RAREFIED BANDS WITHIN THE MODEL OF A ONE-DIMENSIONAL FINITE HEISENBERG FERROMAGNET*

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The distribution of quantum states of a finite one-dimensional Heisenberg ferromagnet, consisting of N spins s, over the discrete Brillouin zone has been analysed by means of stratification of the action of the translation group on the set of all $(2s+1)^N$ magnetic configurations. It is shown that the rarefied bands are associated — through a secular eigenproblem — with irregular orbits, *i.e.* those on which the action is not free. An orthonormal complete basis, involving three exact quantum numbers: quasimomentum, the generalized star in the Brillouin zone, and the total magnetization, has been proposed.

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1. Introduction

Solid state theory is deeply associated with the picture of band structure of energy levels of the system. According to this picture, energy levels of interest (*e.g.* for electrons, phonons, polarons, or other elementary excitations) can be arranged into bands; and each wavevector from the first Brillouin zone, admissible by cyclic Born–von Kármán quantization conditions, is associated within a single non-degenerate band with exactly one quantum state. In the present paper we discuss a feature which is somehow surprising from such a point of view, namely the so called rarefied bands within the Heisenberg model of magnetism in solids, pointed out by Lulek [1]. He considered a one-dimensional magnetic crystal, consisting of N spins

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s, and determined the distribution of all the $(2s+1)^N$ quantum states of the model over the first Brillouin zone. It results that this distribution is inhomogeneous, which yields rarefied bands, *i.e.* such bands for which only some distinguished vectors from those allowed by quantization conditions correspond to some states of the system, whereas all other vectors are not realized, *i.e.* yield some "vacancies" in the space of quantum states. The symmetry of this distribution has been discussed by Florek and Lulek [2] in the light of a general recipe of Weyl ([3], p.138; cf. also Florek *et al.* [4]). They pointed out that this distribution is constant on the so called generalized stars, *i.e.* on orbits of the action of the group $\mathrm{Aut}\, C_N$ of all automorphisms of the translation group C_N of the crystal on the Brillouin zone.

Within the approach used in papers cited above, rarefied bands emerge rather indirectly, as one of consequences of inhomogeneity of the distribution of quantum states of the magnet over the Brillouin zone. This distribution has been determined globally, as the decomposition of the linear representation P, which realizes the action of the translation group C_N . In the present paper we demonstrate existence and origin of rarefied bands in an immediate way, as an obvious result of permutational structure of the action P on the set of all $(2s+1)^N$ magnetic configurations. To this aim, we perform the stratification (Michel [5]; cf. also Kuźma *et al.* [6]) of the representation P, treated as the permutational representation of the symmetric group Σ_N , then take into account the subduction $P \downarrow C_N$ corresponding to embedding $C_N \subset \Sigma_N$, and only in the last step we invoke to the linear structure of the space of quantum states of the magnet. In this way, we are able to account each orbit of the action P separately, and to associate rarefied bands with irregular orbits.

The orbit structure, determined by means of stratification and hierarchy imposed by embedding $C_N \subset \Sigma_N$, allows us also to determine an orthonormal basis in the space of quantum states of the model, which can be useful in some quantum and statistical calculations.

2. Distribution of quantum states of the magnet over the Brillouin zone

Let

$$\tilde{N} = \{j | j = 1, 2, \ldots, N\} \tag{1}$$

be the set of nodes of a finite one-dimensional crystal in a form of a ring, so that N is a regular orbit of the cyclic group C , playing the role of the translation group of the system. Let

$$\tilde{n} = \{i | i = 1, 2, \ldots, n\} \tag{2}$$

be the set of labels of z-projections of the single-node spin s, so that $i \in \tilde{n}$ corresponds to the projection $m_i = i - s - 1$. Then

$$\tilde{n}^{\tilde{N}} = \left\{ f : \tilde{N} \rightarrow \tilde{n} \right\} \tag{3}$$

is the set of all magnetic configurations, and its linear closure over the field of complex numbers constitutes the space of quantum states of the Heisenberg magnet. The unitary structure of this space is imposed by the condition that the set (3) forms an orthonormal complete basis, *i.e.*

$$\langle f | f' \rangle = \langle i_1, \ldots, i_N | i'_1, \ldots, i'_N \rangle = \delta_{i_1 i'_1}, \ldots \delta_{i_N i'_N}, \qquad f, f' \in \tilde{n}^{\tilde{N}}, \tag{4}$$

where

$$|f\rangle \equiv |i_1, \ldots, i_N\rangle \tag{5}$$

is a detailed notation for the configuration $f \in \tilde{n}^{\tilde{N}}$. The action of the symmetric group Σ_N on the set $\tilde{N}$ can be lifted in a natural way to the action P on the set $\tilde{n}^{\tilde{N}}$ by putting

$$P(\sigma) = \begin{pmatrix} F \\ f \circ \sigma^{-1} \end{pmatrix}, \quad f \in \tilde{n}^{\tilde{N}}, \quad \sigma \in \Sigma_N, \tag{6}$$

where

$$|f \circ \sigma^{-1}\rangle = |i_{\sigma^{-1}(1)}, \ldots, i_{\sigma^{-1}(N)}\rangle \tag{7}$$

is the image of the configuration f under the permutation

$$\sigma = \begin{pmatrix} j \\ \sigma(j) \end{pmatrix}, \quad j \in \tilde{N}. \tag{8}$$

P is therefore a permutation representation of Σ_N on the set $\tilde{n}^{\tilde{N}}$. It can be also looked at as a linear representation in the space of states of the magnet, which allows us to write down the decomposition

$$P \downarrow C_N \cong \sum_{k \in B} \oplus \rho(k) \Gamma_k \tag{9}$$

of the subduction $P \downarrow C_N$ of the representation P to the subgroup $C_N \subset \Sigma_N$ into irreducible representations Γ_k of the group C_N. The decomposition (9) establishes the distribution $\rho : B \longrightarrow Z$ of states of the magnet over the Brillouin zone

$$B = \left\{ k = 0, \pm 1, \pm 2, \ldots, \quad \begin{Bmatrix} \pm\frac{N-1}{2} & \text{for} \quad N \quad \text{odd} \\ \pm\frac{N}{2-1}, \ \frac{N}{2} & \text{for} \quad N \quad \text{even} \end{Bmatrix} \right\}, \tag{10}$$

i.e. over the set of all irreducible representations of the translation group C_N (Z is the set of all integers).

Eqs (9)–(10) demonstrate the role of the Brillouin zone as the set of irreducible representations of the translation group. This group is abelian, so that

$$|B| = \tilde{N} = N,$$

and the difference in labelling in Eq. (1) compared with (10) results merely from a shift in the reciprocal space providing that the wavenumber $k = 0 \equiv N \bmod N$ constitutes the center of the Brillouin zone. Equality $|B| = N$ can be also interpreted as a consequence of the fact that the action of the abelian group C_N on the set $\tilde{N}$ is free, *i.e.* that the nodes generated by different translations are mutually different. The situation changes in the case of the action $P \downarrow C_N$ of C_N on the set $\tilde{n}^{\tilde{N}}$. In this case there arise some orbits which are not regular, *i.e.* the action on these orbits is not free. In general, an arbitrary orbit of the group C_N is characterized by the epikernel

$$C_\kappa = \{\bar{\kappa}, 2\bar{\kappa}, \ldots, \kappa\bar{\kappa}\} \triangleleft C_N, \tag{11}$$

where κ is a divisor of the integer N, *i.e.* an element of the lattice $K(N)$ of all divisors of N, and thus of the lattice of subgroups of the translation group C_N , and

$$\bar{\kappa} = \frac{N}{\kappa}, \qquad \kappa \in K(N), \tag{12}$$

is the divisor complementary to κ in the lattice $K(N)$. The orbit with the epikernel $C_\kappa, \kappa \in K(N)$, is thus a carrier of the transitive representation $R^{N:\kappa}$ of the group C_N, and contains $\bar{\kappa}$ elements. In particular, the case $\kappa = 1$ corresponds to the regular orbit, which contains $\bar{\kappa} = N$ elements. The other extreme case is $\kappa = N$, *i.e.* the orbit consisting of the $\bar{\kappa} = 1$ element, or, in other words, an invariant of the permutation representation $P \downarrow C_N$ (a configuration $|ii\ldots i\rangle$, $i \in \tilde{n}$). Other cases correspond to orbits with intermediate number $1 < \bar{\kappa} < N$ of elements, which can be arranged into the partially ordered set according to the lattice $K(N)$ (Lulek and Lulek [7]). Evidently, the space spanned on an irregular orbit, *i.e.* that with the epikernel C_κ, $\kappa \neq 1$, has the dimension $\bar{\kappa} < N$, so that it cannot cover all wavenumbers k of the Brillouin zone (10). It encloses only those values of $k \in B$ which correspond to a "rarefied zone"

$$B/\kappa = \{k = \xi\kappa \bmod N \,|\, \xi \in \tilde{\bar{\kappa}}\} \subset B, \tag{13}$$

where

$$\tilde{\bar{\kappa}} = \{\xi \,|\, \xi = 1, 2, \ldots, \bar{\kappa}\} \tag{14}$$

is a non-symmetric analogue of the Brillouin zone for the quotient groups

$$C_{\bar{\kappa}} = C_N / C_\kappa \tag{15}$$

acting effectively, *i.e.* freely, on the orbit with the epikernel C_κ (note that $C_\kappa = \operatorname{Ker} R^{N:\kappa}$; *cf.* also Sect. 4). Thus the homogeneous component of the distribution ρ emerges from regular orbits ($\kappa = 1$), whereas all the inhomogeneities originate from irregular orbits ($\kappa \neq 1$).

3. Structure of orbits of the cyclic group on the set of magnetic configurations

A simple relation between inhomogeneity of the distribution ρ and epikernels $C_\kappa, k \in K(N)$, of orbits of the group C_N on the set $\tilde{n}^{\widetilde{N}}$ of magnetic configurations suggests us to perform a stratification of this set, in order to achieve a detail recognition of nature of this inhomogeneity. To this aim, it is also convenient to introduce a hierarchy of orbits based on the embedding

$$C_N \subset \Sigma_N, \tag{16}$$

i.e. to determine splitting of an orbit of the symmetric group Σ_N into those of the cyclic subgroup C_N .

We thus first determine the stratification of the set $\tilde{n}^{\widetilde{N}}$ under the action P of the symmetric group Σ_N . Let $\mu : \tilde{n} \to Z$ be a partition of the integer N into n nonnegative integers $\mu(i), i \in \tilde{n}$, so that

$$\sum_{i \in \tilde{n}} \mu(i) = N, \qquad \mu(i) \geq 0 \,. \tag{17}$$

Let $\nu : (\widetilde{N}) \longrightarrow Z$ denote the cyclic structure of the partition μ, *i.e.* the sequence

$$\nu = (\nu_1, \nu_2, \ldots, \nu_N), \tag{18}$$

with ν_l, $l \in \widetilde{N}$, denoting the number of parts of the length l, so that

$$\sum_{l \in \widetilde{N}} l \, \nu_l = N, \tag{19}$$

$$\nu_0 + \sum_{l \in \widetilde{N}} \nu_1 = n \tag{20}$$

and ν_0 is the number of parts of the length 0, *i.e.* such single-node states $i \in \tilde{n}$, for which $\mu(i) = 0$. Let M be the set of all those partitions μ

which satisfy the condition (17), and N — the corresponding set of cyclic structures. Then (a) orbits of the action P of the symmetric group Σ_N on the set $\tilde{n}^{\tilde{N}}$ are classified by partitions $\mu \in M$, so that μ labels the orbit

$$O_\mu = \{P(\sigma)\, f_o | \sigma \in \Sigma_N\}, \tag{21}$$

where

$$\begin{aligned} |f_0\rangle = |11\ldots1 \quad & 22\ldots2 \quad \ldots \quad nn\ldots n\rangle \\ \mu(1) \quad & \mu(2) \quad \ldots \quad \mu(n). \end{aligned} \tag{22}$$

(b) Strata of this action, *i.e.* sets of all orbits with the same epikernel, are classified by cyclic structures $\nu \in N$, so that ν labels the stratum

$$S_\nu = \{O_\mu | \mu \in M(\nu)\}, \tag{23}$$

where $M(\nu) \subset M$ is the set of all partitions with the cyclic structure ν. Usually, the structure ν is replaced by the "standard partition" from the set $M(\nu)$, *i.e.* such a partition $\mu_s \in M(\nu)$,all parts of which are arranged in the non-increasing order, so that

$$\mu_s(i+1) \leq \mu_s(i), \quad i \in \tilde{n}. \tag{24}$$

Correspondingly, the set N of cyclic structures can be replaced equivalently by the subset $M_s \subset M$ of all standard partitions, which is a travers of the action P of the group Σ_N on the set $\tilde{n}^{\tilde{N}}$. The epikernel ν is determined by the class of all subgroups, conjugated in the symmetric group Σ_N with the Young subgroup of the configuration f_0 , *i.e.*

$$\Sigma(\mu, f_0) = \prod_{i\in\tilde{n}} \times \Sigma_{\mu(i)} \tag{25}$$

the outer direct product of symmetric groups on parts of the set $\tilde{N}$ with identical single-node states, as imposed by the partition μ.Thus the stratification of the set $\tilde{n}^{\tilde{N}}$ under the action P of the group Σ_N can be written as

$$\tilde{n}^{\tilde{N}}/P = \bigcup_{\nu\in N} S_\nu. \tag{26}$$

Simple combinatoric arguments (Lulek and Biel [8]) yield

$$|O_\mu| = \frac{N!}{\prod_{i\in\tilde{n}} \mu(i)!}, \quad \mu \in M, \tag{27}$$

$$|S_\nu| = \frac{n!}{\nu_0!\ \prod_{l\in\tilde{N}} \nu_l!}, \quad \nu \in N, \tag{28}$$

$$|\tilde{n}^{\tilde{N}}/P| \equiv |M| = \binom{N+n-1}{N}, \tag{29}$$

respectively for the number of elements of the orbit O_μ , the number of orbits in the stratum S_ν , and the total number of orbits. They also yield the sum rule

$$\sum_{\nu \in N} |S_\nu|\,|O_\mu| = n^N , \tag{30}$$

where the partition μ has the cyclic structure ν.

Now we proceed to perform the subduction of the full permutational symmetry Σ_N to the translational symmetry C_N . Each orbit O_μ , given by Eq. (21), splits into orbits of the subgroup $C_N \subset \Sigma_N$ according to the decomposition

$$R^{\Sigma_N:\Sigma(\mu)} \downarrow C_N = \sum_{\kappa \in K(N)} \oplus m(\mu,\kappa) R^{N:\kappa} \tag{31}$$

of the transitive representation $R^{\Sigma_N:\Sigma(\mu)}$ of the group Σ_N with the Young subgroup $\Sigma(\mu)$ into transitive representations $R^{N:\kappa}$ of the group C_N . Multiplicities $m(\mu,\kappa)$ can be deduced from some combinatoric considerations on "cyclic words" (*cf. e.g.* Hall [9], Eq. (2.1.21)) as

$$m(\mu,\kappa) = \begin{cases} \dfrac{\kappa}{N} \displaystyle\sum_{\kappa' \in K(\mathrm{lcd}(\mu/\kappa))} \tilde{\mu}(\kappa') \frac{\left(\frac{N}{\kappa\kappa'}\right)!}{\prod_{i \in \tilde{n}} \left(\frac{\mu(i)}{\kappa\kappa'}\right)!} & \text{if } \mu(i)/\kappa \in Z,\ i \in \tilde{n}, \\ 0 & \text{otherwise,} \end{cases} \tag{32}$$

where the symbol $\mathrm{lcd}(\mu/\kappa)$ denotes the largest common divisor of all integers $\mu(i)/\kappa$, $i \in \tilde{n}$, and $\tilde{\mu}(\kappa')$ is the standard Möbius function of number theory. The total number of orbits of the group C_N , arising from the orbit O_μ of the group Σ_N, *i.e.* the quantity

$$|O_\mu/(P \downarrow C_N)| = \sum_{\kappa \in K(N)} m(\mu,\kappa), \tag{33}$$

can be evaluated using the cycle index (James and Kerber [10], p. 170). The cycle index $Ci(C_N)$ for the group C_N is a polynom of variables x_κ, $\kappa \in K(N)$, given by

$$Ci(C_N) \equiv Ci\left(C_N; x_\kappa, \kappa \in K(N)\right) = \frac{1}{N} \sum_{\kappa \in K(N)} \varphi(\kappa) x_\kappa^{\bar{\kappa}}, \tag{34}$$

where $\varphi(\kappa)$ is the standard Euler function of number theory. The quantity (33) is the coefficient of the monomial

$$\eta(\mu) = \prod_{i \in \tilde{n}} w_i^{\mu(i)}, \quad \mu \in M, \tag{35}$$

in the polynomial $\omega(C_N)$ of variables ω_i, $i \in \tilde{n}$, given by

$$\omega(C_N) = Ci\left(C_N; \sum_{i\in\tilde{n}} w_i^{\bar{\kappa}}, \quad \kappa \in K(N)\right) \equiv \sum_{\mu\in M} |\frac{O_\mu}{(P \downarrow C_N)}| \eta(\mu). \tag{36}$$

The polynom $\omega(C_N)$ is thus obtained from the cycle index $Ci(C_N)$ by the Polya insertion

$$x_\kappa = \sum_{i\in\tilde{n}} w_i^\kappa, \quad \kappa \in K(N). \tag{37}$$

Putting $w_1 = 1$, $i \in K(N)$, we obtain the total number of all orbits of the group C_N as

$$|\frac{\tilde{n}^{\tilde{N}}}{(P \downarrow C_N)}| = \omega(C_N; 11\ldots1) = \frac{1}{N} \sum_{\kappa\in K(N)} \varphi(\kappa) n^{\bar{\kappa}}. \tag{38}$$

The number of orbits of the group C_N with an epikernel C_κ in the set $\tilde{n}^{\tilde{N}}$ is given by

$$m(P,\kappa) = \sum_{\mu\in M} m(\mu,\kappa) = \sum_{\mu\in M_\bullet} m(\mu,\kappa)|S_\nu| = \frac{1}{\bar{\kappa}} \sum_{\kappa\in K(\bar{\kappa})} \tilde{\mu}(\kappa') n^{\bar{\kappa}/\kappa'}. \tag{39}$$

By the arguments of Sect. 4, it is also the total number of κ-tuply rarefied bands.

In particular, the number of regular orbits ($\kappa = 1$), *i.e.* full bands, is

$$m(P,1) = \frac{1}{N} \sum_{\kappa'\in K(N)} \tilde{\mu}(\kappa') n^{\bar{\kappa}'}. \tag{40}$$

For the other extreme case $\kappa = N$ we obtain

$$m(P,N) = n, \tag{41}$$

so that the number of "N-tuply rarefied bands", consisting of merely one point — the center $k = 0 \equiv N \bmod N$ of the Brillouin zone, coincides with the number of permutational invariants of P, *i.e.* configurations $|ii\ldots i\rangle$, $i \in \tilde{n}$.

In this way, we have described in detail the stratification of orbits of the representation $P \downarrow C_N$ on the set $\tilde{n}^{\tilde{N}}$ of all configurations of our Heisenberg magnet, with taking advantage from the embedding (16).

4. The decomposition of transitive representations of the cyclic group into its irreducible representations

Up to this point, we have exploited essentially only the permutational structure of the representation $P \downarrow C_N$,without resorting to its linear structure (with the exception of a general notion of the Brillouin zone). Now we proceed to use the unitary structure of the space of quantum states of the magnet, imposed by the orthogonality condition (4) of the basis $\widetilde{\tilde{n}^N}$. The key observation is that we can consider separately the linear subspace spanned on each orbit of the group C_N . The decomposition of each such linear space into subspaces irreducible under C_N provides an elementary contribution to the total distribution ρ of quantum states of the magnet over the Brillouin zone. We can thus write down such an elementary contribution, resulting from an orbit with the epikernel C_κ, $\kappa \in K(N)$, in a form

$$R^{N:\kappa} \cong \sum_{k \in B} \oplus m(\kappa.k) \Gamma_k, \tag{42}$$

where the multiplicity $m(\kappa, k)$ is given by

$$m(\kappa, k) = \begin{cases} 1 & \text{if } \operatorname{lcd}(\kappa, k) = \kappa, \\ 0 & \text{in other case,} \end{cases} \tag{43}$$

as an immediate consequence of the Burnside's theorem for the effective translation group $C_{\bar{\kappa}}$ of Eq. (15). According to Eqs (13) and (42), each element of an orbit with epikernel $C_{\bar{\kappa}}$, has the crystallographic interpretation of a configuration composed from κ identical cycles, or "elementary Bravais cells", each of the length $\bar{\kappa}$. As the result, each orbit of the transitive representation $R^{N:\kappa}$ of C_N yields a κ-tuply rarefied band, containing $\bar{\kappa} = N/\kappa$ states instead of N. Such a band corresponds to κ-tuply enlarged elementary cell of the crystal.

5. A classification scheme for the states of a Heisenberg magnet

The considerations given above suggest the following complete orthonormal basis set of states in the space of quantum states of the Heisenberg model of a magnet

$$b = \{|\mu\kappa\alpha k\rangle \ |\mu \in M, \kappa \in K(N), \alpha \in \widetilde{m}(\mu,\kappa), k = \xi\kappa, \ \xi \in \bar{\tilde{\kappa}}\}, \tag{44}$$

where μ is a partition of N into n parts satisfying Eq. (17), κ is an element of the lattice $K(N)$ of divisors of N, $\widetilde{m}(\mu, \kappa)$ is the set of orbits of C_N within the orbit O_μ of Σ_N, *i.e.* the set of repetition indices associated with the multiplicity $m(\mu,\kappa)$ given by Eqs (31)-(32), and $k = \xi\kappa$ is an element of the Brillouin zone, admissible for the orbit with epikernel C_κ along Eqs (13), (14), and (43). The set B/κ of admissible values of k for a given κ (Eq. (13)) constitutes the base for a κ-tuply rarefied band, enclosing $\bar{\kappa}$ states, where $\bar{\kappa}$ is the divisor complementary to κ in the lattice $K(N)$ (Eq. (12)). We proceed to describe the physical meaning of quantum numbers of the basis (44) in the Heisenberg model of magnetism.

The partition μ originates from the symmetric group Σ_N on the set of nodes of the crystal. Even though Σ_N is not the symmetry group of the model, nevertheless the partition μ imposes a convenient quantum number

$$M = \sum_{i\in\tilde{n}} \mu_i(i - s - 1) \tag{45}$$

the projection of the total spin of the system. It is an exact quantum number in the model assuming isotropic exchange interactions, as well as in models admitting single-axis anisotropies. It is worth to observe, however, that a given value of M originates, in general, from various orbits and various strata (23). The minimal value

$$M_{\min} = -Ns = -N(n-1)/2 \tag{46}$$

corresponds to the partition $\mu = (N0\ldots0)$, *i.e.* to the full saturation of the ferromagnet. The corresponding orbit consists from a single configuration $|11\ldots1\rangle$, *i.e.* yields an "extremely rarefied" band, consisting merely of the centre of the Brillouin zone. The value $M_{\min}+1$ is associated with the single orbit, corresponding to the partition $\mu = (N-1, 10\ldots)$. This orbit encloses, according to Eq. (27), N elements and yields thus a regular band — the very well known spin waves (Dyson [11], Mattis [12], Morrish [13]). The case of $M_{\min}+2$ corresponds (for $s > 1/2$) to two orbits: $\mu_1 = (N-2,\ 200\ldots0)$ and $\mu_2 = (N-1,\ 0,\ 10\ \ldots)$. The orbit O_{μ_1} consists of $N(N-1)/2$ configurations, describing two spin deviations at different nodes, whereas the orbit O_{μ_2} consists of N configurations, each with two deviations on a single node. Evidently, the orbit O_{μ_2} appears only for $s > 1/2$ and corresponds to "deep spin waves", which are energetically unstable. We can proceed, in principle, to describe further types of configurations, with increasing combinatoric complications.

The divisor $\kappa \in K(N)$ is also an exact quantum number of the model, and denotes the generalized star of the wavenumber $\kappa \in B$, *i.e.* the set

$$B_\kappa = \{k \in B \mid \mathrm{lcd}\,(k,N) = \kappa\}, \quad \kappa \in K(N), \tag{47}$$

of all elements of the Brillouin zone B, generated from $\kappa \in K(N) \subset B$ by the group $\mathrm{Aut}\,C_N$ of all automorphism of the translation group C_N.

According to a general recipe of Weyl ([3], p. 138; *cf.* also Florek and Lulek [2], Florek *et al.* [4]), the group Aut C_N is related to a "hidden" symmetry of the Heisenberg model. In order to explain the physical implications of this symmetry, we first observe that the natural action of the group Aut C_N on the Brillouin zone B (*cf.* Florek and Lulek [2] for detail) yields the decomposition

$$B = \bigcup_{\kappa \in K(N)} B_\kappa, \quad B_\kappa \bigcap B_{\kappa'} = \emptyset \quad \text{for} \quad \kappa \neq \kappa', \tag{48}$$

which, in turn, imposes a crystallographic interpretation to Aut C_N as a generalized point group for the one-dimensional crystal. The generalized star B_1 is a regular orbit of Aut C_N, which corresponds to the generic stratum, or, in the language of crystallography, to the "general position", whereas the other stars B_κ, $\kappa \neq 1$, correspond to various "special positions", characterized by epikernels C_κ . As shown by Florek and Lulek [2], lifting of the action of the group Aut C_N to the space of quantum states of the system yields a conclusion that the distribution ρ of Eq. (9) is constant on each generalized star B_κ , *i.e.*

$$\rho(k) = \text{const}, \qquad k \in B_\kappa \subset B. \tag{49}$$

This result becomes also evident within the picture of orbits of magnetic configurations, described in Sect. 3. Namely, the action of Aut C_N can be also restricted to the space spanned by an arbitrary orbit of C_N. Each regular orbit of the group C_N is associated with the decomposition (48), with each generalized star B_κ, $\kappa \in K(N)$, occurring exactly once. It yields the homogeneous component of the distribution ρ, equal to $m(P,1)$, as given by Eq. (40). On the other hand, each irregular orbit, *i.e.* $\kappa \neq 1$, corresponds to the rarefied Brillouin zone B/κ, given by Eq. (13), for which

$$\frac{B}{\kappa} = \bigcup_{\kappa' \in K(\bar{\kappa})} B_{\kappa\kappa'}. \tag{50}$$

In other words, the rarefied Brillouin zone B/κ consists of only those generalized stars $B_{\kappa''}$, $\kappa'' = \kappa\kappa'$, for which κ' is an element of the quotient lattice

$$\frac{K(N)}{\kappa} \cong K(\bar{\kappa}).$$

Evidently, B_κ is the generic star in the rarefied zone B/κ (then $\kappa' = 1 \in K(\bar{\kappa})$). Therefore, each orbit with the epikernel C_κ contributes once to every generalized star of the form $B_{\kappa\kappa'}$, $\kappa' \in K(\bar{\kappa})$, and does not contribute (*i.e.* yields "vacancies") to other stars. The global form of the distribution

ρ determined by Lulek [1] is just the sum of such contributions from each orbit. All the symmetry properties of the distribution ρ, discussed by Florek and Lulek [2], in particular the homogeneity (49) on each generalized star, become also evident in the picture of orbits. It is, *e.g.* , evident that each orbit yields once the zeroth star $B_N = \{N \equiv 0 \bmod N\}$, so that the distribution ρ achieves the maximal value at the centre of the Brillouin zone. This value is equal to the total number of all orbits, *i.e.* all elementary bands, given by

$$\rho(0) = \left| \frac{\tilde{n}^{\tilde{N}}}{P \downarrow C_N} \right| \tag{51}$$

(*cf.* Eq. (38)). Similarly, the generic star B_1 enters only regular orbits, so that

$$\rho(1) = m(P, 1) \tag{52}$$

(*cf.* Eq. (40)), the minimal value. Intermediate stars B_κ, $1 < \kappa < N$, have, in addition to the homogeneous component (52), also contributions from appropriate rarefiedd bands, so that the general distribution ρ is given by the formula

$$\rho(k) = \sum_{\kappa' \in K(\bar{\kappa})} m(P, \kappa\kappa') \quad \text{for} \quad k \in B_\kappa \subset B, \tag{53}$$

where $m(P, \kappa)$ is given by Eq. (39).

Evidently, k is also an exact quantum number of the model, for reason of the translational symmetry of C_N — the group of "obvious" symmetry in the terminology of Weyl's recipe. The classification (44) emphasizes the fact that for $\kappa \neq 1$ the wavenumber k does not run over the whole Brillouin zone B (Eqs (10) and (48)), but only the rarefied zone B/κ, Eqs (13) and (50).

The meaning of the label $\alpha \in \widetilde{m}(\mu, \kappa)$ consists in a classification of various orbits of the translation group C_N within the orbit O_μ of the group Σ_N. In general, it is not exact quantum number but merely an index for classification of orthonormal basis states. Hence, each subspace with a given total magnetization M, generalized star $B_\kappa \subset B$, $\kappa \in K(N)$, and $k \in B_\kappa$, yields a secular equation for the Heisenberg Hamiltonian. Let $L(M, \kappa, k)$ be such a subspace. Its dimension is

$$\dim L(M, \kappa, k) = \sum_{\{\mu|(45)\}} \sum_{\kappa' \in K(\kappa)} m(\mu, \kappa'), \tag{54}$$

where the first sum runs over all those partitions $\mu \in M$ which satisfy the condition (45), and $m(\mu, \kappa)$ is given by Eq. (32). The set of all solutions of secular eigenproblems for all possible M, κ, k determines the structure of energy bands of the model.

6. An example

We consider as an example the ring consisting of $N = 12$ nodes, each with the spin $^1/_2$. The set $\widetilde{N}$ resembles thus the set of figures on a clock dial plate. The number of magnetic configurations, *i.e.* the dimension of the space of quantum states is

$$n^N = 2^{12} = 4096. \tag{55}$$

The stratification of the set $\tilde{n}^{\widetilde{N}}$ of magnetic configurations under the action P of the symmetric group Σ_N is given in Table I. Each stratum S_ν, given by the cyclic structure ν of Eq. (18), satisfying conditions (19)–(20), is uniquely determined by the standard bipartition

$$\mu = (\mu(1), \mu(2)), \quad \mu(1) + \mu(2) = 12, \quad \mu(1) \geq \mu(2) \tag{56}$$

(*cf.* (24)), and consists of $|S_\nu|$ orbits, each enclosing $|O_\mu|$ configurations.

TABLE I

Stratification of the set of all magnetic configurations for $N = 12$, $s = {}^1/_2$ under the action P of the symmetric group Σ_{12}.

The cyclic structure ν		The standard partition μ $\mu(1)$	$\mu(2)$	Magnet–ization M	$\lvert S_\nu\rvert$	$\lvert O_\mu\rvert$	$\lvert S_\nu\rvert\lvert O_\mu\rvert$
$\nu_0 = 1$	$\nu_{12} = 1$	12	0	±6	2	1	2
$\nu_1 = 1$	$\nu_{11} = 1$	11	1	±5	2	12	24
$\nu_2 = 1$	$\nu_{10} = 1$	10	2	±4	2	66	132
$\nu_3 = 1$	$\nu_9 = 1$	9	3	±3	2	220	440
$\nu_4 = 1$	$\nu_8 = 1$	8	4	±2	2	495	990
$\nu_5 = 1$	$\nu_7 = 1$	7	5	±1	2	792	1584
$\nu_6 = 2$		6	6	0	1	924	924

$$13 = \binom{12+2-1}{12}, \quad 4096 = 2^{12}$$

Each orbit O_μ is characterized by a definite value of magnetization

$$M = \frac{(\mu(2) - \mu(1))}{2}, \tag{57}$$

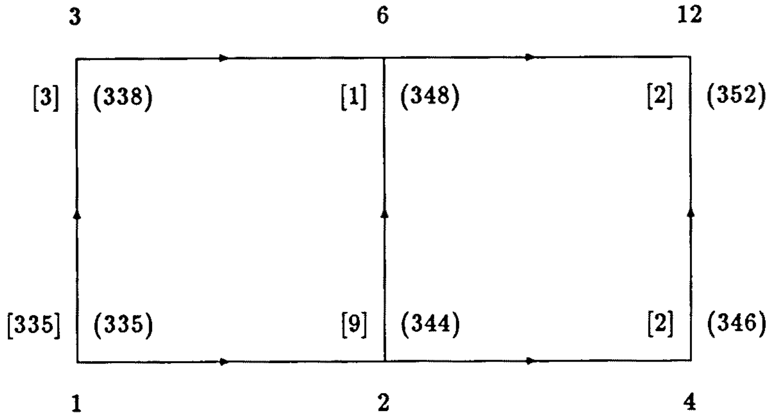

Fig. 1. The lattice $K(12)$ of divisors of the integer 12. Arrows indicate the partial order imposed by the inclusion $C_K \triangleleft C_{12}$. Integers without brackets, in square brackets and parentheses are respectively divisors κ, multiplicities $m(P,\kappa)$ giving number of orbits with the epikernel C_K, and numbers of states ρ_K for the generalized star B_K.

TABLE II

Multiplicities $m(\mu,\kappa)$ in the decomposition (31) of an orbit O_μ of the symmetric group Σ_{12} into orbits of the cyclic subgroup C_{12}. Each entry of the last column is the sum of the corresponding row in accordance with Eq. (33), and the last row exhibits the multiplicity $m(P,\kappa)$, *i.e.* the number of κ-tuply rarefied bands, evaluated according to Eq. (39) — using the column $|S_\nu|$ of Table I. The right bottom corner gives the total number of orbits $|\frac{\tilde{\tilde{n}}^N}{(P\downarrow C_N)}|$.

μ		κ						
$\mu(1)$	$\mu(2)$	1	2	3	4	6	12	$\lvert\frac{O_\mu}{P\downarrow C_{12}}\rvert$
12	0	–	–	–	–	–	–	1
11	1	1	–	–	–	–	–	1
10	2	5	1	–	–	–	–	6
9	2	18	–	1	–	–	–	19
8	4	40	2	–	1	–	–	43
7	5	66	–	–	–	–	–	66
6	6	75	3	1	–	1	–	80
$m(P,\kappa)$		335	9	3	2	1	2	352

and the two orbits within each stratum for $M \neq 0$ differ mutually by the sign of M (this feature holds only for $s = 1/2$; for $s > 1/2$ there appear various absolute values of M within a stratum).

The lattice $K(12)$ of divisors of the integer 12 is given in Fig. 1. The

Brillouin zone for our dial plate is given by

$$B = \{\pm 1, \pm 5\} \cup \{\pm 2\} \cup \{\pm 3\} \cup \{\pm 4\} \cup \{6\} \cup \{12\}\,, \tag{58}$$

where each subset is a generalized star B_κ , generated from the first element — the divisor $\kappa \in K(12)$. In particular, the first star $B_1 = \{\pm 1 \pm 5\} \cong \{1, 5, 7, 11\}$ is the generic one. Splitting of each orbit O_μ of the group Σ_{12} into orbits of the group C_{12} (Eq. (31)) is given in Table II. As seen from this table (*cf.* also Fig. 1), there are 335 regular orbits, *i.e.* full bands, and 17 irregular orbits, leading to rarefied bands.*E. g.* the orbit of the group Σ_{12}, corresponding to the partition $\mu = (9,3)$ splits under the subduction to C_{12} into 18 regular orbits and one irregular orbit, the latter presented in Fig. 2. One can observe from this figure that the irregular orbit has too small number of configurations in order to realize phase shifts of $2\pi/12$, corresponding to $k = \pm 1$ in the Brillouin zone (58). There are only possible the multiplicities of $2\pi/4$, according to the decomposition (42)

$j =$	1	2	3	4	5	6	7	8	9	10	11	12
	+	−	−	−	+	−	−	−	+	−	−	−
$\bar{\kappa} = 4$	−	+	−	−	−	+	−	−	−	+	−	−
	−	−	+	−	−	−	+	−	−	−	+	−
	−	−	−	+	−	−	−	+	−	−	−	+

$\bar{\kappa} = 4$ $\kappa = 3$

Fig. 2. An irregular orbit of the group C_{12}, arising from the orbit O_μ of the group Σ_{12} for the partition $\mu = (9,3)$. The epikernel of this orbit is the subgroup $C_3 = \{4, 8, 12\} \triangleleft C_{12}$. The orbit consists of $\bar{\kappa} = 4$ magnetic configurations (the rows in the figure). Each configuration can be decomposed into $\kappa = 3$ "elementary cells", each of the period $\bar{\kappa} = 4$.

$$R^{12:4} \cong \Gamma_0 \oplus \Gamma_3 \oplus \Gamma_{-3} \oplus \Gamma_6\,. \tag{59}$$

This orbit yields thus the four-fold rarefied band $B/4$, with $k = 0, \pm 3, 6$, *i.e.* with the generalized stars B_3, B_6, and B_{12}, but without B_1, B_2, B_4.

The full list of orbits of the group C_{12} on the set of configurations is given in the last column of Table II and in Fig. 1, the decompositions (50) of rarefied zones B/κ into generalized stars B_κ is given in Table III, and the resulting distribution ρ of quantum states of the magnet is given in terms of generalized stars (*cf.* Eqs (48)–(49)) in Fig. 1, and explicitly in the Brillouin zone B (*cf.* Eqs (9) and (51)–(53)) — in Fig. 3.

Our example demonstrates that the global distribution given in Fig. 3 is the result of a simple summations of elementary contributions from particular orbits of the group C_{12}. Values of $\rho(k)$ are constant on each generalized

TABLE III

The decomposition of a rarefied Brillouin zone B/κ, $\kappa \in K(12)$ (corresponding to an orbit with the epikernel C_κ) into generalized stars B_κ, $\kappa \in K(12)$.

κ	κ' 1	2	3	4	6	12
1	1	1	1	1	1	1
2	–	1	0	1	1	1
3	–	–	1	0	1	1
4	–	–	–	1	0	1
6	–	–	–	–	1	1
12	–	–	–	–	–	1

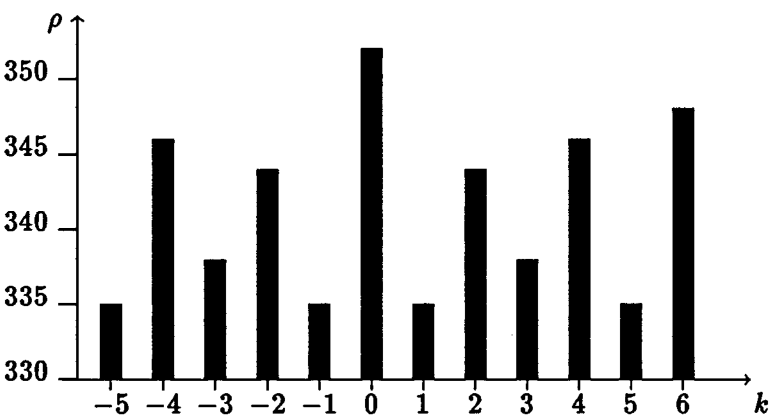

Fig. 3. Distribution ρ of states of the Heisenberg magnet for $N = 12$, $s = {}^1/_2$, over the Brillouin zone.

stars B_κ, *i.e.* on the subsets distinguished in Eq. (58), and the values $\rho(\kappa)$ increase according to the partial order for the lattice $K(12)$, denoted by arrows in Fig. 1. *N. b.* the partial order in $K(N)$ is not necessary consistent with the order of the real axis, so that, *e.g.* , $\rho(3) < \rho(2)$.

The dimensions (54) of secular equations are listed in Table IV. The last row of this table corresponds to saturation states of the ferromagnet ($M = \pm 6$), and the next to the last — to states with a single spin wave ($M = \pm 5$). These are the only cases of one-dimensional secular equations. A relatively small size of secular matrices occurs for the case of two spin waves ($M = \pm 4$): five full bands B and one doubly rarefied $B/2$. The highest size appears for $M = 0$, *i.e.* for the antiferromagnetic ground state: 75 full bands B, 3 doubly rarefied bands $B/2$, one $B/3$ and one $B/6$. In particular, the orbit with the epikernel C_6 yielding the band $B/6$ can be readily recognized as that corresponding to two Neel configurations:

$$|+-+\ldots-\rangle \text{ and } |-+-\ldots+\rangle,$$

used as the starting point in searching for the antiferromagnetic ground state.

TABLE IV

Dimensions of secular equations. The table gives the value of $\dim L(M,\kappa,k)$, $k \in B_\kappa \subset B$ (*cf.* Eq. (54)).

M	κ = 1	2	3	4	6	12
0	75	78	76	78	80	80
±1	66	66	66	66	66	66
±2	40	42	40	43	42	43
±3	18	18	19	18	19	19
±4	5	6	5	6	6	6
±5	1	1	1	1	1	1
±6	0	0	0	0	0	1

7. Final remarks and conclusions

We have discussed the origin of rarefied bands in the model of a finite one-dimensional periodic Heisenberg magnet. Existence of such bands is a natural consequence of irregular orbits of the action $P \downarrow C_N$ of the translation group C_N on the set of all $(2s+1)^N$ magnetic configurations. Orbits of the action $P \downarrow C_N$ yield — through an appropriate secular eigenproblem — to a definite band structure. Each regular orbit, *i.e.* an orbit consisting of N different configurations, yields one full band B, whereas each orbit with a non-trivial epikernel $C_\kappa \triangleleft C_N$, consisting of $\bar{\kappa} = \frac{N}{\kappa} < N$ configurations, yields a κ-tuply rarefied band B/κ. In other words, the component of the action $P \downarrow C_N$ which acts freely on $\widetilde{\tilde{n}^N}$ yields full bands, whereas all inhomogeneities of the distribution ρ, associated with rarefied bands, originate from those orbits of configurations, where the group C_N does not act freely. The global distribution ρ of quantum states of the magnet over the Brillouin zone can be uniquely decomposed into elementary contributions from individual orbits. As a result, generalized stars B_κ , labelled by elements κ of the lattice $K(N)$ of divisors of the integer N, are elementary subsets of the Brillouin zone with the constant values of the distribution ρ, in agreement with the predictions derived by Florek and Lulek [2] from the general recipe of Weyl [3], in a spirit of "action of a group on a set" (Michel [5]).

We have also pointed out that the extension of the translation group C_N to the full symmetric group Σ_N on the set of nodes of the crystal proves to be convenient in quantum and statistical calculations concerning the Heisenberg model. The stratification of the action $P \downarrow C_N$ implied by the embedding $C_N \subset \Sigma_N$ provides an additional important quantum number, the total magnetization M. We are thus able to propose an orthonormal complete basis in the space of quantum states of the magnet, involving altogether three exact quantum numbers: the generalized star $B_\kappa \subset B$, the quasimomentum $k \in B_\kappa$, and the magnetization M. Rows and columns of appropriate secular eigenmatrices of the model Hamiltonian in this basis are labelled by the following pairs of quantum numbers: the partition μ of N into $n = 2s + 1$ parts, and the multiplicity label α for repeated orbits of the translation group C_N in the orbit O_μ of the symmetric groups Σ_N. In particular, this basis is well adjusted to discuss interactions between q ideal spin waves (Dyson [11], Mattis [12], Morrish [13]) in the subspace with $M = Ns - q$. In this context, the rarefied bands can be looked at as a specific effect of "kinematic interactions" of ideal spin waves, which results in "vacancies" of quantum states for some generalized stars in the Brillouin zone.

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