one obtains the "approximate" Hamiltonian

$$H^{\rm PT} = \tfrac{1}{2}p_c^2 + \frac{C^2(0)}{\ell}\,\tan^2\left(\omega\sqrt{\tfrac{\ell}{2}}C(0)q_c\right)\,, \tag{26}$$

where $C(0) = \sqrt{1-\omega\ell/2}$. Here a static potential appears that is essentially the (symmetric) Pöschl–Teller potential [7]:

$$V^{\mathrm{PT}}(q_c) = \frac{C^2(0)}{\ell}\tan^2\left(\omega\sqrt{\tfrac{\ell}{2}}\,C(0)q_c\right)$$
$$= \frac{C^2(0)}{4\ell}\left\{\sin^{-2}\left(\tfrac{\omega}{2}\sqrt{\tfrac{\ell}{2}}\,C(0)q_c - \tfrac{\pi}{4}\right)\right.$$
$$\left. + \cos^{-2}\left(\tfrac{\omega}{2}\sqrt{\tfrac{\ell}{2}}\,C(0)q_c - \tfrac{\pi}{4}\right)\right\} - \frac{C^2(0)}{\ell}\,. \tag{27}$$

For $\ell \to 0$ it approaches the potential of the "classical" harmonic oscillator: $V^{\mathrm{PT}}(q_c) \to \omega^2 q_c^2/2$. If $\ell > 0$ and $C(0)$ is real and $\neq 0$ (*i.e.* $\omega\ell/2 < 1$), this potential is periodic in q_c with the period $\pi/\left(\omega\sqrt{\ell/2}\,C(0)\right)$ and singular at the equidistant poles

$$q_c^{(\nu)} = \frac{2\nu+1}{\omega\sqrt{\ell/2}\,C(0)}\,\frac{\pi}{2} \quad (\nu = 0, \pm1, \pm2, \ldots) \tag{28}$$

(corresponding to the period and the poles of $q = f(q_c, p_c)$, Eq. (21), with p_c formally put equal to zero). In this case the exact spectrum of H^{PT}, calculated *e.g.* in the range $-\pi/2 \leq \omega\sqrt{\ell/2}\,C(0)q_c \leq \pi/2$ between the poles $q_c^{(-1)}$ and $q_c^{(0)}$ (the barriers around the poles $q_c^{(\nu)}$ are impenetrable), is given by formula

$$E_n^{\mathrm{PT}} = \frac{\omega^2\ell C^2(0)}{4}\left(n + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + \left(\tfrac{4}{\omega\ell}\right)^2}\right)^2 - \frac{C^2(0)}{\ell}$$
$$(n = 0, 1, 2, \ldots)\,. \tag{29}$$

For $\ell \to 0$ it tends to the spectrum of the "classical" harmonic oscillator: $E_n^{\mathrm{PT}} \to (n+1/2)\omega$. We can see, however, that for a considerable $\ell > 0$ the spectrum (29) is qualitatively different from the spectrum (16) of the Hamiltonian H of our deformed harmonic oscillator with $\lambda^2 > 1$ (although for $\ell \to 0$ the spectrum of H also tends to the spectrum of the "classical" harmonic oscillator). The reason is that the potential in Eq. (25) for H is then highly nonstatic.

A free deformed particle can be defined as the limit of $\omega \to 0$ of the deformed harmonic oscillator described by its canonical variables p and q. Then $H \to H_{\mathrm{free}}$, $p \to p_{\mathrm{free}}$ and $q \to q_{\mathrm{free}}$, where due to Eqs. (12) and (18)

$$H_{\mathrm{free}} = \tfrac{1}{2}p_{\mathrm{free}}^2\,, \quad p_{\mathrm{free}} = p_c = -i\frac{\partial}{\partial q_c}\,, \quad q_{\mathrm{free}} = f_{\mathrm{free}}(q_c, p_c) \tag{30}$$

and from Eqs (21) and(22)

$$q_{\rm free} \equiv f_{\rm free}(q_c, p_c) = q_c\left(1 + \tfrac{\ell}{2}p_c^2\right). \tag{31}$$

Here, Eq. (14) holds in the limiting form

$$\left[p_{\rm free}, q_{\rm free}\right] = -i\left(1 + \tfrac{\ell}{2}p_{\rm free}^2\right) \tag{32}$$

(cf. Eq. (20) and assume that $\lambda^2 = (1 + \omega\ell/2)/(1 - \omega\ell/2)$ depends on ω, while $\ell = 2(\lambda^2 - 1)/\omega(\lambda^2 + 1)$ does not). Evidently, the spectrum of $H_{\rm free}$ is $E(p_c) = p_c^2/2$, where p_c denotes an eigenvalue of the operator p_c. We can see that in this case the deformation of the Heisenberg "classical" commutation relation (19) is trivial.

Let us mention that the mathematical question may be asked, if our "Bose-type" commutation relation (6), where λ is real (*i.e.* $\lambda^2 > 0$, can be extended to complex λ, giving, a "Fermi-type" commutation relation for imaginary λ (*i.e.* $\lambda^2 < 0$).

Finally, we would like to call the Reader's attention to a paper by Saavedra and Utreras [8], where (several years ago) an extreme conjecture was formulated that, generally, the Heisenberg "classical" commutation relation

$$[p, q] = -i \tag{33}$$

should be (slightly) deformed by multiplication of its rhs by $1 + \ell H$. Here, H is a proper Hamiltonian and ℓ stands for a very distance scale. In the case of our Eq. (14), H is $H - \omega/2$ (with H as given in Eq. (12) and $\ell = 2(\lambda^2 - 1)/\omega(\lambda^2 + 1)$.

Note added in proof: the deformed commutation relation (6) was considered previously, *cf.* Jannusis *et al.*, *Lettera al Nuovo Cimento* **34**, 375 (1982) and references therein; I am indebted to Jerzy Lukierski for his calling my attention to this paper.

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