# FREE NOTOPH THEORY IN LONGITUDINAL GAUGE*

L.C. PAPALOUCAS

Institute of Mathematics, University of Athens
106-79 Athens, Greece

J. REMBIELIŃSKI AND W. TYBOR

Institute of Physics, University of Łódź
Pomorska 149/153, 90-236 Łódź, Poland

*(Received February 28, 1991)*

The description of the free notoph in the noncovariant longitudinal gauge is given. The free propagators of the notoph are calculated.

PACS numbers: 03.50.Kk

## 1. Introduction

To describe, in the simplest way, the free Maxwell theory we choose the Coulomb gauge (*i.e.* the transversal one). Then, from the very beginning, we can put in some Lorentz frame the scalar potential and the longitudinal part of the vector potential equal to zero, and, therefore, we deal with the transversal vector potential only. It describes two independent states of polarization. Of course, the description is not explicitly Lorentz covariant. This necessitates proving the Lorentz covariance, but it does not seem to be too high price for simplicity of the physical picture.

It is interesting to investigate a simple theory of the free notoph (*i.e.* the scalar particle described by the antysymmetric tensor field $B^{\mu\nu} = -B^{\nu\mu} = (\vec{E}, \vec{H})$ [1–3]). It is well known that quantization of the free notoph in the covariant Ogievetsky–Polubarinov gauge [1] presents subtleties (*i.e.* ghosts for ghosts) [3–5]. A possible way to avoid such complications is giving up the explicit Lorentz covariance [5]. Then we show that there exists the simple

* Supported by CPBP 01.03.

description of the free notoph. It corresponds to the longitudinal gauge: we deal with the longitudinal part of the vector $\vec{E}$ only (the other components of $B^{\mu\nu}$ vanish).

The plan of the paper is as follows. In Section 2 we recall the notoph theory in the Ogievetsky–Polubarinov gauge. The description of the free notoph in the longitudinal gauge is given in Section 3. In Section 4 this gauge is used to obtain the Feynman path integral and the notoph propagators are calculated.

## 2. Free notoph in the Ogievetsky–Polubarinov gauge

The notoph [1] is described by the antysymmetric (six component) tensor field $B_{\mu\nu} = -B_{\nu\mu}$. The Lagrangian of the free notoph theory is

$$\mathcal{L} = -\tfrac{1}{2} G_\mu G^\mu \,, \tag{1}$$

where $G^\mu \equiv \partial_\nu B^{\nu\mu}$ is a strength vector. The equations of motion are

$$\partial_\mu G_\nu - \partial_\nu G_\mu = 0 \,. \tag{2}$$

The theory is invariant under the gauge transformation

$$\delta B^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} \partial_\alpha \lambda_\beta \,, \tag{3}$$

where $\lambda^\beta = (\lambda_0, \vec{\lambda})$ is an arbitrary 4-vector function.

For further discussion it is convenient to introduce the 3-dimensional notation

$$B^{k0} \to E_k \,, \qquad B^{kn} \to \varepsilon_{knj} H_j \,.$$

Then the equation of motion and the gauge transformations take the form

$$\Box \vec{E} + \frac{\partial}{\partial t} \operatorname{rot} \vec{H} - \operatorname{rot} \operatorname{rot} \vec{E} = 0 \,, \tag{4a}$$

$$\frac{\partial}{\partial t} \operatorname{rot} \vec{E} + \operatorname{rot} \operatorname{rot} \vec{H} = 0 \,, \tag{4b}$$

$$\delta \vec{E} = \operatorname{rot} \vec{\lambda} \,, \tag{5a}$$

$$\delta \vec{H} = -\frac{\partial}{\partial t} \vec{\lambda} - \operatorname{grad} \lambda_0 \,. \tag{5b}$$

The Ogievetsky–Polubarinov gauge condition has the form [1]

$$\varepsilon^{\mu\nu\alpha\beta} \partial_\nu B_{\alpha\beta} = 0 \,. \tag{6}$$

We point out that Eq. (6), owing to the Bianchi identity, imposes only three independent conditions. In the 3-dimensional notation we get

$$\frac{\partial}{\partial t}\vec{H} - \mathrm{rot}\,\vec{E} = 0\,, \tag{7a}$$

$$\mathrm{div}\,\vec{H} = 0\,. \tag{7b}$$

In this gauge the field equation are

$$\Box\vec{E} = 0\,, \qquad \Box\vec{H} = 0$$

and they remain invariant under the transformations (5) with $\lambda_0$ and $\vec{\lambda}$ obeying now the Maxwell equations

$$\frac{\partial}{\partial t}\mathrm{div}\,\vec{\lambda} + \Delta\lambda_0 = 0\,,$$

$$\Box\vec{\lambda} + \mathrm{grad}\left(\mathrm{div}\,\vec{\lambda} + \frac{\partial}{\partial t}\lambda_0\right) = 0\,. \tag{8}$$

So, three independent gauge conditions do not fix two degrees of freedom of $\lambda^\mu$. Therefore the notoph has only one polarization degree of freedom $(6 - 3 - 2 = 1)$.

Two remarks on the covariant gauge (6) are in order.

*(i)* It seems unusual that the condition (7b) is a restriction imposed on the nondynamical variable $\vec{H}$ (see Section 4). Nevertheless the gauge condition (6) is acceptable: three independent conditions are imposed on three dynamical variables $\vec{E}$.

*(ii)* Eqs (8) are invariant under the gauge transformation

$$\delta\vec{\lambda} = \mathrm{grad}\,\alpha\,, \qquad \delta\lambda_0 = -\frac{\partial}{\partial t}\alpha\,,$$

where $\alpha(\vec{x}, t)$ is an arbitrary function. This is why, when quantizing in the gauge (6), the second generation of ghosts arises.

## 3. Noncovariant gauge

Let us give up the explicit covariance of the notoph description. We discuss Eqs (4) and (5) in some fixed Lorentz frame. Then a decomposition of $\vec{E}$ and $\vec{H}$ can be performed

$$\vec{E} = \vec{E}_\perp + \vec{E}_\parallel\,, \qquad \vec{H} = \vec{H}_\perp + \vec{H}_\parallel\,,$$

where $\perp$ ($\parallel$) indicates transversal (longitudinal) part of vector. The field equations may be rewritten in the form

$$\begin{aligned} &\Box \vec{E}_{\parallel} = 0\,, \\ &\Box \vec{E}_{\perp} + \frac{\partial}{\partial t} \operatorname{rot} \vec{H}_{\perp} - \operatorname{rot} \operatorname{rot} \vec{E}_{\perp} = 0\,, \\ &\Delta \vec{H}_{\perp} - \frac{\partial}{\partial t} \operatorname{rot} \vec{E}_{\perp} = 0\,. \end{aligned} \tag{9}$$

We observe that the variable $\vec{H}_{\parallel}$ does not occur in these equations. The field equations (9) are invariant under the gauge transformations

$$\delta \vec{E}_{\parallel} = 0\,, \quad \delta \vec{E}_{\perp} = \operatorname{rot} \vec{\lambda}_{\perp}\,, \quad \delta \vec{H}_{\perp} = -\frac{\partial}{\partial t} \vec{\lambda}_{\perp}\,, \tag{10}$$

where $\vec{\lambda}_{\perp}$ is the transversal part of $\vec{\lambda}$.

Let us give two examples of noncovariant gauge conditions.

1. The gauge condition

$$\frac{\partial}{\partial t} \vec{H}_{\perp} - \operatorname{rot} \vec{E} = 0\,. \tag{11}$$

Its from is prompted by covariant condition (7). The field equations are

$$\Box \vec{E} = 0\,, \quad \Box \vec{H}_{\perp} = 0\,.$$

They remain invariant under the gauge transformations (10) with $\vec{\lambda}_{\perp}$ obeying the wave equation $\Box \vec{\lambda}_{\perp} = 0$.

2. The gauge condition

$$\operatorname{rot} \vec{E} = 0\,. \tag{12}$$

This longitudinal gauge fixes the gauge freedom (10) completely: $\vec{\lambda}_{\perp} = 0$. The field equations are

$$\Box \vec{E} + \frac{\partial}{\partial t} \operatorname{rot} \vec{H}_{\perp} = 0\,, \qquad \Delta \vec{H}_{\perp} = 0\,. \tag{13}$$

If the fields vanish when $|\vec{x}| \to \infty$, then one gets $\vec{H}_{\perp} = 0$. So, in the longitudinal gauge (12) we get actually

$$\vec{H}_{\perp} = 0\,, \tag{14a}$$

$$\vec{E}_{\perp} = 0\,, \tag{14b}$$

$$\Box \vec{E}_{\parallel} = 0\,, \tag{14c}$$

and the strength vector is

$$G^{\mu} = \left(\operatorname{div}\vec{E}_{\parallel}\,, -\frac{\partial}{\partial t}\vec{E}_{\parallel}\right) . \tag{15}$$

The gauge condition (12) and the resulting one (14a) are not Lorentz invariant conditions. Therefore, to preserve them in the other Lorentz frame, we must allow the field $B^{\mu\nu}$ to transform to the new frame as the Lorentz tensor except for the gauge transformation [6]. In the momentum space we have infinitesimally

$$\begin{aligned} \vec{E}(\vec{k}) \to & \vec{E}(\vec{k}) + \vec{\theta}\times\vec{E}(\vec{k}) - \vec{v}\times\vec{H}(\vec{k}) \\ & - i\vec{k}\times\left(\frac{\vec{v}\times\vec{E}_{\parallel}}{ik_0} + \frac{\vec{v}_{\perp}(\vec{k}\vec{H})}{ik_0^2}\right) , \\ \vec{H}(\vec{k}) \to & \vec{H}(\vec{k}) + \vec{\theta}\times\vec{H}(\vec{k}) + \vec{v}\times\vec{E}(\vec{k}) \\ & - ik_0\left(\frac{\vec{v}\times\vec{E}_{\parallel}}{ik_0} + \frac{\vec{v}_{\perp}(\vec{k}\vec{H})}{ik_0^2}\right) - i\vec{k}a(\vec{k}) , \end{aligned} \tag{16}$$

where $\vec{\theta}(\vec{v})$ are the rotation (boost) parameters and $a(\vec{k})$ is an arbitrary function. Then the strength field (15) transforms covariantly to the new frame (as a Lorentz four vector).

It is worth to compare Eqs (14) and (15) with the description of the Maxwell equations in the transversal gauge:

$$\varphi = 0\,, \qquad \vec{A}_{\parallel} = 0\,, \qquad \Box\vec{A}_{\perp} = 0\,;$$

the electric and magnetic fields are

$$\vec{\mathcal{E}} = -\frac{\partial}{\partial t}\vec{A}_{\perp}\,, \qquad \vec{\mathcal{B}} = \operatorname{rot}\vec{A}_{\perp}\,.$$

## 4. Free notoph propagators in longitudinal gauge

The Lagrangian (1) may be rewritten in the form

$$\mathcal{L} = -\tfrac{1}{2}\left(\operatorname{div}\vec{E}\right)^2 + \tfrac{1}{2}\left(\frac{\partial}{\partial t}\vec{E} + \operatorname{rot}\vec{H}\right)^2 . \tag{17}$$

The standard procedure [7] leads to the Hamiltonian

$$\mathcal{H} = \tfrac{1}{2}\vec{\Pi}^2 + \tfrac{1}{2}\left(\operatorname{div}\vec{E}\right)^2 + \vec{\Lambda}\operatorname{rot}\vec{\Pi}\,, \tag{18}$$

where

$$\vec{\Pi} = \frac{\partial \mathcal{L}}{\partial \partial_0 \vec{E}} = \frac{\partial}{\partial t}\vec{E} + \mathrm{rot}\,\vec{H} \quad \left(\partial_0 \equiv \frac{\partial}{\partial t}\right) \tag{19}$$

is the momentum conjugated to $\vec{E}$, and $\vec{\Lambda}$ is the Lagrange multiplier. So, we deal with a constrained system with $\vec{E}$ and $\vec{\Pi}$ as dynamical variables. The variable $\vec{H}$ does not appear in the canonical formalism. The constraint is

$$\mathrm{rot}\,\vec{\Pi} = 0\,. \tag{20}$$

The Hamiltonian is invariant under the gauge transformations

$$\delta\vec{E} = \mathrm{rot}\,\vec{\lambda}_\perp\,, \qquad \delta\vec{\Pi} = 0\,. \tag{21}$$

They are generated by the constraint (20) and are consistent with Eqs (10) (see Eq. (19)).

The further analysis is performed in the longitudinal gauge (12). We know (see Section 3) that this condition fixes the gauge. In the canonical formalism the possibility to fix a gauge is equivalent to (see, *e.g.* [8])

$$\det\left\{\left(\mathrm{rot}\,\vec{E}(\vec{x},t)\right)_i, \left(\mathrm{rot}\,\vec{\Pi}(\vec{y},t)\right)_j\right\} \neq 0\,, \tag{22}$$

where $\{\ldots,\ldots\}$ denotes the Poisson bracket.

Let us verify the condition (22) because its validity is crucial for writing down the Feynman path integral. The calculation gives

$$\left\{\left(\mathrm{rot}\,\vec{E}(\vec{x},t)\right)_i, \left(\mathrm{rot}\,\vec{\Pi}(\vec{y},t)\right)_j\right\} = \Delta\,\delta^{\mathrm{tr}}_{ij}(\vec{x}-\vec{y})\,, \tag{23}$$

where the transversal $\delta$-function

$$\delta^{\mathrm{tr}}_{ij}(\vec{x}) = \left(\delta_{ij} - \frac{\partial_i\partial_j}{\Delta}\right)\delta(\vec{x})$$

appears because the Poisson bracket is calculated for values depending on the variables $\vec{E}_\perp$ and $\vec{\Pi}_\perp$ from the transversal part of a phase space

$$\{E_{\perp i}(\vec{x},t), \Pi_{\perp j}(\vec{y},t)\} = \delta^{\mathrm{tr}}_{ij}(\vec{x}-\vec{y})\,.$$

The Laplace operator $\Delta$ is reversible, and, therefore, the condition (22) is satisfied.

It is now a simple task to write down the path integral in the longitudinal gauge. We start with the physical degree of freedom only

$$W \sim \int \mathcal{D}\vec{E}_\parallel\,\mathcal{D}\vec{\Pi}_\parallel \exp\left\{i\int dx\left(\vec{\Pi}_\parallel\frac{\partial}{\partial t}\vec{E}_\parallel - \tfrac{1}{2}\vec{\Pi}^2_\parallel - \tfrac{1}{2}\left(\mathrm{div}\,\vec{E}_\parallel\right)^2\right)\right\} \tag{24}$$

and we obtain as result

$$W \sim \int \mathcal{D}B^{\mu\nu}\delta\big(\operatorname{rot}\vec{E}\big)\delta\big(\operatorname{div}\vec{H}\big)\exp\left(i\int dx\big(-\tfrac{1}{2}G_\mu G^\mu\big)\right), \tag{25}$$

where $\mathcal{D}B^{\mu\nu} \sim \mathcal{D}\vec{E}\mathcal{D}\vec{H}$. While the first $\delta$-function represents the gauge condition, the second one arises to integrate over the complete vector $\vec{H}$, not only over its transversal part. The last formula can be rewritten in the form

$$W \sim \int \mathcal{D}B^{\mu\nu}\exp\left\{i\int dx\Big(-\tfrac{1}{2}(G_\mu G^\mu) + \tfrac{1}{2\alpha}\big(\operatorname{rot}\vec{E}\big)^2 + \tfrac{1}{2\beta}\big(\operatorname{div}\vec{H}\big)^2\Big)\right\} \tag{26}$$

and the longitudinal gauge corresponds to $\alpha \to 0$ and $\beta \to 0$ limits. This can be verified analyzing the field equations resulting from the new Lagrangian

$$\mathcal{L}_{\text{new}} = -\tfrac{1}{2}G_\mu G^\mu + \tfrac{1}{2\alpha}\big(\operatorname{rot}\vec{E}\big)^2 + \tfrac{1}{2\beta}\big(\operatorname{div}\vec{H}\big)^2. \tag{27}$$

To obtain the notoph propagators we follow the standard way:

1. We rewrite the action integral in the form

$$\int dx\mathcal{L}_{\text{new}} = \tfrac{1}{2}\int dx\big(E_i A_{ij} E_j + H_i D_{ij} H_j + E_i B_{ij} H_j + H_i C_{ij} E_j\big),$$

where

$$\begin{aligned} A_{ij} &= -\delta_{ij}\Box + \big(1+\tfrac{1}{\alpha}\big)\big(\partial_i\partial_j - \delta_{ij}\Delta\big), \\ C_{ij} &= -B_{ij} = \varepsilon_{ijk}\partial_k\partial_0, \\ D_{ij} &= -\delta_{ij}\Delta + \big(1+\tfrac{1}{\beta}\big)\partial_i\partial_j. \end{aligned}$$

2. We calculate the propagators using the formula

$$\left(\begin{array}{c|c} \mathcal{D}^{(E)}_{ij} & 0 \\ \hline 0 & \mathcal{D}^{(H)}_{ij} \end{array}\right) = \lim_{\substack{\alpha\to 0\\ \beta\to 0}} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1}.$$

In the momentum space we get

$$\mathcal{D}^{(E)}_{ij}(k) = \frac{k_i k_j}{|\vec{k}|^2}\frac{1}{k^2}, \tag{28a}$$

$$\mathcal{D}^{(H)}_{ij}(k) = \left(\delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}\right)\frac{1}{|\vec{k}|^2}, \tag{28b}$$

where $k^2 = \omega^2 - |\vec{k}|^2$.

Some comments on the form of the propagators are needed.

*(i)* If the coupling of the notoph has the form $\frac{1}{2}B_{\mu\nu}J^{\mu\nu}$, then the field equation

$$\partial^\mu G^\nu - \partial^\nu G^\mu = J^{\mu\nu}$$

leads to the conservation low for the current $J^{\mu\nu} = -J^{\nu\mu}$

$$\varepsilon^{\mu\nu\alpha\beta}\partial_\nu J_{\alpha\beta} = 0\,,$$

*i.e.*

$$\frac{\partial}{\partial t}\vec{j}^{(1)} - \mathrm{rot}\,\vec{j}^{(2)} = 0\,,$$
$$\mathrm{div}\,\vec{j}^{(1)} = 0$$

in the 3-dimensional notation $J^{0k} \to j_k^{(2)}$, $J^{ij} \to \varepsilon_{ijk}j_k^{(1)}$. Consequently, in the chosen Lorentz frame, in the longitudinal gauge, the notoph coupling is

$$\vec{H}_\perp \vec{j}^{(1)}_\perp + \vec{E}_\parallel \vec{j}^{(2)}_\parallel\,.$$

This is in agreement with Eqs (28) because adding the source term $\frac{1}{2}B_{\mu\nu}J^{\mu\nu}$ to the Lagrangian (27) we obtain the path integral

$$W[J^{\mu\nu}] = W[0]\exp\left\{-\tfrac{i}{2}\int dk[j_i^{(1)}(k)\mathcal{D}_{ij}^{(H)}(k)j_j^{(1)}(-k) + j_i^{(2)}(k)\mathcal{D}_{ij}^{(E)}j_j^{(2)}(-k)]\right\}\,.$$

*(ii)* While the free notoph has only one polarization degree of freedom (helicity 0 described by $\vec{E}_\parallel$), two additional degrees of freedom (helicities $\pm 1$ described by $\vec{H}_\perp$) arise in notoph interactions. The situation is complementary to that in the Maxwell theory: the free photon has two transversal polarization states, while in interactions the third polarization (helicity 0) arises. The fact that the same number of interacting polarization states arises in both cases is easy to understand: the Maxwell and notoph theories are zero mass limits of two equivalent theories of spin 1 [1].

## 5. Final remarks

The formulation of the notoph theory in the longitudinal gauge seems to be economic and clear, at least in the free case. Investigations of interacting notoph theories and their Lorentz covariance in the longitudinal gauge are being carried on.

We are grateful to Dr M. Majewski and particularly to Dr P. Kosiński for illuminating discussions.

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