SCALING SYMMETRIES FOR LINEAR POLYMER CHAINS*

B. LULEK, T. LULEK

Institute of Physics, A. Mickiewicz University
Matejki 48/49, 60-769 Poznań, Poland

AND

M. KUŹMA

Institute of Physics, Higher Pedagogical School
Rejtana 16A, 35-310 Rzeszów, Poland

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A general Weyl's recipe concerning studies on structural properties of physical models, has been applied to the case of a finite polymer chain, where translational symmetry is described by a cyclic group — the group of the obvious symmetry of the model. It has been shown that the hidden symmetry, imposed by this recipe as the group of all automorphisms of the cyclic group, involves some scaling operations, in addition to simple geometric transformations. Invariance of structure of the chain under non-trivial scaling operations allows us to interpret the chain as a self-similar object, with some fractal-like properties. A model of helical scaling has been proposed as a hypothetical possibility of realization of fractal scaling symmetry for polymer chains in constrained areas of three-dimensional space. General considerations are illustrated on an example of a chain composed of twelve elementary Bravais cells, imitating the clock dial plate.

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1. Introduction

In the present paper we investigate symmetry properties of a polymer, imposed by the structure of a finite closed linear chain, in a spirit of "actions

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of groups on sets" (Michel [1]). We apply a general recipe of Weyl [2] (*c.f.* also [3-6]) in order to determine all symmetries resulting from the structural assumptions. As a result, we find that a linear chain displays the structure which is self-similar under some scalings. This feature of self-similarity is characteristic to a class of fractals (Mandelbrot [7]), so that a linear chain can be looked at — at least to some extent — as a finite analog of a fractal. Correspondingly the appropriate scaling operations can be referred to as fractal symmetries.

In Section 2 we give some physical and structural arguments, motivating the assumptions of the model. The mathematical formulation of the model, based on an elementary number theory, is given in Section 3. In Section 4 we discuss a possibility of physical realization of fractal scaling. All these general structures are demonstrated in Section 5 for the case of a chain consisting of twelve polymer units, corresponding to commonly known clock dial plate. Some final conclusions and remarks are given in Section 6.

2. Physical and structural motivation

When proposing a model of structure of a condensed matter, one usually assumes a rigid solid, *i.e.* Euclidean metrics. Thus the only symmetries, admissible for an infinite rigid body by the Euclidean metrics, are translations and orthogonal transformations (rotations, reflections, etc.), whereas, *e.g.* scaling operations are forbiden. However, it seems plausible that the requirement of a rigid Euclidean metrics is sometimes too restrictive for an adequate description of symmetry. In the case of molecular vibrations in quantum chemistry, this observation is a starting point for introducing the notion of symmetry groups for non-rigid (floppy) molecules [8-11]. Such a molecular symmetry group encloses, besides point isometries (which form a subgroup of the group O(3,R) of three-dimensional orthogonal transformations), also some permutations of identical objects, which are forbidden by the rigid Euclidean metrics, but become admissible in presence of tunneling through some potential barriers. Such permutations are referred to as "feasible operations", for which any transition between two symmetry configurations involves a crowding through a barrier, and thus breaks the Euclidean metrics (this breaking is, in a sense, small). Molecular motions in such description are strongly anharmonic.

Similar considerations can be applied to the case of molecular crystals, which constitute a condensed phase of matter, but do not exhibit a geometrically closed packing. In several cases, complicated structure of such crystals can be approximately described in terms of linear chains or rings [12-14]. In these cases, complex molecular interactions exhibit an anisotropy which leads to a strong coupling of molecules into linear chains, and relatively weak

interactions between the chains. Evidently, an exact description of molecular interactions is very complicated and practically impossible. Instead, one can consider a simplified description within a model, whose principal assumption consists in the observed fact that molecules form the structure of the linear chain. The main aim of the present paper is a full exploitation of an apparently evident assumption that the polymer realises the structure of a finite, closed linear chain. In a spirit of "actions of groups on sets" (Michel [1]) we identify the polymer with a regular orbit of an action of the cyclic group C_N. Our goal is to find such "feasible" symmetry operations which preserve the assumed structure.

The method of a systematic solution of this problem is already known as a general recipe of Weyl [2]. According to this recipe, the invariant properties of a given structure, characterized by a group G (the group of the obvious symmetry in the terminology of Weyl), can be derived by a careful investigation of the group Aut G of all automorphisms of the group G (the group of the hidden symmetry). In particular, one has to discuss the role of every automorphism in its action on the given structure.

In our case, the role of the group G of obvious symmetry is played by the cyclic group C_N, which has a simple structural meaning as the "distributing group", which distributes the polymer units in appropriate nodes of the chain, putting them in a definite cyclic order. The hidden symmetry group in the Weyl's recipe is therefore the group Aut C_N, well known from an elementary number theory ([15]). We show in the sequel that such a formulation yields, somehow surprisingly, symmetries under some scaling operations.

3. The mathematical formulation of the model

Let

$$\tilde{N} = \{j \mid j = 1, 2, \ldots, N\} \tag{1}$$

be the set of all positive integers not larger than N, and

$$C_N = \{\tilde{N}, " + " \bmod N\} \tag{2}$$

the cyclic group in the additive notation, *i.e.* the group with the addition modulo N as the group multiplication. The set $\tilde{N}$ constitutes an orbit of the regular representation of the group C_N. In our model the set $\tilde{N}$ represents a polymer chain, whose one-dimensional elementary Bravais cells are labelled by the index j.

The scaling operation on the chain $\tilde{N}$, or, equivalently, on the group C_N, is an endomorphism $\eta_l : C_N \to C_N$, given by the formula

$$\eta_l = \begin{pmatrix} j \\ l_j \bmod N \end{pmatrix}. \tag{3}$$

Each endomorphism η_l consists therefore in the l-fold magnification of the lattice constant of the chain $\bar{N}$ in agreement with the modular condition (3). The set of all inequivalent endomorphisms of the group C_N,

$$\mathrm{End}\, C_N = \{\eta_l \mid l \in \bar{N}\}, \tag{4}$$

has a natural mathematical structure of a commutative ring over the group C_N.

An important characteristics of a single scaling η_l is its kernel

$$\mathrm{Ker}\, \eta_l = \{j \in C_N \mid l_j \bmod N = N \bmod N\}, \tag{5}$$

consisting of all those elements of the group C_N which are mapped onto the unit element $j' = N$ under this scaling, and its image

$$\mathrm{Im}\, \eta_l = \{lj \bmod N \mid j \in C_N\}. \tag{6}$$

It is easy to observe that these sets are both normal subgroups of the group C_N, namely

$$\mathrm{Ker}\, \eta_l = C_\kappa =< \bar{\kappa} >= \{\bar{\kappa}, 2\bar{\kappa}, \ldots, \kappa\bar{\kappa}\} \lhd C_N, \tag{7}$$

and

$$\mathrm{Im}\, \eta_l = C_{\bar{\kappa}} =< \kappa >= \{\kappa, 2\kappa, \ldots, \bar{\kappa}\kappa\} \lhd C_N, \tag{8}$$

where

$$\kappa = \mathrm{lcd}(l, N) \tag{9}$$

is the largest common divisor of integers l and N,

$$\bar{\kappa} = \kappa/N \tag{10}$$

is the divisor complementary to κ with respect to N, *e.g.* $< \kappa >$ denotes the cyclic group, generated from the element $\kappa \in C_N$ (so that, in particular, $C_N =< 1 >$). The mathematical description of scaling as given above is accompanied by an evident structural interpretation. As the result of a scaling η_l, $l \in \bar{N}$, the initial chain $\bar{N}$ is transformed to a new chain $\bar{\bar{\kappa}}$, consisting of $\bar{\kappa}$ elementary Bravais cells. Each elementary cell j' of the new chain $\bar{\bar{\kappa}}$ ($j' = 1, 2, \ldots, \bar{\kappa}$) corresponds to κ elementary cells of the initial chain $\bar{N}$, namely to those cells $j \in \bar{N}$ which satisfy the modular equation

$$lj \bmod \bar{\kappa} = j' \ , \ j \in \tilde{N} \, . \tag{11}$$

We proceed to point out some analogies between scaling of finite linear chains and similar symmetry operations on fractals. The term "fractal" has been introduced by Mandelbrot [7] to describe a class of irregular subsets of an Euclidean space, for which various mathematical definitions of set-theoretic and topological dimension yield different results. An important feature, characteristic for the most of fractal structures considered in physical and chemical applications [16-21], is self-similarity under scalings. Just this feature is shared by finite polymer rings, subjected to scalings by automorphisms $\eta_l \in$ Aut C_N. Thus the polymer ring can be looked at as a finite analogy of a self-similar fractal, which possesses only a finite number of different cyclic orders, obtained by a repeated application of a given scaling η_l, and imitating an infinity of fractal configurations. For this reason, nonrigid scalings imposed by automorphisms $\eta_l \in$ Aut C_N can be interpreted as fractal symmetries of the structure of a linear chain. In cases, when the elementary cell of the chain $\tilde{N}$ consists of a single structural element, such a scaling can be interpreted as a condensation of κ elementary cells of the initial chain $\tilde{N}$ into a complex elementary cell of the new chain $\tilde{\bar{\kappa}}$. It is important to observe, however, that such a condensation, defined by the modular condition (11), exhibits global (non-local) character since it implies that a collection of κ initial structural elements distributed uniformly over $\tilde{N}$ (separated by κ lattice constants) concentrates into a single cell of the new chain $\tilde{\bar{\kappa}}$. Such kind of condensation differs therefore from an ordinary, local phase transition where the elementary cell of a chain is, *e.g.* doubled as the result of mutual alternate displacements (towards and apart) of neighbor structural units (dimerization [12-14], Peierls phase transitions [22-25].

In cases, when the elementary cell of the initial chain encloses two or more structural units, the scaling η_l distributes these units over different units of the new chain $\tilde{\bar{\kappa}}$ (*c.f.* an example in Section 5).

We thus conclude that, from the structural point of view, each scaling $\eta_l \in$ End C_N carries the initial polymer chain $\tilde{N}$ into a new chain $\tilde{\bar{\kappa}}$, where κ and $\bar{\kappa}$ are divisors of N, given by Eqs (9) and (10). New chains can differ mutually either by the length, *i.e.* the number $\bar{\kappa}$ of elementary Bravais cells, or by the cyclic order. It is therefore reasonable to divide the set End C_N of all scalings into subsets yielding the chains of the same length $\bar{\kappa}$. In this way the ring End C_N, or, equivalently, the set $\tilde{N}$, decomposes into disjoint subsets

$$\tilde{N} = \bigcup_{\kappa \in K(N)} \tilde{N}_\kappa \, , \tag{12}$$

given by

$$\widetilde{N}_\kappa = \{l \in \widetilde{N} \mid \mathrm{lcd}\,(l, N) = \kappa\} \tag{13}$$

and the set-theoretic sum in Eq. (12) runs over elements of the set

$$K(N) = \{\kappa \in \widetilde{N} \mid \mathrm{lcd}\,(N, \kappa) = \kappa\} \tag{14}$$

of all divisors of N. All scalings $\eta_l, l \in \widetilde{N}_\kappa$, yield chains with the same length $\bar{\kappa} = N/\kappa$, and differ mutually by a cyclic order.

The set $K(N)$, providing the classification (12) of scalings of the chain $\widetilde{N}$, constitues a mathematical structure called lattice ([15, 5]), *i.e.* a partially ordered set (by the relation of divisibility), with unique maximal ($\kappa_{max} = N$) and minimal ($\kappa_{min} = 1$) elements. According to an elementary number theory, Eq. (12) implies the sum rule

$$N = \sum_{\kappa \in K(N)} \varphi(\bar{\kappa})\,, \tag{15}$$

where $\varphi(\bar{\kappa})$ is the value of the Euler function, *i.e.* the number of elements of the class $\widetilde{N}_\kappa$ given by Eq. (13), or, equivalently, the number of all such elements of the new chain $\tilde{\bar{\kappa}}$, which are relatively prime with $\bar{\kappa}$. In general, the Euler function is given by the formula

$$\varphi(N) = \begin{cases} 1 & \text{for } N = 1, \\ N \prod\limits_{p \in \pi(N)} (p-1)/p & \text{for } N > 1, \end{cases} \tag{16}$$

where $\pi(N) \subset K(N)$ is the socle of N, *i.e.* the set of all prime divisors of N (*c.f.* [5] and references therein for detail).

The case of minimal divisor $\kappa = 1$ is of particular importance since then each scaling η_l yields a new chain with the same length N as the initial chain. All such scalings form the group

$$\mathrm{Aut}\ \mathrm{C}_N = \{\eta_l \in \mathrm{End}\ \mathrm{C}_N \mid \mathrm{lcd}\,(l, N) = 1\} \tag{17}$$

of the hidden symmetry of the structure of linear chain in the terminology of the Weyl's recipe. Scalings belonging to this group yield only a change of cyclic order of a polymer chain, but preserve its structure. All images $\eta_l(\widetilde{N})$, $\mathrm{lcd}\,(l, N) = 1$, are mutually similar configurations, so that they are finite analogies of fractals.

We proceed to evaluate the number of essentially different fractal variants of a given chain $\widetilde{N}$. It is easy to observe that the group $\mathrm{Aut}\ \mathrm{C}_N$ contains for $N > 2$ a subgroup

$$C_{1h} = \{\eta_1, \eta_{N-1}\} \triangleleft \text{Aut } C_N , \tag{18}$$

consisting of the unit element η_1 (the trivial scaling 1 : 1) and the scaling

$$\eta_{N-1} = \begin{pmatrix} j \\ (N-1)\, j \bmod N \end{pmatrix} = \begin{pmatrix} j \\ -j \bmod N \end{pmatrix}, \; j \in \tilde{N} , \tag{19}$$

corresponding to the one-dimensional inversion of the chain $\tilde{N}$ in the node $j = N$. The group C_{1h} has the evident meaning of the geometric point symmetry group of the one-dimensional chain. All non-trivial, *i.e.* non-geometric fractal symmetry operations are therefore given by the quotient group

$$Q = \text{Aut } C_N / C_{1h} . \tag{20}$$

Each non-trivial element of the group Q determines (with the accuracy up to the one-dimensional inversion η_{N-1}) a "fractal" cyclic order, essentially different from that imposed by the natural order in the set $\tilde{N}$. Thus the required number of fractal variants of the chain $\tilde{N}$ is given by the formula

$$|Q| = \begin{cases} 1 & \text{for } N = 1 \text{ or } 2, \\ \Phi(N)/2 & \text{for } N > 2. \end{cases} \tag{21}$$

Appropriate values for $N < 12$ are listed in Table I. A scaling η_l, $l \in \tilde{N}_\kappa$, $\kappa > 1$, is no longer fractal symmetry operation, but it imposes a breaking of translational symmetry of the distributing group C_N to its subgroup $C_{\bar{\kappa}}$. In particular, for $\kappa = N$ we have $\bar{\kappa} = 1$, *i.e.* the absolute destruction of the translational symmetry of the chain.

TABLE I

Values of the Euler function $\varphi(N)$ and of the number $|Q|$ of essentially different fractal variants of the chain $\tilde{N}$ for $N \leq 15$.

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\varphi(N)$	1	1	2	2	4	2	6	4	6	4	10	4	12	6	8
$\|Q\|$	1	1	1	1	2	2	3	2	3	2	5	2	6	3	4

4. Helical scaling in limited areas of space

A finite linear chain $\widetilde{N}$ can be looked at as the set of N points, distributed regularly over a circle. In this picture the set $\widetilde{N}$ is embedded into a covering space — a two-dimensional Euclidean space. The scaling operation η_l, $l \in \widetilde{N}$, corresponds in this picture to the angular magnification, so that an arbitrary point A on the circle, having the angle α with a fixed point $N \equiv 0$ with respect to the center of the circle (*c.f.* Fig. 1), is transformed to the point determined by the angle $l\alpha$ modulo 2π. A more transparent interpretation of such a scaling can be obtained by introducing the third dimension. The scaling η_l can be then treated as a transformation of the circle of the length Na into a regular closed helix of the length lNa, consisting of N convolutions, scrolled over a torus. The position of a point on helix is determined by the angle α, $0 \leq \alpha < 2\pi$, and by the number of the convolutions, *i.e.* the winding number. The final result of the operation η_l is obtained as the result of reduction of the helix to a single circle, by neglecting the winding number. This procedure will be referred hereafter to as the helix scaling.

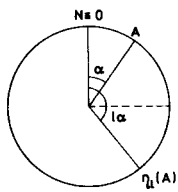

Fig. 1. Scaling on a circle (here $l = 3$).

It is easy to propose a possibility of physical realization of helix scaling. We assume that our chain is placed in a limited area of space, and that the volume of this area undergoes some fluctuations, *e.g.* as a result of complex motions of the surrounding matter. We also assume that the main structure of the linear chain is assured by strong short-range interactions, *e.g.* attraction between nearest structural units of the chain, and repulsing between next-nearest ones, whereas the interaction between third, fourth, etc. units has a tendency — in the presence of a fairly large free volume — to scroll the chain into a helix. Then in the case of presence of a volume large enough to form a torus the chain becomes a helix, and under lowering of the free volume this helix collapses to a new chain, with a new cyclic order. Such a kind of helix scaling provides a channel for virtual transitions between different equivalent fractal configurations, or for phase transitions,

associated with a global breaking of the translational symmetry of the chain — from the initial group C_N to one of its subgroups C_K, given by the lattice $K(N)$.

5. An example: the clock dial plate

We proceed now to demonstrate all possible fractal configurations, as well as symmetry-breaking helical scalings for the case $N = 12$, *i.e.* for a chain corresponding to the clock dial plate. All scalings η_l for this case, determined by means of Eq. (3), are collected in Table II. They all form the ring End C_{12}.

TABLE II

Scalings on the clock dial plate.

j / l	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	2	4	6	8	10	12
3	3	6	9	12	3	6	9	12	3	6	9	12
4	4	8	12	4	8	12	4	8	12	4	8	12
5	5	10	3	8	1	6	11	4	9	2	7	12
6	6	12	6	12	6	12	6	12	6	12	6	12
7	7	2	9	4	11	6	1	8	3	10	5	12
8	8	4	12	8	4	12	8	4	12	8	4	12
9	9	6	3	12	9	6	3	12	9	6	3	12
10	10	8	6	4	2	12	10	8	6	4	2	12
11	11	10	9	8	7	6	5	4	3	2	1	12
12	12	12	12	12	12	12	12	12	12	12	12	12

Fig. 2 presents the lattice $K(12)$ of divisors of the integer 12. We observe that the partial order, imposed by the divisibility relations (arrows in Fig.2) is not consistent with the linear order. The lattice $K(12)$ classifies scalings of the ring End C_{12} according to Eqs (12)-(13). This classification is given in Table III, together with kernels Ker η_κ and images Im η_κ for each class $\kappa \in K(12)$ of scalings (*c.f.* Eqs (5)-(10)).

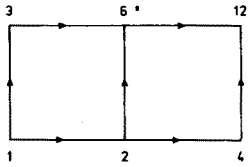

Fig. 2. The lattice $k(12)$ of divisors of the number 12. The minimal and maximal elements are respectively $\kappa_{min} = 1$ and $\kappa_{max} = 12$. Arrows indicate the partial order.

TABLE III

Classification of scalings of the clock dial plate.

κ	$\bar{\kappa}$	$\widetilde{N}_\kappa$	Ker $\eta_\kappa = C_\kappa = \langle\bar{\kappa}\rangle$	Im $\eta_\kappa = C_\kappa^- = \langle\kappa\rangle$
1	12	$\{1,5,7,11\} \equiv \{\pm1,\pm5\}$	$C_1 = \{12\}$	C_{12}
2	6	$\{2,10\} \equiv \{\pm2\}$	$C_2 = \{6,12\}$	C_6
3	4	$\{3,9\} \equiv \{\pm3\}$	$C_3 = \{4,8,12\}$	C_4
4	3	$\{4,8\} \equiv \{\pm4\}$	$C_4 = \{3,6,9,12\}$	C_3
6	2	$\{6\}$	$C_6 = \{2,4,6,8,10,12\}$	C_2
12	1	$\{12\}$	$C_{12} = \{1,2,\ldots,12\}$	C_1

In particular, the set $\widetilde{N}_\kappa$ for $N = 12$, $\kappa = 1$, labels the elements of the group

$$\text{Aut } C_{12} = \{\eta_1, \eta_5, \eta_7, \eta_{11}\} \tag{22}$$

of reversible scalings, *i.e.* the group of the hidden symmetry of the clock dial plate according to the recipe of Weyl. The group multiplication for Aut C_{12} is given in Table IV. We find that Aut C_{12} is an abelian group, isomorphic with the dihedral point group D_2.

TABLE IV

The multiplication table for the group Aut C_{12}.

η_1	η_5	η_7	η_{11}
η_5	η_1	η_{11}	η_7
η_7	η_{11}	η_1	η_5
η_{11}	η_7	η_5	η_1

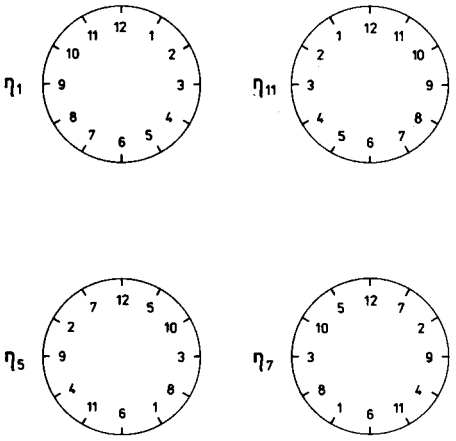

Fig. 3. The action of the group Aut C_{12} of the hidden symmetry on the clock dial plate.

The action of the group Aut C_{12} on the clock dial plate is given in Fig. 3 (*c.f.* also Table II). This figure shows us that η_1 is the unit scaling 1:1, and η_{11} corresponds to the one-dimensional inversion in the fixed point $j = 12$, which is equivalent to the reversal of cyclic order of the plate (*c.f.* Eq. (19)). These two scalings are trivial in a sense, since they constitue the group

$$C_{1h} = \{\eta_1, \eta_{11}\} \triangleleft \text{Aut } C_{12} \tag{23}$$

of operation of geometric symmetry of the chain (*c.f.* Eq. (18)). Thus the first row in the Fig. 3 presents the geometric symmetries of the dial plate.

The corresponding quotient group, *i.e.*

$$Q = \text{Aut } C_{12}/C_{1h} = \{\{\eta_1, \eta_{11}\}, \{\eta_5, \eta_7\}\} \equiv \{\{\eta_{\pm 1}\}, \{\eta_{\pm 5}\}\} \qquad (24)$$

consists of two elements, corresponding to two essentially different fractal configurations on the clock dial plate (the first and second row in Fig. 3). Transitions between these two configurations are realized either by the scaling η_5, *i.e.* 5:1, or by η_7, *i.e.* $7:1 \equiv (-5):1$. These scalings provide a non-trivial, fractal symmetry of the clock dial plate. We observe in Fig. 3 that these scalings change essentially the cyclic order, but preserve the structure of the dial plate.

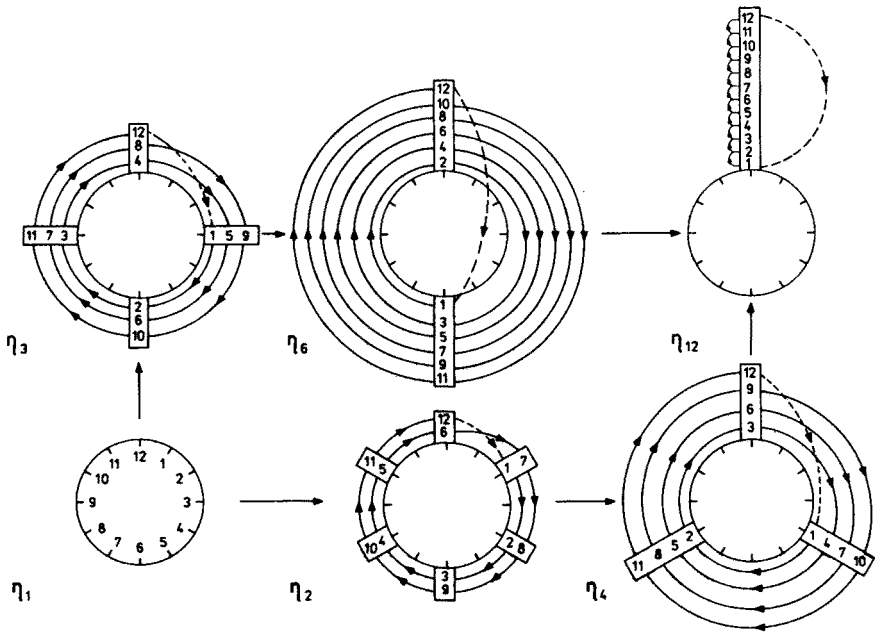

Fig. 4. Helical scaling on the clock dial plate for divisors of 12. Arrows on arc denote the projection of the orientation of the helix onto the initial circle. Direct arrows between clocks point out the partial order in the lattice K(12).

The remaining scalings of $\text{End}C_{12}$ (*c.f.* Table II) break the translational symmetry of the clock dial plate. The scalings $\eta_\kappa, \kappa \in K(12)$, *i.e.* scalings representing each class of Table III, are presented in Fig. 4, with the location (of each scaling) consistent with the structure of the lattice $K(12)$ (*c.f.*

Fig. 2). We can observe that each scaling η_κ, $\kappa \in K(12)$, $\kappa \neq 1$, yields a new chain $\bar{\bar{\kappa}}$, in which the clock figures $1, 2, \ldots, 12$, are no longer distributed uniformly over the dial plate, but they are grouped on $\bar{\kappa}$ places, each group having κ figures. In particular, for $\kappa = 12$ we obtain a total "chaos", *i.e.* the complete lack of any translational symmetry. We also observe that the transition from the full translational symmetry C_{12} of the clock dial plate to the full chaos C_1 can be performed by intermediate stages on different, inequivalent ways, defined by the structure of the lattice $K(12)$. The minimal element $\kappa_{min} = 1$ and maximal $\kappa_{max} = 12$ of this lattice correspond respectively to the maximal and minimal translational symmetry, whereas intermediate symmetries are described by subgroups of C_{12}.

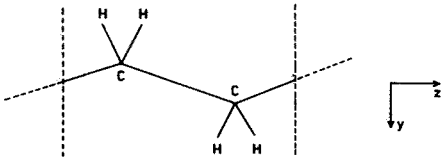

Fig. 5. The elementary cell of a polyethylene crystal. Structural units CH_2 lay in planes parallel to xy.

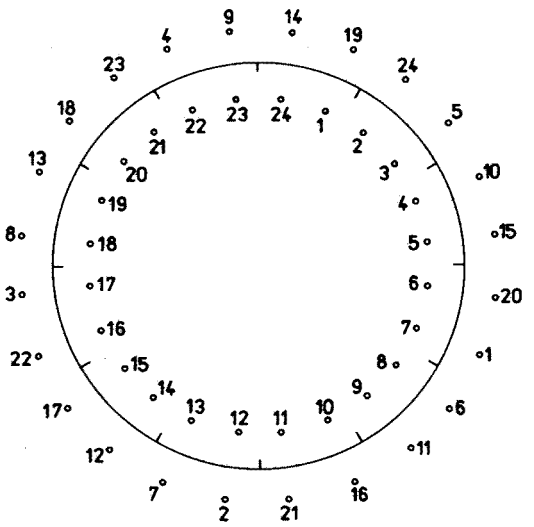

Fig. 6. The fractal scalings η_5 for the $(C_2H_4)_{12}$ polyethtlene chain.

Each figure of the clock represents an elementary cell of the chain. As an example of a chain with composite elementary Bravais chain, we consider

the polyethylene chain $(C_2H_4)_{12}$ [24]. Each cell of this chain consists of two structural units CH_2 (Fig. 5). The center of the cell in Fig. 5 coincides with the center of the carbon bond, whereas the hydrogen scissors are placed in the plane perpendicular to the figure and to the chain. The helix scaling η_l yields l-fold magnification along the axis of the chain (assumed to be the z-axis), whereas the perpendicular dimensions (x along), in particular the hydrogen scissors, remain unchanged. The effect of the helix scaling η_5 (fractal symmetry) is given in Fig. 6. The figures 1, 2, ..., 24 inside the dial plate denote the labels of consecutive structural units CH_2 in the initial cyclic order, and outside — those after scaling. This example clearly shows that the fractal symmetry provides — contrary to a purely geometric symmetry — an essential structural reconstruction of the chain. Let us, *e.g.* consider the structural units with labels 1 and 2, which constitute the first elementary cell of the initial polyethylene chain. After the helix scaling η_5 we find the unit 1 in the fourth elementary cell, together with the unit 6 from the third cell of the initial chain, whereas the unit 2 constitutes, together with the unit 21 from eleventh cell, the sixth cell of the new chain. We like to stress at this point that the picture outside the dial plate in Fig. 6 is exactly determined by the procedure of helix scaling η_5, despite an apparent impression of a chaotic mixture.

6. Final remarks and conclusions

We have considered the structural properties of a finite polymer linear chain [11-14,25], using the recipe of Weyl [15-19]. The role of the obvious symmetry of the chain is played by the cyclic group C_N, the distributing group for elementary cells of the chain. According to recipe of Weyl, the hidden symmetry is given by the group Aut C_N of all automorphisms of the group C_N, which is well known from elementary number theory [15]. We have proposed in the present paper a physical interpretation of these automorphisms as structural symmetries of a linear chain. This group is composed of geometric symmetries, combined with some new symmetry operations involving scalings. In particular, for the case of a helical scaling, we obtain some new, non-geometric symmetries which preserve the structure of the linear chain, but imply its strong reconstruction. It involves a global (*i.e.* non-local) exchange of structural units, followed by an essential change of the cyclic order. Invariance of structure of the chain under the scalings of the group Aut C_N allows us to interpret this chain as a finite analog of a fractal, *i.e.* a structure which is self-similar under such operations.

Fractal symmetries imposed by the group Aut C_N are candidates for the description of generalized symmetry of finite linear chains in an analogy to molecular symmetry groups for non-rigid, floppy molecules [8-11], since

they preserve the structure of the chain. Evidently, a practical realization of a particular fractal symmetry operations is associated with such dynamic parameters as the height of the potential barrier between two cyclic orders. One has to expect that within some appropriate energy scales only some subgroups of the group Aut C_N will be important, in accordance with the dynamics of the chain.

The purely fractal symmetry is — contrary to the geometric one — highly non-local, for reason of an essential change of the cyclic order, which yields a shuffling of structural units of the chain. As a result, a non-rigid linear chain cannot be dynamically treated as a solid body even approximately, but one has to take into account the "floppy" degrees of freedom, realizing admissible permutations of structural elements.

Not every scaling of a linear chain is reversible. We have shown that irreversible scalings lead to breaking of translational symmetry of the chain. All possible ways of breaking are classified by the lattice $K(N)$ of subgroups of the group C_N, *i.e.* of divisors of the number N of elementary cells of the chain.

It is worth to notice an important role of the dimension of the space in our finite analogons of fractals, which is quite different than in the case of the "fractal" dimension (Mandelbrot [7]). We observe that (*i*) a line crystal is, by definition, a finite or discrete subset of a one-dimensional space; (*ii*) a linear chain is treated as a subset of a circle, *i.e.* a figure associated with a two-dimensional space; (*iii*) fractal symmetries can be realized by helical scalings, *i.e.* transformations of a circle to a helix, which involve the third dimension for virtual intermediate stages in realization of the scaling. The one-dimensional linear chain "engages" therefore at least three dimensions for realization of a fractal symmetry.

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