EXTERNAL CURRENT IN THEORY OF NOTIVARG*

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The interaction of the notivarg with an external current is discussed. The "continuity" equations for the external current are obtained.

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There exist gauge theories of the scalar massless particle. The earliest one is the theory of the Ogievetsky–Polubarinov notoph [1]. Deser, Siegel and Townsend [2] and Tybor [3, 4] have given other examples of the free scalar gauge theory — the theory of the notivarg.

Let us recall some facts about the notoph:

(i) its description is complementary to the one of the photon: in the Maxwell theory the potential is the 4-vector A^μ and the strength tensor is the antisymmetric tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, while in the notoph theory we have the reverse situation: the potential is the antisymmetric tensor and the strength is the 4-vector;

(ii) as in every gauge theory, additional degrees of freedom (except the physical ones in the free case) are needed to describe interactions in the consistent way. In the case of the notoph except the scalar state two additional (transversal) degrees of freedom must be taken into account [5];

(iii) the theory of the selfinteracting notoph is equivalent to the nonlinear σ — model [6].

In comparison with the notoph theory, the theory of notivarg is on an opening stage: the classical descriptions of the free notivarg are known only [2, 4]. In the present paper we make the first step to investigate the notivarg interactions, namely we introduce the interaction of the notivarg

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with an external source. The conditions on the source protecting us from inconsistency of the theory are analysed.

Let us start with recalling the most essential points of the description of the free notivarg [3, 4]. The action integral has the form

$$I = \int d^4x \left(-\left(\partial_\sigma K^{\sigma\nu\alpha\beta}\right)^2 + \left(\partial_\sigma K^{\sigma\nu\alpha}{}_\nu\right)^2\right) \tag{1}$$

and the Lagrangian density reads

$$\mathcal{L} = -\left(\partial_\sigma K^{\sigma\nu\alpha\beta}\right)^2 + \left(\partial_\sigma K^{\sigma\nu\alpha}{}_\nu\right)^2 . \tag{2}$$

The 20-component field $K^{\mu\nu\alpha\beta}$ has the symmetry of the Riemann tensor, *i.e.* $K^{\mu\nu\alpha\beta} = K^{\alpha\beta\mu\nu} = -K^{\mu\nu\beta\alpha}$, $\varepsilon_{\mu\nu\alpha\beta}K^{\mu\nu\alpha\beta} = 0$.

The action (1) has been obtained (by the $m^2 \to 0$ limit) from the action describing a massive spin 2 particle with the help of the 4-th rank tensor [3, 4]. The description in the massive case is equivalent to the well known theory of Pauli and Fierz for a spin 2 particle.

The physical contents of the theory given by the action (1) have been analysed in Ref. [4] and it has been proved that there is only one physical degree of freedom: the state with helicity 0.

The action (1) is invariant under the gauge transformation

$$\begin{aligned}
\delta K^{\mu\nu\alpha\beta} &= \varepsilon^{\mu\nu\sigma\lambda}\varepsilon^{\alpha\beta\varphi\kappa}\partial_\sigma\partial_\varphi\omega_{\lambda\kappa} \\
&+ g^{\mu\alpha}\left(\partial^\nu\eta^\beta + \partial^\beta\eta^\nu\right) + g^{\nu\beta}\left(\partial^\mu\eta^\alpha + \partial^\alpha\eta^\mu\right) \\
&- g^{\mu\beta}\left(\partial^\nu\eta^\alpha + \partial^\alpha\eta^\nu\right) - g^{\nu\alpha}\left(\partial^\mu\eta^\beta + \partial^\beta\eta^\mu\right) \\
&- 2\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\right)\partial_\sigma\eta^\sigma ,
\end{aligned} \tag{3}$$

where $\omega_{\alpha\beta} = \omega_{\beta\alpha}$ and η_α are gauge tensors. Two remarks are necessary:

(i) not all components $\omega^{\alpha\beta}$ and η^α act effectively in the gauge transformation. It is connected with the invariance of Eq. (3) under the transformations

$$\delta\omega^{\alpha\beta} = \partial^\alpha\lambda^\beta + \partial^\beta\lambda^\alpha ,$$

$$\delta\omega^{\alpha\beta} = g^{\alpha\beta}\Lambda , \quad \delta\eta^\alpha = -\tfrac{1}{2}\partial^\alpha\Lambda ,$$

where λ^α and Λ are arbitrary functions;

(ii) in the papers [3, 4] we have discussed another form of the gauge transformation. Instead of the part with $\omega^{\alpha\beta}$, the transformation with the 3-rd rank tensor $\omega^{\alpha\beta\nu}$ has been used. The tensor $\omega^{\alpha\beta\nu}$ obeys the symmetry relations $\omega^{\alpha\beta\nu} = -\omega^{\beta\alpha\nu}$, $\varepsilon_{\mu\nu\alpha\beta}\omega^{\alpha\beta\nu} = 0$ and the constraint $\partial_\alpha\omega^{\alpha\beta\nu} = 0$. The gauge transformation (3) corresponds to the solution of this constraint.

While the action (1) is invariant, the Lagrangian (2) changes by 4 — divergence under the gauge transformation (3). So, the Lagrangian (2) cannot be written in terms of the strength tensor (see Appendix D)

$$H^{\mu\nu} = \tfrac{1}{2}\left(\varepsilon^{\mu}{}_{\sigma\alpha\beta}\partial^{\sigma}\partial_{\kappa}K^{\kappa\nu\alpha\beta} + \varepsilon^{\nu}{}_{\sigma\alpha\beta}\partial^{\sigma}\partial_{\kappa}K^{\kappa\mu\alpha\beta}\right) \tag{4}$$

only. The strength tensor (4) is invariant under the gauge transformation (3) and obeys the relations $H^{\mu\nu} = H^{\nu\mu}$, $H^{\mu}_{\mu} = 0$, $\partial_{\mu}H^{\mu\nu} = 0$. We see that the description of the free notivarg is complementary to the massless theory of Pauli and Fierz, where the symmetric tensor $h^{\mu\nu}$ is the potential and the (linearized) Riemann tensor

$$R^{\mu\nu\alpha\beta} = \tfrac{1}{2}\left(\partial^{\nu}\partial^{\alpha}h^{\mu\beta} + \partial^{\mu}\partial^{\beta}h^{\nu\alpha} - \partial^{\mu}\partial^{\alpha}h^{\nu\beta} - \partial^{\nu}\partial^{\beta}h^{\mu\alpha}\right)$$

(invariant under the gauge transformation $\delta h^{\mu\nu} = \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu}$) is the strength tensor.

We introduce the external source $j^{\mu\nu\alpha\beta}$ with the symmetry properties of a Riemann tensor. The standard form of the action is assumed

$$I = \int d^4x \left(-\left(\partial_{\mu}K^{\mu\nu\alpha\beta}\right)^2 + \left(\partial_{\mu}K^{\mu\beta\alpha}{}_{\beta}\right)^2 + \tfrac{1}{4}K^{\mu\nu\alpha\beta}j_{\mu\nu\alpha\beta}\right) . \tag{5}$$

In the case of the 20-component current $j^{\mu\nu\alpha\beta}$ the action (5) is the most general one (up to the transformation

$$\begin{aligned} j^{\mu\nu\alpha\beta} \to j^{\mu\nu\alpha\beta} &+ A\left(g^{\mu\alpha}j^{\nu\beta} + g^{\nu\beta}j^{\mu\alpha} - g^{\mu\beta}j^{\nu\alpha} - g^{\nu\alpha}j^{\mu\beta}\right) \\ &+ B\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\right)j , \end{aligned}$$

where $j^{\mu\alpha} \equiv j^{\mu\nu\alpha}{}_{\nu}$, $j \equiv j^{\mu}_{\mu}$ and $1 + 2A \neq 0$, $1 + 6A + 12B \neq 0$).

The action (5) is invariant under the gauge transformation (3) if the current $j^{\mu\nu\alpha\beta}$ obeys the following relations

$$\partial_{\mu}\partial_{\alpha}j^{\mu\nu\alpha\beta} = \Box j^{\nu\beta} - \tfrac{1}{2}\partial^{\nu}\partial^{\beta}j , \tag{6a}$$

$$\partial_{\mu}j^{\mu\nu} = \tfrac{1}{2}\partial^{\nu}j . \tag{6b}$$

It is not hard to prove that the field equations following from the action (5)

$$\begin{aligned} &\partial^{\mu}K^{\nu\alpha\beta} - \partial^{\nu}K^{\mu\alpha\beta} + \partial^{\alpha}K^{\beta\mu\nu} - \partial^{\beta}K^{\alpha\mu\nu} \\ &- \tfrac{1}{2}\left(g^{\mu\alpha}\left(\partial^{\nu}K^{\beta} + \partial^{\beta}K^{\nu}\right) + g^{\nu\beta}\left(\partial^{\mu}K^{\alpha} + \partial^{\alpha}K^{\mu}\right)\right. \\ &\left.-g^{\mu\beta}\left(\partial^{\nu}K^{\alpha} + \partial^{\alpha}K^{\nu}\right) - g^{\nu\alpha}\left(\partial^{\mu}K^{\beta} + \partial^{\beta}K^{\mu}\right)\right) = j^{\mu\nu\alpha\beta} \\ &\qquad\left(K^{\nu\alpha\beta} \equiv \partial_{\sigma}K^{\sigma\nu\alpha\beta} , \qquad K^{\alpha} \equiv K^{\nu\alpha}{}_{\nu}\right) \end{aligned} \tag{7}$$

are consistent if the relations (6) are fulfilled. We note that the conditions (6) restrict to 11 the number of the independent components of the current $j^{\mu\nu\alpha\beta}$.

Let us perform a reduction of the gauge freedom imposing a gauge condition. We must care for consistency of the gauge condition with the field equations. The gauge conditions must not be too strong. For example, the following ones [3, 4]

$$K^{\mu\nu\alpha}{}_{\nu} = 0\,, \quad \partial_\mu \partial_\alpha K^{\mu\nu\alpha\beta} = 0 \tag{8}$$

are consistent with the free equations, but they lead to the undesirable constraint $j^{\mu\nu} = 0$ on the current.

The gauge condition

$$K^{\mu\alpha} - \tfrac{1}{4} g^{\mu\alpha} K = 0 \tag{9}$$

(where $K^{\mu\alpha} \equiv K^{\mu\nu\alpha}{}_{\nu}$ and $K \equiv K^{\mu}_{\mu}$) is consistent with the field equation (7). It restricts to 11 the number of the independent components of the field $K^{\mu\nu\alpha\beta}$. In this gauge we can rewrite the field equations in the form

$$\begin{aligned}
\Box C^{\mu\nu\alpha\beta} &= j^{\mu\nu\alpha\beta} - \tfrac{1}{2}\left(g^{\mu\alpha} j^{\nu\beta} + g^{\nu\beta} j^{\mu\alpha} - g^{\mu\beta} j^{\nu\alpha} - g^{\nu\alpha} j^{\mu\beta}\right) \\
&\quad + \tfrac{1}{6}\left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}\right) j\,,
\end{aligned}$$

$$\Box K = -2j,$$

$$2\partial_\nu \partial_\beta C^{\mu\nu\alpha\beta} + \tfrac{1}{6}\left(\tfrac{1}{4} g^{\mu\alpha} \Box - \partial^\mu \partial^\alpha\right) K = j^{\mu\alpha} - \tfrac{1}{4} g^{\mu\alpha} j\,,$$

where $C^{\mu\nu\alpha\beta}$ is the Weyl part of $K^{\mu\nu\alpha\beta}$. (To prove that we use the dual properties of the tensor $K^{\mu\nu\alpha\beta}$ — see Appendix A). The gauge freedom is not removed completely but only restricted according to $\delta K^{\mu\alpha} = \frac{1}{4} g^{\mu\alpha} \delta K$.

Let us finish with the following remark. In the free case the physical degree of freedom is described by a component of the Weyl part of $K^{\mu\nu\alpha\beta}$. Therefore it is attractive to restrict the consideration to the Weyl current ($j^{\mu\nu} = 0$) only. The current conditions have now the form

$$\partial_\mu \partial_\alpha j^{\mu\nu\alpha\beta} = 0 \tag{10}$$

and the gauge conditions (8) do not contradict the equations of motion (7).

For further applications it is convenient to rewrite Eq. (10) in terms of current components. Using the decomposition of the Weyl tensor

$$j^{\mu\nu\alpha\beta} = (\tau^{ij},\ \sigma^{ij})\,,$$

where

$$\begin{aligned}
j^{0i0j} &= \tau^{ij}\,, \\
j^{0ijk} &= \varepsilon^{jkm} \sigma^{i}_{m}\,,
\end{aligned}$$

$$j^{mnab} = -\left(g^{ma}\tau^{nb} + g^{nb}\tau^{ma} - g^{mb}\tau^{na} - g^{na}\tau^{mb}\right)$$

and

$$\tau^{ij} = \tau^{ji}\,,\ \tau^{i}_{i} = 0\,,\ \sigma^{ij} = \sigma^{ji},\ \sigma^{i}_{i} = 0\,,\quad (i,\, m,\, a, = 1,\, 2,\, 3\,)$$

we get

$$\begin{aligned}
&\partial_m\partial_n\tau^{mn} = 0\,,\\
&\partial^0\partial_m\tau^{mi} - \varepsilon^{imk}\partial_m\partial_n\sigma^{n}_{k} = 0\,,\\
&\left((\partial^0)^2 + \Delta\right)\tau^{ij} + \partial^i\partial_m\tau^{mj} + \partial^j\partial_m\tau^{mi}\\
&\qquad\qquad -\partial^0\left(\varepsilon^{imk}\partial_m\sigma^{j}_{k} + \varepsilon^{jmk}\partial_m\sigma^{i}_{k}\right) = 0\,.
\end{aligned}$$

Using the helicity decomposition (Appendix C) we obtain

$$\begin{aligned}
&\tau_{\mathrm{L}} = 0\,,\\
&\partial^0\tau^{i}_{\mathrm{T}} - \varepsilon^{imk}\partial_m\sigma_{\mathrm{T}k} = 0\,,\\
&\left((\partial^0)^2 + \Delta\right)\tau^{ij}(\pm 2) - \partial^0\left(\varepsilon^{imk}\partial_m\sigma^{j}_{k}(\pm 2) + \varepsilon^{jmk}\partial_m\sigma^{i}_{k}(\pm 2)\right) = 0\,.
\end{aligned}$$

We observe that there is no restriction on σ_{L}. Inspecting the interaction term

$$\tfrac{1}{4}K^{\mu\nu\alpha\beta}j_{\mu\nu\alpha\beta} = \left(T^{ab} - K^{ab}\right)\tau_{ab} - 2\sigma_{ab}S^{ab}$$

we note that current σ_{L} is coupled to the field S_{L} (*i.e.* to the field describing the physical degree of freedom in the free case — see Appendix D).

The "continuity" equations (6) (or (10)) are necessary for hamiltonization and quantization of the theory. Our investigations in that directions will be presented in the next papers.

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APPENDIX A

The decomposition of the tensor $K^{\mu\nu\alpha\beta} = K^{\alpha\beta\mu\nu} = -K^{\mu\nu\beta\alpha}$ obeying $\varepsilon_{\mu\nu\alpha\beta}K^{\mu\nu\alpha\beta} = 0$ in the irreducible Lorentz parts is:

$$K^{\mu\nu\alpha\beta} = C^{\mu\nu\alpha\beta} + E^{\mu\nu\alpha\beta} + G^{\mu\nu\alpha\beta}\,,$$

where

$$\begin{aligned}
C^{\mu\nu\alpha\beta} &= K^{\mu\nu\alpha\beta} - \tfrac{1}{2}\left(g^{\mu\alpha}K^{\nu\beta} + g^{\nu\beta}K^{\mu\alpha} - g^{\mu\beta}K^{\nu\alpha} - g^{\nu\alpha}K^{\mu\beta}\right)\\
&\qquad + \tfrac{1}{6}\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\right)K\,,\\
E^{\mu\nu\alpha\beta} &= \tfrac{1}{2}\left(g^{\mu\alpha}K^{\nu\beta} + g^{\nu\beta}K^{\mu\alpha} - g^{\mu\beta}K^{\nu\alpha} - g^{\nu\alpha}K^{\mu\beta}\right)\\
&\qquad - \tfrac{1}{4}\left(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\right)K\,,\\
G^{\mu\nu\alpha\beta} &= \tfrac{1}{12}\left(g^{\mu\alpha}g^{\nu\beta} - g^{\beta\mu}g^{\nu\alpha}\right)K\,.
\end{aligned}$$

The dual properties of these parts are:

$$
\begin{aligned}
{}^{\sim}C^{\mu\nu\alpha\beta} &= C^{\sim\mu\nu\alpha\beta}\,,\\
{}^{\sim}E^{\mu\nu\alpha\beta} &= -\,E^{\sim\mu\nu\alpha\beta}\,,\\
{}^{\sim}G^{\mu\nu\alpha\beta} &= G^{\sim\mu\nu\alpha\beta}\,,
\end{aligned}
$$

where left-handed and right-handed dual tensors are respectively

$$
\begin{aligned}
{}^{\sim}A^{\mu\nu\alpha\beta} &\equiv \tfrac{1}{2}\varepsilon^{\mu\nu}{}_{\sigma\lambda}A^{\sigma\lambda\alpha\beta}\,,\\
A^{\sim\mu\nu\alpha\beta} &\equiv \tfrac{1}{2}\varepsilon^{\alpha\beta}{}_{\sigma\lambda}A^{\mu\nu\sigma\lambda}\,.
\end{aligned}
$$

APPENDIX B

Let us consider the tensor $K^{\mu\nu\alpha\beta} = K^{\alpha\beta\mu\nu} = -K^{\mu\nu\beta\alpha}$ obeying the condition $\varepsilon_{\mu\nu\alpha\beta}K^{\mu\nu\alpha\beta} = 0$. We introduce the new variables

$$
T^{ij} = T^{ji}\,,\ K^{ij} = K^{ji}\,,\ A^{i}\,,\ S^{ij} = S^{ji}\,,\ S^{i}_{i} = 0 \quad (i,\,j = 1,\,2,\,3)
$$

defined by

$$
\begin{aligned}
K^{0i0j} &= T^{ij}\,,\\
K^{mnij} &= \varepsilon^{mnk}\varepsilon^{ijl}K_{kl} + \tfrac{1}{2}\left(g^{mi}g^{nj} - g^{mj}g^{ni}\right)K^{l}_{l}\,,\\
K^{0ijk} &= \varepsilon^{jkm}S^{i}_{m} + g^{ij}A^{k} - g^{ik}A^{j}\,.
\end{aligned}
$$

So, we get the following decomposition of $K^{\mu\nu\alpha\beta}$

$$
K^{\mu\nu\alpha\beta} = \left(T^{ij}, K^{ij}\,, S^{ij}\,, A^{i}\right).
$$

APPENDIX C

The well known decomposition of a vector into transversal and longitudinal parts is

$$
V^{i} \equiv V^{i}_{\mathrm{T}} + V^{i}_{\mathrm{L}}\,,
$$

where

$$
V^{i}_{\mathrm{T}} = V^{i} + \tfrac{1}{\Delta}\partial^{i}\partial_{j}V^{j}\,,\quad V^{i}_{\mathrm{L}} = -\tfrac{1}{\Delta}\partial^{i}\partial_{j}V^{j}\,,\quad \Delta = -\partial_{i}\partial^{i}\,.
$$

The analogous decomposition of a symmetric traceless tensor a^{ij} is

$$
a^{ij} \equiv a^{ij}(\pm 2) + a^{ij}(\pm 1) + a^{ij}(0)\,,
$$

where

$$a^{ij}(\pm 1) = -\tfrac{1}{\Delta}\left(\partial^i a^j_{\mathrm{T}} + \partial^j a^i_{\mathrm{T}}\right),$$
$$a^{ij}(0) = \tfrac{3}{2}\left(\tfrac{1}{\Delta}\partial^i\partial^j + \tfrac{1}{3}g^{ij}\right)a_{\mathrm{L}},$$
$$a^i_{\mathrm{T}} = a^i + \tfrac{1}{\Delta}\partial^i\partial_j a^j, \quad a_{\mathrm{L}} = \tfrac{1}{\Delta}\partial_i a^i,$$
$$a^i = \partial_j a^{ji}.$$

APPENDIX D

To construct the strength tensor $H^{\mu\nu}$ (Eq. (4)) we have used the derivatives of the field $K^{\mu\nu\alpha\beta}$ of the lowest (as it is possible) order.

Using the duality properties of $K^{\mu\nu\alpha\beta}$ (Appendix A) we see that the strength tensor depends only on the Weyl part of the field $K^{\mu\nu\alpha\beta}$

$$H^{\mu\nu} = \varepsilon^{\mu}{}_{\sigma\alpha\beta}\partial^{\sigma}\partial_{\kappa}C^{\kappa\nu\alpha\beta}.$$

The components of $H^{\mu\nu}$ are ($i = 1, 2, 3$)

$$H^{00} = 2\partial^i\partial^j S_{ij},$$
$$H^{0i} = -2\partial^0\partial_m S^{mi} - \varepsilon^{imp}\partial_m\partial^s(T_{ps} - K_{ps}),$$
$$H^{ij} = 2\left[(\partial^0)^2 + \Delta\right]S^{ij} - 2g^{ij}\partial_m\partial_n S^{mn}$$
$$+2\left(\partial^i\partial_n S^{nj} + \partial^j\partial_n S^{ni}\right) + \partial^0\left[\varepsilon^{isb}\partial_s\left(T^j_b - K^j_b\right) + \varepsilon^{jsb}\partial_s\left(T^i_b - K^i_b\right)\right],$$

where the tensors S^{ij} and $T^{ij} - K^{ij} - \frac{1}{3}g^{ij}\left(T^k_k - K^k_k\right)$ are the components of the Weyl tensor $C^{\mu\nu\alpha\beta}$ (Appendix B).

If the gauge is fixed completely (for example, by imposing the noncovariant gauge conditions

$$T^{ij} - \tfrac{1}{3}g^{ij}T^m_m = 0, \quad \partial_m A^m = 0, \quad K^m_m = 0, \quad \varepsilon^{ilk}\partial_l S^j_k + \varepsilon^{jlk}\partial_l S^i_k = 0\,)$$

we obtain

$$H^{00} = 2\Delta S_{\mathrm{L}},$$
$$H^{0i} = 2\partial^0\partial^i S_{\mathrm{L}},$$
$$H^{ij} = (\partial^0)^2\left(g^{ij} + 3\tfrac{\partial^i\partial^j}{\Delta}\right)S_{\mathrm{L}} - \left(\partial^i\partial^j + g^{ij}\Delta\right)S_{\mathrm{L}},$$

where the field S_{L} is the helicity 0 part of S^{ij} (see Appendix C) and it describes the physical degree of freedom.

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