CONDENSATION OF MATTER AND FORMATION OF TRAPPED SURFACES IN NONSPHERICAL CONVEX GEOMETRIES*

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We prove that the concentration of matter in a small volume is sufficient for the formation of trapped surfaces in a class of initial data for the Einstein equations. We consider momentarily static nonspherical initial data and formulate a condition for the existence of averaged trapped surfaces. The results are obtained in terms of the total rest mass M and the largest proper radius $\sup L$ of a smallest convex equipotential surface that encloses a nonspherical body.

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1. Introduction

There were several attempts to justify the folk belief that if a finite amount of matter is squeezed into a sufficiently small volume, then a black hole should form. This informal statement can be expressed in precise terms as the *trapped surface conjecture* (TSC) which states: "Any mass that is concentrated in a region of sufficiently small diameter can be surrounded by a trapped surface" [1]. The existence of a trapped surface would imply the existence of a black hole, as suggested by the famous Penrose–Hawking singularity theorems [2].

The TSC was recently precisely formulated and proved for particular simple geometries of initial data [3, 4]. It happens that in spherically symmetric case the appropriate measure of the degree of concentration of matter inside a two-sphere is the ratio M/L where M is the total rest mass and L is the proper radius of the sphere [3].

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The first step towards the generalization of the above results to non-spherical systems was done in [4]. For a momentarily static conformally flat ellipsoidal 3-geometry the total rest mass M and the proper (largest) radius $\sup L$ of a configuration remain good characteristics. If the ratio $M/\sup L$ exceeds certain critical value, then the existence of an averaged trapped surface (ATS; to be defined below) is inevitable. More information was required to make statements about point-wise trapped surfaces.

In this paper we continue the investigation of formation of ATS's in material systems with no gravitational waves. The absence of gravitational waves is understood in the sense that the geometry of space-time becomes flat if there is no matter in a system. There are two classes of interest:

(i) spherically symmetric geometries;
(ii) conformally flat geometries with time symmetry.

The first case has been already solved [3], hence we will restrict our investigation to the second case, of time-symmetric and conformally flat initial data of Einstein equations. Below we will show that if to pack enough matter in a small convex volume, then trapped surfaces will occur.

From the technical point of view, there are two new ideas in comparison to the previous research [3, 4]. The main trick is to employ a special foliation (Eq. (10)), to allow for the use of ordinary differential inequalities. This point is discussed in Section 3. The aforementioned convexity condition would be related to the global existence of the foliation. The second input is the Gauss–Bonnet theorem.

The notation of this paper is literally taken from [4]. Let Σ be a three-manifold with a Riemannian metric g_{ab}. Let S be a closed two-surface the embedding of which in Σ is described by the induced metric h_{ab} and the extrinsic curvature p_{ab}. The first derivative of the area of S with respect to the uniform normal deformation is equal to the total mean curvature of S, denoted here by $H(S)$:

$$H(S) = \int_S p dS\,, \tag{1}$$

where $p = p_{ab}h^{ab}$.

Let us consider Σ as the time-symmetric Cauchy surface for the Einstein equations. Then the sign of $H(S)$ says whether the area of the light front outgoing orthogonally from S is decreasing or increasing. A surface of non-positive $H(S)$ is called the *averaged trapped surface*, as distinct from the *trapped surface* for which the mean curvature p is non-positive pointwise. The metric g_{ab} must satisfy the Hamiltonian constraint

$${}^{(3)}R = 16\pi\rho\,, \tag{2}$$

where as usual we assume that the matter density ρ is non-negative. We assume also that the metric g_{ab} is in some sense asymptotically Euclidean. Following Lichnerowicz–York approach we introduce another asymptotically Euclidean metric $\widehat{g}_{ab}$ conformally related to g_{ab}

$$g_{ab} = f^4 \widehat{g}_{ab} . \tag{3}$$

Since from (2) we have ${}^{(3)}R \geq 0$, it is always possible to choose $\widehat{g}_{ab}$ so that the scalar curvature of $\widehat{g}_{ab}$ vanishes. Then the conformal factor f satisfies the Lichnerowicz equation

$$\widehat{\Delta} f = -2\pi \rho f^5 \tag{4}$$

with the boundary condition $f = 1$ at infinity, where $\widehat{\Delta}$ is the laplacian with respect to $\widehat{g}_{ab}$ and we have replaced ${}^{(3)}R$ by ρ using (2). Let us multiply Eq. (4) by f and integrate over the volume V enclosed by the closed two-surface S. Using Stokes' theorem we obtain

$$\int_V f \widehat{n}^a \widehat{\nabla}_a f d\widehat{S} = \int_S (\widehat{\nabla}_a f)^2 d\widehat{V} - 2\pi M(S) , \tag{5}$$

where $d\widehat{S} = f^{-4} dS$ and $d\widehat{V} = f^{-6} dV$ are surface and volume elements with respect to $\widehat{g}_{ab}$; $\widehat{n}$ is a unit normal (in $\widehat{g}_{ab}$) to S. By $M(S)$ we have denoted the rest mass enclosed by S

$$M(S) = \int_V \rho dV . \tag{6}$$

It is easy to check that

$$p = f^{-2} (\widehat{p} + 4 f^{-1} \widehat{n}^a \widehat{\nabla}_a f) , \tag{7}$$

where $\widehat{p}$ denotes the mean curvature of S as embedded in $(\Sigma, \widehat{g}_{ab})$. Putting this into the l.h.s. of (5) and dividing by 2π we finally obtain

$$1/(8\pi) H(S) = D(S) - M(S) , \tag{8}$$

where

$$2\pi D(S) \equiv \int_V (\widehat{\nabla}_a f)^2 d\widehat{V} + \tfrac{1}{4} \int_S f^2 \widehat{p} d\widehat{S} . \tag{9}$$

Identity (8), which is the central point of our analysis, expresses the total mean curvature of a closed two-surface S in terms of the total mass inside

S and a quantity $D(S)$, the interpretation of which is the main difficulty. In spherically symmetric geometry a quantity $D(S)$ can be easily related to the proper radius $L(S)$ of the two-sphere S [3]. In [4] $D(S)$ has been interpreted in terms of the "size" for a class of conformally flat geometries. In what follows I will deal with a more general geometry.

2. Main results

Suppose that the metric g_{ab} is conformally flat and the level surfaces of the conformal factor f (i.e. the equipotential surfaces of the gravitational field) are homeomorphic to a sphere. The line element reads

$$ds^2 = f^4(\sigma)\left[\widehat{g}_{\sigma\sigma}d\sigma^2 + \widehat{g}_{\tau\tau}d\tau^2 + 2\widehat{g}_{\tau\phi}d\tau d\phi + \widehat{g}_{\phi\phi}d\phi^2\right] , \qquad (10)$$

where $\sigma \geq 0$, σ foliates levels of the conformal factor f, $\widehat{g}_{ab}$ is a flat metric and the quasi-angle variables τ and ϕ are of finite range. The metric $\widehat{g}_{ab}$ is supposed to be flat to guarantee the absence of gravitational waves; in the case without matter fields, Eq. (4) possesses only the flat solution $f = 1$, while for nonflat $\widehat{g}_{ab}$ it would admit nontrivial solutions f.

Now let $S(\sigma) \equiv \{x: \sigma(x) = \text{const.}\}$. We will need an inequality that bounds from below the largest geodesic radius of the surface S. We state it in the form of a conjecture.

Conjecture 1

$$\sup L(S) \equiv \sup_{(\tau,\phi)} \int_0^{t(\sigma,\phi,\tau)} ds\ f^2(s)\left(\widehat{g}_{ik}u^iu^k\right)^{\frac{1}{2}} \geq \int_0^{\sigma} ds\ f^2(s) \sup\left(\widehat{g}_{\sigma\sigma}\right)^{\frac{1}{2}} , \quad (11)$$

where $u^i \equiv \partial x^i/\partial t$ is tangent to a geodesic whose (flat) length is equal to t, $\{x^i\} \equiv (\sigma, \tau, \phi)$. At present I am not able to prove the estimation (11), but it is necessary to point out that (11) holds in a few specific nonspherical geometries [4]. Moreover, it is easy to show that the inequality (11) follows from another conjecture which refers to a property of convex foliations in the flat space.

Conjecture 2

Define a set $\Gamma_e = \bigcup_\sigma [x \in S(\sigma): n^i\partial_i f(x) = \min]$ (in which $g_{\sigma\sigma}$ achieves its maximal values for each fixed value of the coordinate σ). Let surfaces $S(\sigma)$ be convex and f be a solution of Eq. (5) with a nonnegative matter density ρ. Then Γ_e contains a geodesic that starts from the set $S(0)$.

The validity of the above statements is under investigation. In (11) the equality may be attained in cases when this particular level surface of f which corresponds to $\sigma = 0$, $S(0)$, is a single point. (This is the case of spherical symmetry, for instance.)

Lemma. Assume the geometry (10), with level surfaces of f being convex (in the flat metric $\widehat{g}$) and homeomorphic to a sphere. Assume also that $S(0)$ consists of a single point. Let S be a level surface of the conformal factor. If all levels of the conformal factor f inside S are not ATS then

$$\sup L(\sigma) \geq D(\sigma)\,; \tag{12}$$

$D(\sigma)$ is defined by (9).

Proof

At $\sigma = 0$, we need $\sup L(\sigma = 0) \geq D(\sigma = 0)$; in the case when $S(\sigma = 0)$ is a single point, it is trivial since then both sides vanish. Now, assuming Conjecture 1, it is sufficient to prove that

$$\partial_\sigma \int_0^\sigma ds\; f^2(s) \sup\left(\widehat{g}_{\sigma\sigma}\right)^{\frac{1}{2}} \geq \partial_\sigma D(\sigma)\,. \tag{13}$$

Since the left hand side of (13) is equal to $f^2(\sigma)\sup(\widehat{g}_{\sigma\sigma})^{\frac{1}{2}}$ we have to show the inequality

$$f^2(\sigma)\sup(\widehat{g}_{\sigma\sigma})^{\frac{1}{2}} \geq \partial_\sigma D(\sigma)$$

$$\equiv \frac{1}{2\pi}\int\int d\tau d\phi \sqrt{\widehat{g}}\widehat{g}^{\sigma\sigma}\left(\partial_\sigma f\right)^2 + \frac{1}{8\pi}\int\int d\tau d\phi \partial_\sigma\left(f^2 \widehat{p}(\widehat{g}\widehat{g}^{\sigma\sigma})^{\frac{1}{2}}\right)$$

$$= \frac{1}{2\pi}\int\int d\tau d\phi 2(\partial_\sigma f)(\widehat{g}^{\sigma\sigma})^{\frac{1}{2}}\left((\widehat{g}\widehat{g}^{\sigma\sigma})^{\frac{1}{2}}\partial_\sigma f + f\partial_\sigma(\widehat{g}\widehat{g}^{\sigma\sigma})^{\frac{1}{2}}/4\right)$$

$$-\frac{1}{2\pi}\int\int d\tau d\phi (\widehat{g})^{\frac{1}{2}}\widehat{g}^{\sigma\sigma}\left(\partial_\sigma f\right)^2 + C(\sigma) \leq H(\sigma)\partial_\sigma f/(4\pi f) + C(\sigma)\,; \tag{14}$$

here $H(\sigma)$ is defined in formula (1) and $C(\sigma)$ reads

$$C(\sigma) = \frac{1}{8\pi}f^2(\sigma)\int\int d\tau d\phi \partial_\sigma\left((\widehat{g}^{\sigma\sigma})^{\frac{1}{2}}\partial_\sigma(\widehat{g}\widehat{g}^{\sigma\sigma})^{\frac{1}{2}}\right)\,. \tag{15}$$

The right hand side of (14) is smaller or equal to $C(\sigma)$ since by assumption $H(\sigma$ is positive inside S (this is equivalent to saying that ATS's are absent inside S) while $\partial_\sigma f \leq 0$ (this is because of the definition of σ and also of the fact that f satisfies an elliptic equation (4) with a positive matter density ρ). A lengthy but straightforward calculation and the Gauss–Bonnet theorem allow us to get the desired estimation

$$C(\sigma) \leq f^2(\sigma) \sup(\widehat{g}_{\sigma\sigma})^{\frac{1}{2}} . \tag{16}$$

In that place it is necessary to make use of the fact that level surfaces of f are convex, *i.e.* their Gauss curvature is nonnegative. A more detailed calculation is presented in the Appendix.

The above lemma and the identity (8) allow us to conclude that if inside a surface S (which is a level surface of the conformal factor f) ATS's are absent, then $\sup L(S) > M(S)$. By contradiction, if at S the inequality

$$\sup L(S) \leq M(S) \tag{17}$$

holds, then inside S do exist ATS's. Thus, the following is true.

Theorem 1. Assume the conditions of Lemma. Then if the total rest mass $M(S)$ is greater than the largest proper radius $\sup L(S)$, there must exist ATS inside a level surface S.

This result becomes more transparent in case of geometries generated by compact bodies whose outer boundary is a level surface of the conformal factor. Then the above theorem says that in conformally flat convex geometries (10), (*i.e.* with nonnegative Gauss curvature of levels of the conformal factor), the inequality (17) at the outer boundary of a compact body is sufficient to guarantee the existence of ATS's inside the body.

The sufficient condition is an analogue of the result proved in [3], with the important exception that in spherical geometries all results are formulated in terms of pointwise trapped surfaces. Now, similarly as in [4] we cannot conclude that if $M > \sup L(\sigma)$ then a black hole has to develop; the existence of an ATS is not sufficient to make use of the Hawking–Penrose singularity theorems [2]. In [4] were formulated certain additional conditions which guaranteed that a nonspherical ATS is a trapped surface. A similar analysis would be done also for the geometry (10), with the expected result that if the ratio of the total rest mass to the largest proper radius is large enough and the ATS is not too nonspherical, then it must be pointwise trapped. However, let us remark that even if an ATS is not a trapped surface, then still the geometry could contain trapped surfaces of a different shape. The point is that in our case the shape of an apparent horizon is not necessarily compatible with the shape of the level surface of the conformal factor.

3. Generalizations

Let us comment on some of the assumptions in the above investigation. I conjecture that the metric (10) is generic in the class of conformally flat metrices. One meets the following problem: is it possible to find the background foliation (σ, τ, ϕ) to transform (Eq. (4)) into an ordinary differential equation? That the answer is affirmative, follows from the Mini–Max principle, since the matter density ρ was assumed to be nonnegative. Thus the analysis leading to the inequality (14) can be done, at least locally. More restrictive is the condition that the Gauss curvature of level surfaces of f is nonnegative (*i.e.* they are convex). Intuitively, this seems to imply even the global existence of foliation (10) and I do not expect that generic flat geometries would satisfy it.

A new difficulty, which appears now, is that in the generic case the set $S(0)$ (*i.e.* the set of which the conformal factor f achieves its maximal values) is not a single point. If one assumes the convexity condition, then $S(0)$ would be a flat deformed disc or a rod with a convex boundary. The inspection of the proof of the Lemma shows that we need a bound

$$\lim_{\sigma\to 0}\frac{1}{8\pi}\int\int_{S(\sigma)} d\widehat{S}\widehat{p} \equiv \lim_{\sigma\to 0}\frac{1}{8\pi}\int\int_{S(\sigma)} d\tau d\phi(\widehat{g}^{\sigma\sigma})^{\frac{1}{2}}\partial_\sigma(\widehat{g}\widehat{g}^{\sigma\sigma})^{\frac{1}{2}}$$

$$\leq \lambda \sup l(S(0)), \tag{18}$$

where $\sup l(S)$ is the largest flat radius of the volume enclosed by S and λ is a constant. The inequality (18) is given in [5] with the value of a constant equal to $\lambda = \pi/4$.

The next step of the proof of Lemma requires, however, a modification. As demonstrated in an explicit example [4] the rate of growth of $\sup L(\sigma)$ can be smaller than the rate of growth of the geodesic contained in Γ_ϵ (see the definition below the formula (11)), *i.e.* we cannot claim any longer that

$$\partial_\sigma \sup L(\sigma) \geq f^2(\sigma)\sup(\widehat{g}_{\sigma\sigma})^{\frac{1}{2}}, \tag{19}$$

if $S(0)$ is not a single point. An obvious modification is to prove the inequality

$$\sup L(\sigma) + \pi f^2(0)\sup l(0)/4 \equiv \sup L(\sigma) + \pi \sup L(0)/4 \geq D(\sigma) \tag{20}$$

instead of (12); using (11) we reduce our task to the already proven inequality (14). As the conclusion, we may infer the following:

Theorem 2. Assume a conformally flat and convex geometry (10). Then if

$$\sup L(S) + \pi \sup L(0)/4 \leq M(S), \tag{21}$$

there must exist ATS inside a level surface S. Theorem 2 may be formulated in a more elegant way noticing that $\sup L(S) \geq \sup L(0)$:

Theorem 2'. Under conditions as in Theorem 2, if

$$M(S) \geq (1 + \pi/4) \sup L(S), \tag{22}$$

there is ATS inside S.

The primary intention of this work is to prove that the concentration of matter alone is sufficient to form ATS (and perhaps TS); the last theorem is entirely satisfactory from this point of view. Nevertheless, I do not regard it sufficient for reasons which are explained below. First, a specific case of nonspherical geometries elaborated in [4] gave precisely the same criterion as in Theorem 1, although it was a situation with $S(0)$ being a disc. This gives incentive to conjectured that the statement of Theorem 1 should be true also in general geometries. Next, more serious reservation is that there would exist an upper limit on the degree of condensation of matter. Arnowitt, Deser and Misner [6] conjecture that the largest value of the ratio $M(S)/\sup L(S)$ cannot exceed 2. This was proven in [7] for spherically symmetric geometries and in [4] for a class of nonspherical geometries. The number $1 + \pi/4$ is smaller than 2, so that certainly the content of Theorem 2' is not empty, but my feeling is that the coefficients need improvement.

4. Summary

This paper gives a criterion for the formation of trapped surfaces for a large class of nonspherical geometries whose metrics are conformal to convex and flat matrics. It is given in terms of the total rest mass and the largest proper radius of a body. The set of time-symmetry and conformally flat geometries is likely to be the largest one in which the notion of the total rest mass remains a good characteristic. In general geometries the result analogous to the presented above would presumably use a different quasilocal measure of the gravitational energy.

Let us point out that a different criterion for trapped surfaces has been proven by Schoen and Yau [8]. Their analysis does not use any symmetry assumptions but it is not quite satisfying [3] because of a nonstandard definition of the measure of the size of bodies.

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Appendix

We shall show that

$$C(\sigma) \leq \alpha\, f^2(\sigma)\,, \tag{A1}$$

where $C(\sigma)$ is defined in formula (15) and $\alpha = \alpha(\sigma) = \sup(\widehat{g}_{\sigma\sigma}(\sigma,\tau,\phi))^{\frac{1}{2}}$. In fact, since the background geometry is flat and $R_{\sigma\sigma} = 0$, the integrand of the expression (15) can be written as follows, after a lengthy but simple calculation

$$\partial_\sigma \left((\widehat{g}^{\sigma\sigma})^{\frac{1}{2}} \partial_\sigma (\widehat{g}\widehat{g}^{\sigma\sigma})^{\frac{1}{2}} \right) = 2(\widehat{g})^{-\frac{1}{2}} \left(\widehat{\Gamma}^\sigma_{\tau\tau} \widehat{\Gamma}^\sigma_{\phi\phi} - (\widehat{\Gamma}^\sigma_{\tau\phi})^2 \right)$$

$$+\partial_\tau \left((\widehat{g})^{\frac{1}{2}} \widehat{g}^{\sigma\sigma} \widehat{\Gamma}^\tau_{\sigma\sigma} \right) + \partial_\phi \left((\widehat{g})^{\frac{1}{2}} \widehat{g}^{\sigma\sigma} \widehat{\Gamma}^\phi_{\sigma\sigma} \right)\,, \tag{A2}$$

where $\widehat{\Gamma}$'s denote the Christoffel's symbols. The divergence term in (A2) does not contribute to $C(\sigma)$, so we are left with the integral of the first term in (A1), which is just the Gauss curvature K multiplied by $2\ (\widehat{g})^{\frac{1}{2}}$. Thus the integral of the right hand side of (A2) is equal to

$$\int\!\!\int d\tau d\phi (\widehat{g}^{\sigma\sigma}\widehat{g})^{\frac{1}{2}} K 2 (\widehat{g}_{\sigma\sigma})^{\frac{1}{2}}$$

which in turn can be estimated from above by

$$2\alpha \int\!\!\int d\tau d\phi (\widehat{g}\widehat{g}^{\sigma\sigma})^{\frac{1}{2}} K = 8\Pi\alpha$$

if K is nonnegative and $\widehat{g}_{\sigma\sigma}(\sigma) \leq \alpha$, as assumed in Lemma in the main text. The last equality is the content of the Gauss–Bonnet theorem for surfaces homeomorphic to a sphere.

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