NOTES ON THE EINSTEIN EQUATIONS FOR LOCALLY HERMITE–EINSTEIN SPACES

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It is shown that for every one-sided type-D Euclidean $\mathcal{HH}$-space with $\Lambda \neq 0$ for which the local fundamental 2-form Φ is nowhere vanishing, the Einstein equations can be locally reduced to a single, second-order nonlinear partial differential equation for one real function of three real variables.

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1. Introduction

Recently there has been a large amount of interest in "non-Lorentzian relativities" ($\equiv$ "non-hyperbolic relativities" [1–4]). In particular "Euclidean relativity" (ER) plays an important role in the path-integral formulation of quantum gravity [5–8]. "Complex relativity" (CR) appears in the natural manner in the twistor program of Penrose [9–12] and in the $\mathcal{H}$-space theory of Newman [13,14] and Plebański [15]. Then CR has been intensively explored by Plebański and co-workers [2], [15–31], [58–63]. One of the fundamental results in this subject has been found by Plebański and Robinson [20] and it states that for every vacuum complex space-time of an algebraically degenerate self-dual or/and anti-self-dual part of the Weyl tensor ($\equiv$ $\mathcal{HH}$-space), Einstein's vacuum equations can be locally, almost everywhere reduced to a single, second-order nonlinear partial differential equation for one holomorphic function. Analogous statement has been proved to hold true in the case of any $\mathcal{HH}$-space with Λ [21]. One can expect that a better insight into the structure of $\mathcal{HH}$-space with Λ and also into more general complex space-times enables us to obtain new results in the "Lorentzian relativity" ($\equiv$ the "hyperbolic relativity" or HR). This is the Plebański program (see Refs [32–34]).

It is well known that CR plays a distinguished role in the generation technics (compare the "complex coordinate transformations" of Newman [35–37]).

Very recently CR and ER have found their important applications in Ashtekar's Hamiltonian formulation of general relativity [38–44]. As it has been shown by Capovilla, Jacobson and Dell [44], the Ashtekar's Hamiltonian formalism is closely related to a separation of complex Einsteinian substructures given by Plebański in his pioneer work published in 1977 [30]. Also very recently an unexpected relation has been found between some result in CR or ER [22,45–48] and $sl(\infty)$-Toda equations or $(2+1)$-dimensional Einstein–Weyl geometry [49,50].

Main results of the $\mathcal{HH}$-space theory can be easily specialized to be true in the "ultrahyperbolic relativity" (UR), *i.e.*, for the case of a real metric of the signature $(++--)$. To this end one deals with real coordinates and real functions instead of complex coordinates and holomorphic functions. But an analogous transition from CR to ER is not so automatic and the natural question arises whether for every Euclidean $\mathcal{HH}$-space with Λ one can locally reduce ten Einstein equations to a single, second-order nonlinear partial differential equation for one real function.

First it has been shown that every Euclidean $\mathcal{HH}$-space with Λ appears to be a locally Hermite–Einstein space [51,52,58]. (Notice that in the case of Euclidean $\mathcal{H}$-space with $\Lambda \neq 0$ we are able to prove this fact under the assumption that the space is of the class C^ω, *i.e.* real analytic, rather than C^∞. Nevertheless we expect that the proof can be given for the space of class C^∞.)

Conversely, given a four-dimensional locally Hermite–Einstein space with a metric ds^2, and a point p of this space, there exist a neighbourhood U of p and a complex structure J on U such that (U, ds^2, J) is a *Hermite–Einstein space* and then, the anti-self-dual part (with respect to the natural orientation determined by J [53]) of the Weyl tensor at p is algebraically degenerate. We consider two distinct cases:

A. There exists a neighbourhood $U' \subset U$ of p such that (U', ds^2, J) appears to be *Kähler–Einstein space.*

B. There exists a neighbourhood $V \subset U$ of p such that the exterior derivative of the fundamental 2-form on V is nowhere vanishing. (The fundamental 2-form Φ on V is defined by [51,53]

$$\Phi(X,Y) = ds^2(X, JY) \tag{1.1}$$

for any vector fields X, Y on V). Then, for the case A, the anti-self-dual part of the Weyl tensor appears to be of the following types on U':

(A1) [-] if $\Lambda = 0$ (Euclidean $\mathcal{H}$-space, the hyper-Kähler space [40]),

or

(A2) D if $\Lambda \neq 0$.

For the case B the anti-self-dual part of the Weyl tensor takes the following form on V:

(B1) It is of the type D if $\Lambda = 0$.

or

(B2) If $\Lambda \neq 0$ then on some open set $V' \subset V$ it is of the type D and on the set $V - V'$ it is of the type [-]. As it has been shown the Einstein equations for the cases (A1), (A2), (B1) and (B2) with $V' = \phi$ can be locally reduced to a single, second-order nonlinear partial differential equation for one real function [51,54–57]. The aim of this note is to execute the similar reduction for the case (B2) with $V' = V$ (Sect. 2). Thus one arrives at the conclusion that for *every non-Lorentzian $\mathcal{HH}$-space with Λ the Einstein equations can be locally, almost everywhere reduced to a single second-order nonlinear partial differential equation for one real (for ER or UR) or complex (for CR) function.*

2. The reduction of the Einstein equations

Let M be a 4-dimensional differentiable manifold endowed with a positive definite metric ds^2. Then (M, ds^2) is said to be a locally Hermite–Einstein space if (M, ds^2) is an Einstein space and for each point $p \in M$ there exist a neighbourhood U of p and a complex structure J on U such that (U, ds^2, J) is a Hermitian space [51]. Consequently, (M, ds^2) is a locally Hermite–Einstein space iff for each point $p \in M$ there exist a neighbourhood U_1 of p and complex coordinates z^1, z^2 on U_1 such that

$$ds^2 = g_{\alpha\bar{\beta}} \left(dz^\alpha \otimes dz^{\bar{\beta}} + dz^{\bar{\beta}} \otimes dz^\alpha \right) \text{ on } U_1 ,$$

$$\alpha, \beta = 1, 2 ; \; dz^{\bar{\beta}} := \overline{dz^\beta} ; \; g_{\alpha\bar{\beta}} = \overline{g_{\beta\bar{\alpha}}} , \tag{2.1}$$

where the bar stands for the complex conjugation, and

$$R_{\alpha\beta} = 0 = R_{\bar{\alpha}\bar{\beta}} , \; R_{\alpha\bar{\beta}} = -\Lambda g_{\alpha\bar{\beta}} \text{ on } U_1 , \tag{2.2}$$

where Λ is assumed to be constant on M, and $R_{\alpha\beta} dz^\alpha \otimes dz^\beta + R_{\bar{\alpha}\bar{\beta}} dz^{\bar{\alpha}} \otimes dz^{\bar{\beta}} + R_{\alpha\bar{\beta}} \left(dz^\alpha \otimes dz^{\bar{\beta}} + dz^{\bar{\beta}} \otimes dz^\alpha \right)$ is the Ricci tensor field on U_1. Assume that there exists a neighbourhood $V_1 \subset U_1$ of p such that for each point $q \in V_1$

$$d \left(g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}} \right) \neq 0 \text{ at } q . \tag{2.3}$$

Thus we deal with the case B (see the Introduction).

Now from the Einstein equations (2.2) with (2.3) one infers [56] that there exist a neighbourhood $V \subset V_1$ of p, complex coordinates z^1, z^2 on V and a function $F = F(z^\alpha, z^{\bar\alpha})$ such that

$$g_{1\bar 1} = F_{,\bar 1}\ , \ \ g_{1\bar 2} = F_{,\bar 2}\ , \ \ g_{2\bar 1} = \bar F_{,2} \tag{2.4}$$

and

$$(\ln H)_{,1} = \left(\frac{1}{2}\dot C^{(3)} + \frac{\Lambda}{3}\right) F\,, \tag{2.5a}$$

$$(\ln H)_{,\alpha\bar\beta} - \frac{2}{H}\delta^2_\alpha \delta^{\bar 2}_{\bar\beta} = -\left(\dot C^{(3)} - \frac{\Lambda}{3}\right) g_{\alpha\bar\beta}\,, \tag{2.5b}$$

$$g_{2[\bar 2,\bar 1]} = \frac{g}{F}\,, \tag{2.5c}$$

where

$$H := \frac{F\bar F}{g}\,, \tag{2.6}$$

$g := \det \|g_{\alpha\bar\beta}\|$; $\dot C^{(3)}$ is the only identically nonvanishing component of the anti-self-dual part (with respect to the natural orientation of V defined by the 4-form $dz^1 \wedge dz^{\bar 1} \wedge dz^2 \wedge dz^{\bar 2}$) of the Weyl tensor; δ^2_α and $\delta^{\bar 2}_{\bar\beta}$ are the Kronecker deltas; the coma ", " denotes the partial derivative, and the square bracket [...] stands for the antisymmetrization.

We start with an unexpected and important result. Namely, we prove that Eq. (2.5c) appears to be a consequence of (2.4), (2.5a) and (2.5b) with H defined by (2.6). To this end one finds that from (2.5a) and (2.5b) for $\alpha = 1$, with (2.4), it follows

$$F^2 (\ln H)_{,1} - \frac{\Lambda}{3} F^3 = f(z^\alpha)\,, \tag{2.7}$$

where $f = f(z^\alpha)$ is some holomorphic function. Then from (2.5a) and (2.7) one gets

$$\dot C^{(3)} = \frac{2f}{F^3}\,. \tag{2.8}$$

Substitute now (2.8) into (2.5a) and (2.5b) for $\alpha = 2$, $\bar\beta = \bar 2$. Differentiating the first equation (*i.e.*, (2.5a)) with respect to z^2 and $z^{\bar 2}$ and the second one (*i.e.*, (2.5b) for $\alpha = 2$, $\bar\beta = \bar 2$) with respect to z^1, and then subtracting the results we obtain

$$\frac{2g}{F\bar{F}}\left(\frac{f}{F^2}+\frac{\Lambda}{3}F\right) = -\left(\frac{2f}{F^3}-\frac{\Lambda}{3}\right)(g_{2\bar{2},1}-g_{1\bar{2},2})$$
$$+\frac{2}{F^3}\left(3f\frac{F_{,1}}{F}-f_{,1}\right)g_{2\bar{2}}-\frac{2}{F^3}\left(3f\frac{F_{,2}}{F}-f_{,2}\right)g_{1\bar{2}}. \tag{2.9}$$

As $g_{2\bar{2}}$ is a real positive function from (2.5b) for $\alpha = 2$ and $\bar{\beta} = \bar{2}$ it follows that $\dot{C}^{(3)}$ is a real function *i.e.*,

$$\frac{f}{F^3} = \frac{\bar{f}}{\bar{F}^3}. \tag{2.10}$$

Hence

$$\frac{2}{F^3}\left(3f\frac{F_{,\alpha}}{F}-f_{,\alpha}\right) = \frac{6f}{\bar{F}F^3}\bar{F}_{,\alpha}. \tag{2.11}$$

Then from (2.9), (2.11) and (2.4) (remember that $g_{1\bar{1}}$ is also a real positive function, and consequently $F_{,\bar{1}} = \bar{F}_{,1} > 0$) one gets

$$g_{[2|\bar{2}|,1]} = \frac{g}{\bar{F}}. \tag{2.12}$$

Finally, the complex conjugation of (2.12) yields (2.5c). Thus we have proved that Eq. (2.5c) is a consequence of (2.4), (2.5a) and (2.5b) with H defined by (2.6).

It is also evident that Eqs (2.5b) for $\alpha = 1$ and $\bar{\beta} = \bar{1}, \bar{2}$ or $\alpha = 2$ and $\bar{\beta} = \bar{1}$, with (2.8) assumed, follow from (2.4) and (2.5a).

Consider now the case:

$$\dot{C}^{(3)} \neq 0 \text{ on } V. \tag{2.13}$$

Then without any loss of generality, for sufficiently small V, one can put (compare with Ref. [56])

$$f = 1 \tag{2.14}$$

and

$$F = \bar{F}. \tag{2.15}$$

As $F_{,\bar{1}} = \bar{F}_{,1} = g_{1\bar{1}}$ the relation (2.15) yields $F_{,1} = F_{,\bar{1}}$ and, consequently, the function F is of the following form

$$F = F\left(z^1 + z^{\bar{1}}, z^2, z^{\bar{2}}\right) = \bar{F}. \tag{2.16}$$

With (2.14) and (2.15) fixed, only the following coordinate transformations

$$(z^1, z^2) \mapsto (z^1 + m(z^2), n(z^2)) \,, \quad n_{,2} \neq 0 \,, \tag{2.17}$$

are admitted, where $m = m(z^2)$ and $n = n(z^2)$ are any holomorphic functions of z^2.

Define

$$K = K\left(z^1 + z^{\bar{1}}, z^2, z^{\bar{2}}\right) := \ln H \,. \tag{2.18}$$

From Eqs (2.5b) with (2.6), (2.8), (2.14), (2.15) and (2.18) one infers that

$$-\frac{2}{F^2} + \frac{\Lambda}{3}F = \varepsilon e^{\frac{1}{2}K}\left[K_{,11}\left(K_{,2\bar{2}} - 2e^{-K}\right) - K_{,12}K_{,1\bar{2}}\right]^{\frac{1}{2}} \,, \quad \varepsilon = \pm 1 \,. \tag{2.19}$$

Consequently, from (2.7), (2.14), (2.18) and (2.19) we get

$$\Lambda F = 2K_{,1} + \varepsilon e^{\frac{1}{2}K}\left[K_{,11}\left(K_{,2\bar{2}} - 2e^{-K}\right) - K_{,12}K_{,1\bar{2}}\right]^{\frac{1}{2}} \,. \tag{2.20}$$

If $\Lambda = 0$ (the case (B1)) then (2.20) leads to the well known equation

$$K_{,11}K_{,2\bar{2}} - K_{,12}K_{,1\bar{2}} - 2e^{-K}\left[K_{,11} + (K_{,1})^2\right] = 0 \,. \tag{2.21}$$

In this case (2.19) and (2.20) give

$$\varepsilon = -1 \,, \quad K_{,1} = \frac{1}{F^2} > 0 \,. \tag{2.22}$$

(For further details see Ref. [56]).

Assume now that $\Lambda \neq 0$. Thus we deal with the case (B2) (see the Introduction) for $V' = V$. Then one can find F from (2.20), and substituting the result into (2.19) one gets the final equation for K

$$(2K_{,1} + \varepsilon L)^2 (K_{,1} - \varepsilon L) - 3\Lambda^2 = 0 \,,$$

$$L := e^{\frac{1}{2}K}\left[K_{,11}\left(K_{,2\bar{2}} - 2e^{-K}\right) - K_{,12}K_{,1\bar{2}}\right]^{\frac{1}{2}} \,. \tag{2.23}$$

From (2.5b), (2.8), (2.14), (2.18)–(2.20) one finds the metric (2.1) in terms of K. Namely, one has

$$
\begin{aligned}
g_{1\bar{1}} &= \frac{1}{\Lambda} K_{,11}\left(1+\varepsilon\frac{2K_{,1}}{L}\right), \\
g_{1\bar{2}} &= \frac{1}{\Lambda} K_{,1\bar{2}}\left(1+\varepsilon\frac{2K_{,1}}{L}\right), \\
g_{2\bar{1}} &= \overline{g_{1\bar{2}}} = \frac{1}{\Lambda} K_{,12}\left(1+\varepsilon\frac{2K_{,1}}{L}\right), \\
g_{2\bar{2}} &= \frac{1}{\Lambda}\left(K_{,2\bar{2}} - 2e^{-K}\right)\left(1+\varepsilon\frac{2K_{,1}}{L}\right).
\end{aligned}
\tag{2.24}
$$

From (2.18) and (2.24) it follows that

$$
i\left(\frac{\partial}{\partial z^1} - \frac{\partial}{\partial z^{\bar{1}}}\right) \tag{2.25}
$$

is the Killing vector field on V.

Evidently one can obtain the similar results in CR or UR. To this end one should merely treat the objects with bar as complex (for CR) or real (for UR) quantities *a priori* independent of the ones without bar. Notice that in CR (and analogously in UR) the cases A or B as defined in our introduction correspond exactly to the nonexpanding or expanding spaces, respectively [20, 21]. (It is worth pointing out that by (2.8) the subset of V for which $\dot{C}^{(3)} = 0$, *i.e.*, the Weyl tensor is self-dual, is determined by an analytic set in C^2 defined by $f(z^\alpha) = 0$). As yet we have been unable to find any solution of Eq. (2.23). A further analysis of this equation is needed.

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