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INTEGRABLE HAMILTONIAN SYSTEM IN $2N$ DIMENSIONS

K. Golec–Biernat *

Institute of Nuclear Physics
Radzikowskiego 152, 31-342 Cracow, Poland

AND

Th.W. Ruijgrok

Institute of Theoretical Physics
Rijksuniversiteit, Princetonplein 5
P.O.Box 80.006, 3508 TA Utrecht, The Netherlands

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An integrable hamiltonian system in $2N$ dimensions is constructed. It describes N interacting particles on a plane. Solutions to the equations of motion are also presented.

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1. Introduction

Integrable classical hamiltonian systems are very important since it is possible to obtain the solutions of the equations of motion for such systems by quadrature. Practically all systems for which the equations of motion have been solved explicitly are integrable hamiltonian systems.

Let us recall the definition of the classical integrable system [1, 2]. A hamiltonian system with n degrees of freedom, Poisson bracket $\{\ ,\ \}$ and Hamiltonian H is said to be integrable if it possesses n independent integrals of motion K_i $(i = 1, 2, \ldots, n)$ in involution, i.e. :

$$\{H,\ K_i\} = 0\,,$$

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$$\{K_i, K_j\} = 0,$$

where H is not independent of K's.

Liouville's theorem states that the solution of the equations of motion of an integrable system is obtained by quadrature [1, 2].

In this paper we present an example of the classical integrable system with $2N$ degrees of freedom. The example is nontrivial, since it describes N interacting particles on a plane. The interaction is described by a N-body potential, which is an arbitrary function of a variable characterizing the spatial size of the system of particles. The geometrical interpretation of the variable depends on N and is discussed in the next Section. The example we are going to present is also interesting from the formal point of view. It illustrates all important features of integrable hamiltonian systems. Let us add that the case $N = 3$ was considered earlier in paper [3]. Our example generalizes these considerations.

The paper is organized as follows. In Section 2 we define our system in the lagrangian formulation and find integrals of motion. In Section 3 we present the hamiltonian formulation of the dynamics in order to define the hamiltonian system in our example and check whether the integrals of motion from Section 2 are in involution. In Section 4 we integrate the equations of motion of the system, whereas in the last Section we present another simple example of the integrable system with the potential proportional to the area spanned by N particles on the plane.

2. Lagrangian formulation

Let us represent the positions of N particles on the plane by the complex numbers

$$r_k = x_k + i\, y_k\,, \quad k = 1, 2, \ldots, N\,, \tag{1}$$

where the pairs $(x_k\,, y_k)$ are the Cartesian coordinates of the particles. We treat formally the r_k's and their complex conjugate $r_k^* = x_k - i\, y_k$ as independent variables.

With the help of these coordinates we define the following Lagrangian leading to the integrable hamiltonian system

$$L = \frac{m}{2}\left\{|\dot{r}_1|^2 + \ldots + |\dot{r}_N|^2\right\} - V(R)\,, \tag{2}$$

where the potential V is an arbitrary function of the variable

$$R = \frac{1}{N}|r_1 + \epsilon\, r_2 + \epsilon^2\, r_3 + \ldots + \epsilon^{N-1}\, r_n| \tag{3}$$

and $\epsilon = \exp(2\pi i/N)$. The symbol $|\cdots|$ denotes the complex modulus, the dots above the r's mean the differentiation with respect to time and m is the mass of the particles.

The kinetic energy in the Lagrangian written in terms of the Cartesian coordinates x and y has the usual form. The variable R needs clarification. It has a clear geometrical interpretation for N equal 2 and 3, being proportional to the shortest distance between two or three particles on the plane [3]. For N greater then 3 the R is less then the size of the system $\langle r \rangle$

$$R \leq \langle r \rangle \,,$$

where $\langle r \rangle$ is defined as the minimal radius of a sphere which contains all particles. The proof of this statement bases on the Schwartz inequality and observation that the R does not depend on a position of the coordinate system with respect to which the coordinates r's are defined. Thus, the origin of the coordinate system can be placed in the origin of the minimal sphere containing all particles.

Let us stress that the system described by the Lagrangian (2) is integrable for any form of the potential $V = V(R)$. The specific form of the variable R allows for sufficiently enough symmetries to produce integrals of motion *via* Noether's theorem. Recall that if the infinitesimal coordinate transformation

$$r'_k = r_k + \alpha_k \,, \quad k = 1, 2, \ldots, N \,, \tag{4}$$

where α_k's are infinitesimal variations, does not change the Lagrangian L, then the quantity

$$K = \sum_{k=1}^{N} \left(\frac{\partial L}{\partial \dot{r}_k} \alpha_k + \frac{\partial L}{\partial \dot{r}^*_k} \alpha^*_k \right) \tag{5}$$

is an integral of motion.

In our case the variable R transforms under the transformation (4) as follows:

$$R' = R + \frac{1}{2N} \{ (\alpha_1 + \epsilon\, \alpha_2 + \ldots + \epsilon^{N-1}\, \alpha_N) \\ \times (r^*_1 + \epsilon\, r^*_2 + \ldots + \epsilon^{N-1}\, r^*_N) + \text{complex conjugate} \} \,,$$

up to terms linear in the variations α's.

It is easy to see that the infinitesimal rotation

$$\alpha_k = i(\phi/2) r_k \,, \quad k = 1,\ 2, \ldots, N \,, \tag{6}$$

where the real parameter ϕ is an infinitesimal angle, does not change the R and the kinetic part of the Lagrangian (2). Thus, the transformation (6) is a symmetry of our system which leads to the angular momentum with respect to the center of coordinates system

$$J = i\frac{m}{2}\sum_{k=1}^{N}(r_k \dot{r}_k^* - r_k^* \dot{r}_k), \tag{7}$$

as a conserved quantity.

All other symmetries can be obtained assuming that the functions α_k's are constant numbers which fulfill the equation

$$\alpha_1 + \epsilon\,\alpha_2 + \ldots + \epsilon^{N-1}\,\alpha_N = 0. \tag{8}$$

These transformations are infinitesimal translations of the particles of our system.

It is easy to prove that there exist $N-1$ linearly independent solutions of Eq. (8). We write down the solutions in the form of the vector $\alpha^{(j)}$ whose components are the variations α_k's

$$\alpha^{(j)} = (\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \alpha_N^{(j)}),$$

where the upper index j enumerates the solutions. Let us define the unitary $N \times N$ matrix

$$\alpha_{jk} = N^{-1/2}\,\epsilon^{(j-1)(k-1)}, \tag{9a}$$

where $j,\ k = 1,\ 2, \ldots, N$. The first $N-1$ rows of the matrix α_{jk} form the solutions of Eq. (8):

$$\alpha_k^{(j)} = \alpha_{jk}, \tag{9b}$$

where $j = 1,\ 2, \ldots, N-1$ and $k = 1,\ 2, \ldots, N$, since substituting (9b) into (8) and keeping in mind that $\epsilon^N = 1$ we have

$$N^{1/2}\sum_{k=1}^{N}\alpha_{jk}\,\epsilon^{k-1} = \sum_{k=1}^{N}\epsilon^{j(k-1)} = \frac{1-\epsilon^{jN}}{1-\epsilon^{j}} = 0.$$

The last equality was written provided $\epsilon^j \neq 1$, which holds true for $j = 1,\ 2, \ldots, N-1$.

The solutions (9b) are linearly independent because the determinant of the unitary matrix α_{jk} is different from zero. Thus, we have found the $N-1$ independent translations in the complex plane which are symmetries of the Lagrangian (2).

Formula (5) gives independent integrals of motion generated by these translations

$$K_j = m\sum_{k=1}^{N}\alpha_{jk}\,\dot{r}_k^*, \quad j = 1, 2, \ldots, N-1. \tag{10a}$$

The complex conjugate of K's

$$K_j^* = m \sum_{k=1}^{N} \alpha_{jk}^* \cdot \dot{r}_k \,, \quad j = 1, 2, \ldots, N-1 \,. \tag{10b}$$

are also independent integrals of motion. Thus, Eqs (10a) and (10b) define $2(N-1)$ linearly independent integrals of motion generated by the infinitesimal translations.

We end the discussion on the integrals of motion of our system writing probably the most obvious one: the energy

$$E = \frac{m}{2} \left\{ |\dot{r}_1|^2 + \ldots + |\dot{r}_N|^2 \right\} + V(R) \,, \tag{11}$$

which is conserved since the potential V does not depend on time.

In the next Section we will define the hamiltonian system for our example in order to discuss the problem of its integrability. The $2N$ algebraically independent integrals of motion (7), (10a), (10b) and (11) will play a crucial role in the discussion.

3. Hamiltonian formulation

In order to define the hamiltonian system we must pass from the lagrangian to the hamiltonian formulation of the dynamics of our system. Applying the Legendre transformation to the Lagrangian (2) we obtain the following canonical momenta:

$$p_k = p_{xk} + i\, p_{yk} = m\, \dot{r}_k \,,$$

$$p_k^* = p_{xk} - i\, p_{yk} = m\, \dot{r}_k^* \,, \tag{12}$$

$(k = 1, 2, \ldots, N)$ and the Hamiltonian

$$H = \frac{1}{2m} \left\{ p_1\, p_1^* + \ldots + p_N\, p_N^* \right\} + V(R) \,. \tag{13}$$

The fundamental Poisson brackets (FPB) for the canonical variables $(r_k\,, r_k^*\,, p_k\,, p_k^*)$ are of the form

$$\{r_k\,, p_j^*\} = \{r_k^*\,, p_j\} = 2\delta_{kj} \,, \tag{14a}$$

while all other FPB are equal to zero

$$\{r_k\,, r_j\} = \{r_k^*\,, r_j^*\} = \ldots = 0 \,. \tag{14b}$$

The Hamiltonian (13) and the fundamental Poisson brackets (14) define the hamiltonian system in our case. In the previous Section we have found $2N$ independent integrals of motion. We express them in terms of canonical momenta instead of velocities and check whether they are in involution.

The Hamiltonian (13) (equal to the energy (11) expressed in terms of momenta) and the integrals of motion

$$K_j = \sum_{k=1}^{N} \alpha_{jk}\, p_k^* \quad \text{and} \quad K_j^* = \sum_{k=1}^{N} \alpha_{jk}^*\, p_k \tag{15}$$

$(j = 1, 2, \ldots, N-1)$ are in involution :

$$\{H, K_j\} = \{H, K_j^*\} = \{K_i, K_j^*\} = 0\,.$$

The only problem arises with the angular momentum

$$J = \frac{i}{2} \sum_{k=1}^{N} (r_k\, p_k^* - r_k^*\, p_k)\,. \tag{16}$$

Its Poisson brackets with K_j's and K_j^*'s are different from zero :

$$\{J\,, K_j\} = iK_j\,,$$

$$\{J\,, K_j^*\} = -iK_j^*\,.$$

However, it is possible to define a new independent integral of motion which replaces the angular momentum (16)

$$M = J + \frac{i}{2N} \sum_{j=1}^{N-1} \sum_{k=1}^{N} \left\{ \alpha_{jk}\, r_k^*\, K_j^* - \alpha_{jk}^*\, r_k\, K_j \right\}\,, \tag{17}$$

and is in involution with the other integrals of motion :

$$\{M\,, H\} = \{M\,, K_j\} = \{M\,, K_j^*\} = 0\,.$$

Summarizing, the hamiltonian system in our example is integrable since there exist the $2N$ independent integrals of motion in involution (13), (15) and (17).

4. Solutions to the equations of motion

The integrals of motion from the previous Sections suffice to integrate the equations of motion of our system. In order to facilitate this procedure let us perform the transformation leading from the old variables (r_k, r_k^*) to new ones (z_j, z_j^*) ,

$$r_k = \sum_{j=1}^{N} \alpha_{kj}\, z_j \,, \tag{18}$$

where $k = 1, 2, \ldots, N$ and α_{jk} is defined by Eq. (9a). The equations for the r_k^*'s and z_j^*'s are obtained by taking the complex conjugate of Eq. (18). It is important for the later considerations that the inverse relation

$$z_j = \sum_{k=1}^{N} \alpha_{jk}^*\, r_k \,, \tag{19}$$

written for $j = N$ gives:

$$z_N = N^{-1/2} \sum_{k=1}^{N} \epsilon^{-(N-1)(k-1)}\, r_k = N^{-1/2} \sum_{k=1}^{N} \epsilon^{k-1}\, r_k = N^{1/2} R\, e^{i\phi} \,,$$

where R is defined by Eq. (3) and ϕ is the argument of the complex number z_N .

Expressing the constants of motion (7), (10a, b) and (11) in terms of the new coordinates we get

$$J = \frac{im}{2} \sum_{k=1}^{N} (z_k\, \dot{z}_k^* - z_k^*\, \dot{z}_k) \,, \tag{20}$$

$$K_j = m\, \dot{z}_j^* \quad , \quad K_j^* = m\, \dot{z}_j \,, \tag{21}$$

where in the last row $j = 1, 2, \ldots, N-1$ and

$$E = \frac{m}{2} \sum_{k=1}^{N} \dot{z}_k\, \dot{z}_k^* + V(R) \,. \tag{22}$$

We can immediately integrate Eqs (21)

$$z_j(t) = (K_j^*/m)\, t + z_0 \,, \tag{23a}$$

$$z_j^*(t) = (K_j/m)\, t + z_0^* \,, \tag{23b}$$

for $j = 1, 2, \ldots, N-1$.

Now we are left with the integration of the variable z_N (R and ϕ equivalently). The most convenient way to do this is to make use of the integral of motion M (Eq. (17)) instead of J, since in the new coordinates

$$M = \frac{im}{2} (z_N\, \dot{z}_N^* - z_N^*\, \dot{z}_N) = m\, N\, R^2\, \dot{\phi} \,. \tag{24}$$

Substituting relations (19) and (21) into formula (22) we obtain

$$E = \frac{1}{2m}\sum_{j=1}^{N-1}|K_j|^2 + \frac{mN}{2}(\dot{R}^2 + R^2\dot{\phi}^2) + V(R)\,. \tag{25}$$

Computing $\dot{\phi}$ from Eq. (24): $\dot{\phi} = (M/mN)R^{-2}$, and inserting this into the formula for the energy (25) we obtain the final equation

$$\dot{R}^2 + \frac{(M/mN)^2}{R^2} + \frac{2}{mN}V(R) = \kappa\,, \tag{26}$$

which can be solved by quadrature. The constant κ denotes

$$\kappa = \frac{2E}{mN} - \frac{1}{N}\sum_{j=1}^{N-1}|K_j|^2\,.$$

Hence we get the relation $R = R(t)$ and also the solution for ϕ

$$\phi(t) = \frac{M}{mN}\int\frac{dt}{R^2(t)} + \phi_0\,. \tag{27}$$

We have integrated the equations of motion for z_j and z_j^*. This was possible since we constructed $2N$ integrals of motion in involution. The solutions for $r_k = r_k(t)$ and $r_k^* = r_k^*(t)$ can be obtained from relation (18), whereas the trajectories of motion $(x_k(t)\,, y_k(t))$ for N particles on the plane are defined by the relations resulting from Eq. (1):

$$x_k = \tfrac{1}{2}(r_k + r_k^*) \text{ and } y_k = -\tfrac{i}{2}(r_k - r_k^*)\,.$$

5. The area-interaction

This interaction is proportional to the area of a polygon defined by the N order points $r_1, r_2, \ldots, r_N$, where the positive ordering on the plane is defined. It is easy to prove that in this case the system is described by the Lagrangian (2) with the potential:

$$V = k\frac{i}{4}\sum_{m=1}^{N} r_m\, r_{m+1}^* + \text{c.c.}\,, \tag{28}$$

where $r_{N+1} = r_1$ and k is some real constant. The potential does not depend on the orientation and position of the coordinate system in the plane, thus, the angular momentum and the total momentum of the system are conserved. In fact, in this case we do not need symmetries in order to integrate the equations of motion of the system since the potential (28) is a hermitian bilinear form which can always be diagonalized.

The transformation which diagonalizes V is defined by Eq. (18). Thus, in the new coordinates z_j the Lagrangian L has the form:

$$L = \frac{m}{2}\sum_{j=1}^{N}\left\{\dot{z}_j\,\dot{z}_j^* - \left(\frac{k}{m}\sin\frac{2\pi(j-1)}{N}\right) z_j\, z_j^*\right\}. \tag{29}$$

The Euler-Lagrange equations are very simple

$$\ddot{z}_j + \left(\frac{k}{m}\sin\frac{2\pi(j-1)}{N}\right) z_j = 0\,, \tag{30}$$

for $j = 1,\ 2, \ldots, N$ and can be solved immediately.

The $2N$ integrals of motion in involution can be written without any difficulty looking on the Lagrangian (28). They are of the form:

$$E_{jc} = |\dot{z}_{jc}|^2 + \left(\frac{k}{m}\sin\frac{2\pi(j-1)}{N}\right)|z_{jc}|^2\,,$$

for $j = 1,\ 2, \ldots, N$ and $c = x,\ y$, where $z_j = z_{jx} + i\,z_{jy}$. The total energy of the system is the sum of these integrals of motion.

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REFERENCES

[1] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer Verlag, Berlin 1978.

[2] A.J. Lichtenberg, M.A. Lieberman, *Regular and Stochastic Motion*, Springer Verlag, Berlin 1983.

[3] Th.W. Ruijgrok, *Eur. J. Phys.* **5**, 21 (1984).