STATISTICAL PROPERTIES OF A DYNAMIC PROCESS PERTURBED BY A CHAOS-GENERATOR

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We investigate the problem of discrete in time regular perturbations affecting a deterministic process with a generic relaxation time. The effect on the moments of a kinetic process caused by chaotic changes in the deterministic forcing is calculated. Our discussion is carried out in two stages. First, direct calculations determining statistical properties of the process driven by a chaotic motion are presented. Second, the long time predictions of the model are analyzed in a perspective of a properly taken limit of continuous-time idealization of a regular forcing. Some comparisons are made with the known properties of continuous time stochastic processes.

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1. Introduction

Mechanical processes are known to posses domains of a "chaotic motion" which may influence other mechanical processes, especially in complex many-body systems. At the level of the theories of the Generalized Langevin Equation [1] or the Exact Master Equation [2,3], one can derive evolution equations for relevant mechanical variables (or "low observables" projected from the relevant part of the density matrix) which take forms of nonuniform differential equations. An extra nonuniform term appearing there comes from averaging over "fast variables" and has usually an interpretation of a thermal noise [3]. It would be therefore natural to ask whether a chaotic character of underlying mechanical processes can imply a "stochastic" (or noisy) behaviour of a physical system at hand. When does a chaotic motion become a source of a random behaviour and under what conditions its influence on the dynamics can be approximated by noise-generators? What are possible routes of unification of mechanics and the theory of stochastic processes [4]? A different rationale for these studies is the concept of thermal bath in the theory of open systems. The bath is usually described as a stochastic set of harmonic oscillators. A possible, and maybe more natural, extension of this description would be a set of coupled nonlinear oscillators with a chaotic behaviour. As the investigation of influence of a chaos generated by such a system is rather cumbersome, one can simplify the problem by using, instead of physical oscillators, model generators of nonlinear behaviour. An example can be a logistic map or its topological conjugate (also, an appropriate set of the maps) [5,6].

The above way of reasoning relates closely to the problem of ergodicity of dynamic systems [7]. Further, we will concern mostly "semi-ergodic hypothesis" by studying trajectories of non-ergodic systems influenced by noise and investigating their asymptotic properties (non-ergodic trajectories may become ergodic after a noise being imposed on the dynamic system [8–10]).

In this paper we try to go deeper in the understanding of non-ergodic and noise-like behaviours of chaotic generators. Our interest will be devoted to investigation of non-ergodic and stochastic behaviours of logistic mapping and linear systems perturbed by chaotic generators of this type.

Section 2 discusses ergodic *vs* non-ergodic properties of the logistic map in the range of either fully, or partially developed chaos. Statistical properties of the transformation are studied in terms of correlation functions. In particular, non-ergodic behaviour of the logistic transformation is investigated in the range of $3 \leq A \leq 3.449\ldots$ (this choice of A produces, by a pitchfork bifurcation, two stable fixed points x_1, x_2 which form an attractor of period two). Some numerical results are also presented for the range of A-values which lead to subsequent period-doubling bifurcations of the map.

Section 3 is devoted to statistical analysis of the linear system perturbed by an additive term representing a chaos-generator. Long time behaviour of the system is compared with expectations calculated by ensemble averaging. As a natural extension of these studies, we present results for the linear system driven by a "shot chaos" (Section 4).

Further complicacy of the asymptotic behaviours comes from multiplicative perturbations imposed on the system driven by, otherwise fully "deterministic", forcing. Section 5 discusses time-evolution of such systems by use of partial differential equations of the Fokker-Planck type.

2. Non-ergodic behaviour and stochastic properties of logistic mapping

Time evolution of dynamic systems depends frequently on externally perturbed model parameters. In the variety of recent studies [8–11] it has been observed that inclusion of either chaotic or stochastic perturbations of these parameters can induce quantitatively new phenomena. Different concepts [12] have been used to distinguish between perturbation-induced chaotic and stochastic behaviours. The possibility of the existence of fractal attractors or (and) diffusive motions has been pointed out [13] by studying the sensitivity of a given dynamics to the strength, frequency and time correlations of external perturbations. It has been shown [14], that in the limit when the dynamics of the perturbations converges to a stationary Gaussian diffusion process, the long-time behaviour of the dynamic system coincides with the predictions afforded by a stochastic analysis. Particular attention has been focused on the class of phenomena induced by the presence of regular perturbations created by the mapping:

$$y_{n+1} = 2y^2 - 1\,, \tag{2.1}$$

$$\alpha_{n+1} = \lambda\alpha_n + \tau_{1/2}y_{n+1} \qquad y \in [-1,1]\,. \tag{2.2}$$

Here τ, $\lambda = \exp(-\gamma\tau)$ are positive constants and the logistic transformation may be substituted by any other mapping conjugated to the Bernoulli shift. Decreasing τ the system (2.1)–(2.2) exhibits a transition from non-Gaussian chaotic to Gaussian stochastic behaviour [15] provided the mapping in (2.1) has the so called φ-mixing property. It has been rigorously proved [14] that for $\tau \to 0$ and $Ty = 2y^2 - 1$ a "classical" stochastic process is generated in the α-variable, namely the Ornstein-Uhlenbeck velocity process [16]. This transition has been recently studied [17] in terms of r-point correlation functions. The authors have developed a systematic method to calculate correlation functions of arbitrary order for the class of systems characterized by the Bernoulli-shift. The method allows to analyze meaningful differences in the structure of r-point correlation functions when the dynamics of the system evolves from chaotic to a stochastic-Gaussian behaviour.

In our analysis we use an equivalent form of T (2.1) which is:

$$x_{n+1} = Ax_n(1 - x_n)\,, \qquad x \in (0,1)\,. \tag{2.3}$$

For $A = 4.0$, the above map is ergodic (even mixing, [5]) and there exists the absolute continuous invariant density $\varrho(x)$,

$$\varrho(x) = \frac{1}{\pi\sqrt{x(1-x)}}\,, \qquad \int_0^1 \varrho(x)\,dx = 1\,. \tag{2.4}$$

Given a map T, a phase space X, the ergodic invariant measure $\varrho(x)$ and a test function $f(x)$, an ensemble average $\langle f(x)\rangle$ is defined as

$$\langle f(x)\rangle = \int_0^1 f(x)\varrho(x)\,dx\,. \tag{2.5}$$

From (2.5) one easily gets *e.g.* :

$$\langle x^m \rangle = \frac{2m-1}{2m} \langle x^{m-1} \rangle = \frac{(2m-1)!!}{(2m)!!}, \tag{2.6}$$

$$\langle x[x(1-x)]^m \rangle = \frac{2m-1}{8m} \langle x[x(1-x)]^{m-1} \rangle . \tag{2.7}$$

We further define correlation functions:

$$g(n, n+m) = \langle x(x_n - \langle x \rangle)(x_{n+m} - \langle x \rangle) \rangle = \langle x_n x_{n+m} \rangle - \tfrac{1}{4} . \tag{2.8}$$

In the case of a stationary dynamic process, $g(n, n+m) = g(m)$. The evolution generated by a logistic map T in the domain of "fully developed chaos" possesses this property. In fact, for $A = 4.0$, $g(m)$ is trivially zero[1] as

$$\langle x_n x_{n+m} \rangle = \tfrac{1}{4} . \tag{2.9}$$

This proves authomatically mixing-property[2] of the logistic map [5,6,18] for $A = 4.0$.

Existence of $\varrho(x)$ for $A = 4.0$ means that distribution of any set of initial points $\{x_0\}$ leads to the distribution of $\{x_n\}$ predicted by $\varrho(x)$, so that averaging with the invariant measure $\varrho(x)$ yields ensemble-averaging. On the other hand, $\varrho(x)$ can be obtained directly from a rigorous "time-solution" of (from iterates of) a trajectory $x_n(x_0)$ with a given starting value x_0. This implies, in turn, that averaging with $\varrho(x)$ is effectively equivalent to the time average and

$$\langle f(x) \rangle = \overline{f(x)} . \tag{2.10}$$

All of the above statements need not to be true for $A \leq 4$. In fact, for $A \leq 4$, the process (2.3) is not ergodic as a trajectory $\{x_0, x_1, \ldots, x_n, \ldots\}$ does not cover the whole phase space $(0,1)$. In the region of regular dynamics all trajectories become attracted by either one stable fixed point or by a finite family of cycles emerging by the "period-doubling" transformation. In the chaotic domain, the trajectory covers the domain of a fractal dimension. Moreover, the logistic process is not then ergodic even in a "weaker sense", *i.e.*

$$\langle f(x) \rangle \neq \overline{f(x)} , \tag{2.11}$$

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[1] We refer to the paper of Beck [17], where it has been shown that with T defined by (2.1), $\langle y_n y_m \rangle = 0$ for $n \neq m$.

[2] A map T is called mixing if for every pair of measurable sets A and B

$$\lim_{n \to \infty} \varrho[T^n(A) \cap B] = \varrho(A)\varrho(B)$$

which requires correlation functions decay to zero in the infinite time limit.

what can be easily shown numerically, especially in the region of two-cycles.[3]

2.1. Statistical properites of logistic map in the region of two-cycles

In the domain of appearence of a two-cycle, we get asymptotically (*i.e.* for a stationary process):

$$x_\infty = \begin{cases} x_1 = \bar{x} - \alpha\,, \\ x_2 = \bar{x} + \alpha\,, \end{cases} \tag{2.12}$$

and

$$\bar{x} = \frac{A+1}{2A}\,, \qquad \alpha^2 = \frac{(A+1)(A-3)}{4A^2}\,. \tag{2.13}$$

Let us assume conventionally an initial value $x_{-\infty}$, so that asymptotic dynamics leads to a set $\{x_0, x_1, \ldots, x_n, \ldots\}$ and, in particular, for a two-cycle:

$$x_{2n} = \begin{cases} x_1\,, & \text{with a frequency} \quad \nu_1 \equiv \nu\,, \\ x_2\,, & \text{with a frequency} \quad \nu_2 \equiv 1-\nu\,, \end{cases} \tag{2.14a}$$

and

$$x_{2n+1} = \begin{cases} x_1\,, & \text{with a frequency} \quad \nu_2\,, \\ x_2\,, & \text{with a frequency} \quad \nu_1\,. \end{cases} \tag{2.14b}$$

From the above formula, it can be easily shown that:

$$\langle x_{2n} \rangle = \nu_1 x_1 + \nu_2 x_2\,, \qquad \langle x_{2n+1} \rangle = \nu_2 x_1 + \nu_1 x_2\,. \tag{2.15}$$

And the appropriate values of the correlation functions are:

$$\langle x_{2n_1} x_{2n_2} \ldots x_{2n_k} \rangle = \nu_1 x_1^k + \nu_2 x_2^k \equiv \langle x_{2n}^k \rangle\,,$$

$$\langle x_{2n_1+1} x_{2n_2+1} \ldots x_{2n_k+1} \rangle = \nu_2 x_1^k + \nu_1 x_2^k \equiv \langle x_{2n+1}^k \rangle\,. \tag{2.16}$$

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[3] As the logistic transformation is an example of a "phase-space shrinking" process, it cannot be strictly ergodic (except from the case of $A = 4$). This would suggest studying its ergodicity in a weaker sense:

$$\langle f(x) \rangle \to \overline{f(x)} \qquad \text{asymptotically.}$$

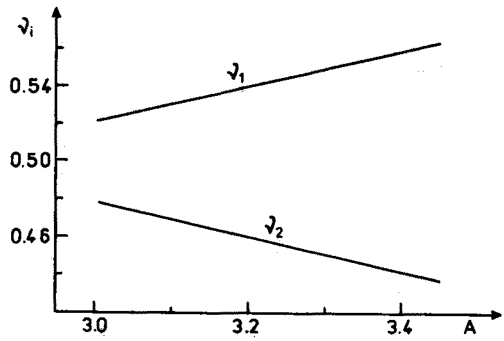

Fig. 1. Two-Cycle. A-dependence of ν_i, a relative frequency of occurrence of a stationary state x_i after an even number of steps. Iterations have been done for $N = 10^4$ different values of initial points x_0.

In particular,

$$\langle x_{2n} x_{2m+1} \rangle = x_1 x_2 \,,$$

$$\langle x_{2n_1} \dots x_{2n_k} x_{2m_1+1} \dots x_{2m_l+1} \rangle = \nu_1 x_1^k x_2^l + \nu_2 x_2^k x_1^l \,. \tag{2.17}$$

Furthermore, if $\tilde{x} = x - \langle x \rangle$, one gets:

$$\langle \tilde{x} \rangle = 0 \,, \qquad \langle \tilde{x}_n^2 \rangle = \nu(1-\nu)(x_1 - x_2)^2 \,,$$

$$\langle \tilde{x}_{2n} \tilde{x}_{2n+1} \rangle = \langle x_{2n} x_{2n+1} \rangle - \langle x_{2n} \rangle \langle x_{2n+1} \rangle = -\nu(1-\nu)(x_1 - x_2)^2 \,, \tag{2.18}$$

where for each pair of n, m

$$\tilde{x}_{2n} + \tilde{x}_{2m+1} \equiv 0 \,. \tag{2.19}$$

In order to achieve

$$\langle x \rangle = \overline{x} = \tfrac{1}{2}(x_1 + x_2) \,, \tag{2.20}$$

i.e. for the weak (asymptotic) ergodicity condition to be fulfilled, the frequency of realization in the n-th step any of the values x_1, x_2 has to be the same ($= \frac{1}{2}$) when starting with various initial values x_0.

Numerical calculations display discrepancy to the requirement (2.20), (*c.f.* Fig. 1). Iterations of the trajectories have been started from a set (10^4 points) of various $\{x_0\}$, distributed either regularily or randomly over the interval $(0,1)$. We have calculated numbers of trajectories $N(x_0 \to x_i)$ which after n steps of iteration have occupied x_1 (x_2, respectively):

$$\nu_i = \frac{N(x_0 \to x_i)}{N}. \tag{2.21}$$

It can be seen that lower states ($\nu_1 > \nu_2$, $x_1 < x_2$) are more frequently attracting the trajectories, so that the ensemble average of any quantity $f(x)$ will asymptotically approach

$$\langle f(x) \rangle = \nu_1 f(x_1) + \nu_2 f(x_2), \tag{2.22}$$

wheras its time-average will be

$$\overline{f(x)} = \frac{1}{2} f(x_1) + \frac{1}{2} f(x_2). \tag{2.23}$$

The difference between $\langle f \rangle$ and $\overline{f}$ can be easily estimated by direct use of[4] ν_i:

$$\delta = \left| \frac{\langle x \rangle - \overline{x}}{\overline{x}} \right| = \frac{(1 - 2\nu_2)(x_2 - x_1)}{x_1 + x_2}. \tag{2.24}$$

The same tendency is observed for other multicycles. Fig. 2 displays relevant frequencies for the 8-cycle ($A = 3.550$); see also Fig. 4. In the chaotic region, differences between $\langle x \rangle$ and $\overline{x}$ become less prominent. We have constructed a coarse-grained type density $\tilde{\varrho}(x)$ (*c.f.* Fig. 3, $A = 3.7$) which serves as a measure of time the system spends in a direct neighbourhood of a given point. To get the histogram of $\tilde{\varrho}(x)$, we have started from the set of $\{x_0\}$ ($N \sim 10^4$ points regularly or randomly distributed over $(0, 1)$) and calculated the number of trajectories which after m steps fall into the interval $[m\,dx, (m+1)\,dx)$, where[5] $dx = 0.01$.

Before closing up this section, let us summarize statistical properties of quantities that can be calculated numerically for the logistic mapping with some (finite) accuracy. Dispersion of the correlation function $g(m)$ is defined as:

$$\sigma^2_{g(m)} = \left\langle (z_n z_{n+m})^2 \right\rangle - \left\langle z_n z_{n+m} \right\rangle^2, \tag{2.25}$$

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[4] The estimate of δ gives values in the range $0.18\% \sim 3.6\%$.

[5] The solid line in Fig. 3 has been drawn through centers of the histogram, the dashed line represents the shape of $\varrho(x)$ for $A = 4.0$.

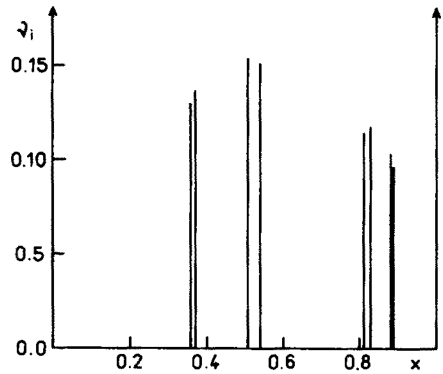

Fig. 2. Eight-Cycle. Intensity of occurrence of various stationary states. $N = 10^4$ randomly sampled initial points have been chosen for iteration.

where $z = x - \langle x \rangle$. For a fully developed chaos, one finds $\sigma^2_{g(1)} = (\sigma^2_x)^2 = 1/64$, which prompts that relatively big fluctuations can be met in numerical calculations of the correlation functions for $A < 4.0$.

For a two-cycle, a dispersion of the time-average $\overline{x}$ is

$$\sigma^2_{\overline{x}} = \alpha^2 = \frac{(A+1)(A-3)}{4A^2}, \tag{2.26}$$

whereas a dispersion of the ensemble average $\langle x \rangle$ gives:

$$\sigma^2_{\langle x \rangle} = 4\nu_1\nu_2\alpha^2 = \langle z_n z_{n+2} \rangle . \tag{2.27}$$

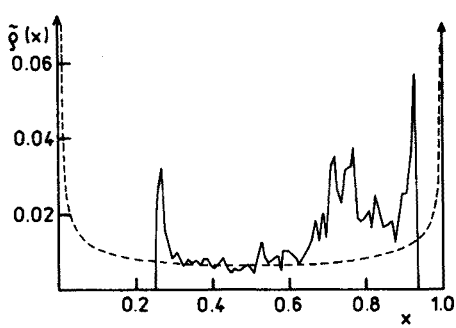

Fig. 3. Coarse-grained density $\tilde{\varrho}(x)$ (*c.f.* the text) calculated for the dynamic system (2.3) with $A = 3.7$. Solid line links centers of the histogram, the dashed line depicts the form of the invariant measure $\varrho(x)$ for $A = 4.0$.

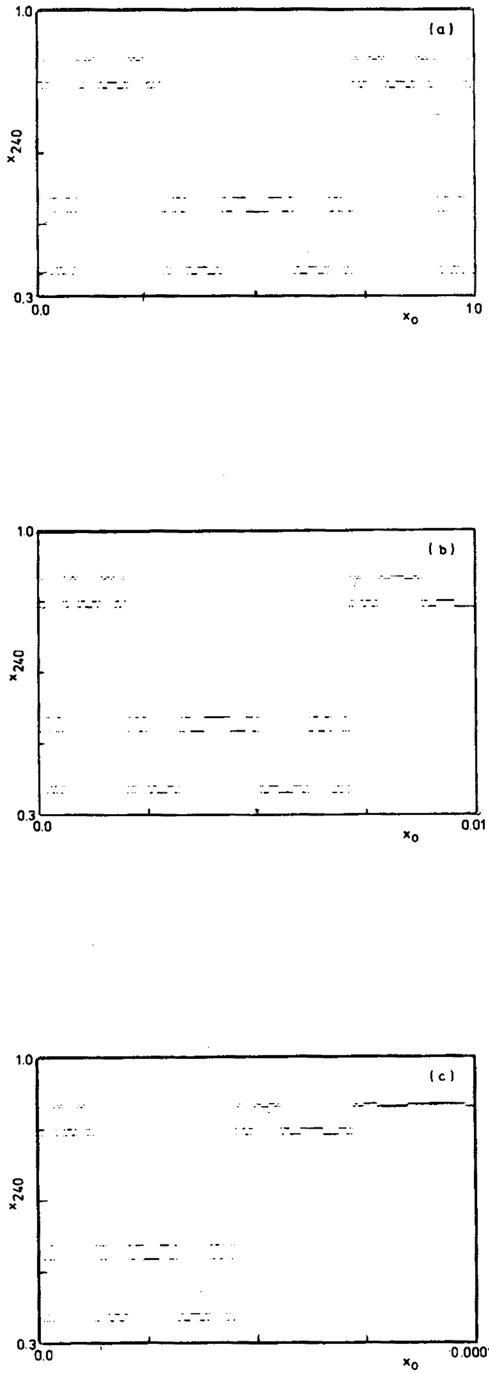

Fig. 4. "Basins of attraction" for the eight-cycle ($A = 3.550$). To study their "quasi-fractal" structure, we have refined the region of starting points x_0.

For the correlation functions one gets:

$$\sigma^2_{g(1)} = 0 \tag{2.28}$$

and

$$\sigma^2_{g(2)} = 16\nu_1\nu_2(\nu_1 - \nu_2)^2\alpha^4 , \tag{2.29}$$

which for a uniform distribution of the stationary states $\nu_1 = \nu_2 = 1/2$ gives identically zero. In conclusion, the chaotic regime of the dynamical system (2.3) exhibits features of a "random-noise generator" only for a particular value of $A = 4.0$. Other systems conjugated to the Bernoulli shift could be chosen as potential noise-generators, but as it has been proved recently [17,18], the mapping $Tx = 4x(1 - x)$ has a very special status among all of them. Strong properties of randomness possessed by this map can be viewed as the consequence of the fact that the system has less higher order correlations than any other generic system conjugated to the Bernoulli shift.

Chaos and noise-like behaviour can be also generated in systems driven by a delayed force [19]. If the delayed force is chosen to be periodic with respect to the amplitude of the motion and the period is much smaller than the response time of the system, the chaotic solutions are shown to be noise-like, *i.e.* they have statistical properties similar to those of random chaos resulting from nonlinear Langevin equations. Large delay in the periodic feedback is known to generate random chaos, but only large intensity of the feedback produces Gaussian noise [20].

Due to richness in variety of dynamic systems realized in nature all of the phenomena mentioned above can be displayed by physical dynamic processes. It seems thus plausible to investigate closely averages of quantities whose time evolution is governed by equations leading to both regular and chaotic dynamics.

3. Chaos-driven linear process

Let us discuss a linear process perturbed by an additive "noise" χ represented by a chaos generator:

$$\dot{z}(t) \equiv \frac{dz}{dt} = -\gamma z(t) + \lambda x(t) , \tag{3.1}$$

where evolution of $x(t)$ is given by a logistic mapping (2.3) and

$$x(t) = x_n \quad \text{for} \quad n\Delta < t < (n+1)\Delta . \tag{3.2}$$

As it stands, the process $\{z(t)\}$ resembles closely an Ornstein-Uhlenbeck process (except that χ is not of the standard form given by time-derivative of the Wiener process [16]) extensively used in the theory of Brownian motion. Our task will be to reconsider all of the characteristic features of this "Ornstein-Uhlenbeck" process.

First, let us note that the trajectory $z(t)$ is given by:

$$z(t) = \exp\{-\gamma(t - n\Delta)\}(z_n - \lambda\frac{x_n}{\gamma}) + \lambda\frac{x_n}{\gamma}, \qquad n\Delta < t < (n+1)\Delta, \tag{3.3}$$

so that

$$z_n = G^n z_0 + B\sum_{l=0}^{n-1} G^{n-l-1} x_l, \tag{3.4a}$$

$$G = \exp\{-\gamma\Delta\} < 1, \qquad B = \frac{\lambda}{\gamma}(1 - \exp\{-\gamma\Delta\}). \tag{3.4b}$$

To proceed further, we assume that the process $\{z(t)\}$ starts from a given fixed value z_0 and is perturbed by a stationary noise[6] (3.2). Ensemble averages of the process (taken over all realizations of the path) are then expressed by:

$$\langle z(t)\rangle = \exp\{-\gamma(t - n\Delta)\}\langle z_n\rangle + \frac{\lambda}{\gamma}[1 - \exp\{-\gamma(t - n\Delta)\}]\langle x_n\rangle, \tag{3.5}$$

$$\begin{aligned}\langle z^2(t)\rangle = {} & \exp\{-2\gamma(t - n\Delta)\}\langle z_n^2\rangle \\ & + 2\frac{\lambda}{\gamma}\exp\{-\gamma(t - n\Delta)\}[1 - \exp\{-\gamma(t - n\Delta)\}]\langle z_n x_n\rangle \\ & + \left(\frac{\lambda}{\gamma}\right)^2[1 - \exp\{-\gamma(t - n\Delta)\}]^2\langle x_n^2\rangle,\end{aligned} \tag{3.6}$$

$$\begin{aligned}\langle z(t)z(t+\tau)\rangle = {} & \exp\{-\gamma\Theta_1\}\exp\{-\gamma\Theta_2\}\langle z_n z_{n+m}\rangle \\ & + \frac{\lambda}{\gamma}\exp\{-\gamma\Theta_1\}(1 - \exp\{-\gamma\Theta_2\})\langle z_n x_{n+m}\rangle \\ & + \frac{\lambda}{\gamma}\exp\{-\gamma\Theta_2\}(1 - \exp\{-\gamma\Theta_1\})\langle z_{n+m} x_n\rangle \\ & + \left(\frac{\lambda}{\gamma}\right)^2(1 - \exp\{-\gamma\Theta_1\})(1 - \exp\{-\gamma\Theta_2\})\langle x_n x_{n+m}\rangle,\end{aligned} \tag{3.7}$$

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[6] For this requirement to be met, the process $\{x(t)\}$ has to be started at $t \to -\infty$ and $\langle x_0\rangle = \langle x_n\rangle$.

where

$$t = n\Delta + \Theta_1 , \qquad t + \tau = (n+m)\Delta + \Theta_2 , \qquad m \geq 0 . \tag{3.8}$$

In the case of a fully developed chaos, from (3.4) one gets:

$$\langle z_n \rangle = G^n z_0 + \tfrac{1}{2} B \sum_{l=0}^{n-1} G^{n-l-1} = G^n z_0 + \frac{\gamma}{2\lambda} (1 - G^n) , \tag{3.9a}$$

$$\langle z_n^2 \rangle = \langle z_n \rangle^2 + \tfrac{1}{8} \left(\frac{\lambda}{\gamma}\right)^2 \left(1 - G^{2n}\right) , \tag{3.9b}$$

$$\langle z_n x_{n+m} \rangle = \langle z_n x_n \rangle = \tfrac{1}{2} G^n z_0 + \frac{\lambda}{4\gamma} (1 - G^n) , \tag{3.9c}$$

$$\langle z_{n+m} x_n \rangle = \tfrac{1}{2} G^{n+m} z_0 + \frac{\lambda}{4\gamma} \left(1 - G^{n+m}\right) + \tfrac{1}{8} B G^{m-1} , \tag{3.9d}$$

$$\begin{aligned} \langle z_n z_{n+m} \rangle = {} & G^{2n+m} z_0^2 + z_0 G^{n+m} \frac{\lambda}{2\gamma} (1 - G^n) \\ & + z_0 \frac{\lambda}{2\gamma} G^n \left(1 - G^{n+m}\right) + \tfrac{1}{4} (1 - G^n) \left(1 - G^{n+m}\right) \left(\frac{\lambda}{\gamma}\right)^2 \\ & + \tfrac{1}{8} \left(\frac{\lambda}{\gamma}\right)^2 G^m \left(1 - G^{2n}\right) . \end{aligned} \tag{3.9e}$$

From the above formula and an explicit form of G (*c.f.* (3.4b)) it can be easily seen that $z(t)$ are exponentially correlated (so that $\{z(t)\}$ can model a non-white noise).

Direct evaluation of the dispersion yields:

$$\sigma_{z_n}^2 = \tfrac{1}{8} \left(\frac{\lambda}{\gamma}\right)^2 (1 - G^{2n}) = \left(\frac{\lambda}{\gamma}\right)^2 \left(1 - G^{2n}\right) \sigma_x^2 . \tag{3.10}$$

From (3.5) and (3.6) we deduce that

$$\sigma_{z(t)}^2 = \exp\{-2\gamma(t - n\Delta)\} \sigma_{z_n}^2 + \left(\frac{\lambda}{\gamma}\right)^2 [1 - \exp\{-\gamma(t - n\Delta)\}]^2 \sigma_x^2 , \tag{3.11}$$

which allows to evaluate dispersion of the trajectory at any instant of time.

Long-time limit leads to the following formula for the averages:

$$\langle z_n \rangle \to \frac{\lambda}{\gamma} \langle x_n \rangle = \frac{\lambda}{2\gamma}, \tag{3.12a}$$

$$\langle z_n^2 \rangle \to \left(\frac{\lambda}{\gamma}\right)^2 \langle x_n^2 \rangle, \tag{3.12b}$$

$$\langle z_n x_n \rangle = \langle z_n x_{n+m} \rangle \to \frac{\lambda}{\gamma} \langle x_n \rangle^2 = \frac{\lambda}{4\gamma}, \tag{3.12c}$$

$$\langle x_n z_{n+m} \rangle \to \frac{\lambda}{4\gamma} \left[1 + \tfrac{1}{2}(1-G)G^{m-1}\right], \tag{3.12d}$$

$$\langle z_n z_{n+m} \rangle \to \tfrac{1}{4} \left(\frac{\lambda}{\gamma}\right)^2 \left(1 + \tfrac{1}{2}G^m\right), \tag{3.12e}$$

$$\sigma_{z_n}^2 \to \left(\frac{\lambda}{\gamma}\right)^2 \sigma_x^2, \tag{3.12f}$$

$$\sigma_{z(t)}^2 \to \left[\exp\{-2\gamma\Theta\} + (1 - \exp\{-\gamma\Theta\})^2\right] \left(\frac{\lambda}{\gamma}\right)^2 \sigma_x^2, \tag{3.12g}$$

with $\Theta = t - n\Delta$. (Note that for $t \to \infty$, $n \to \infty$, $t - n\Delta$ remains finite and time-dependent quantities become periodic: $\Theta = t - \mathrm{Ent}\,(t/\Delta) = \Delta\,\mathrm{Frac}(t/\Delta)$, $\Theta \in [0, \Delta]$). By repeating the same type of calculations for $z(t)$, we arrive at:

$$\langle z(t) \rangle \to \frac{\lambda}{\gamma} \langle x \rangle, \tag{3.13a}$$

$$\langle z^2(t) \rangle \to \tfrac{1}{8} \left(\frac{\lambda}{\gamma}\right)^2 (3 - 2\exp\{-\gamma\Theta\} + 2\exp\{-2\gamma\Theta\}), \tag{3.13b}$$

$$\langle z(t)z(t+\tau) \rangle \to \tfrac{1}{4} \left(\frac{\lambda}{\gamma}\right)^2 \Big\{1 + \tfrac{1}{2}G^{m-1}\exp\{-\gamma\Theta_2\} \times \left[G\exp\{-\gamma\Theta_1\} + (1-G)\big(1 - \exp\{-\gamma\Theta_1\}\big)\right]\Big\}. \tag{3.13c}$$

If the transformation (2.3) enters (3.1) with A covering the range of partial chaos (*i.e.* $3.57\ldots < A < 4.0$), the above presented results can be reproduced only *via* numerical evaluation. We can, however, estimate averages of interest in case when the noise source is given by a chaos-generator (2.3) activated in the region of two-cycles ($3.0 < A < 3.449\ldots$):

$$\langle z_{2n}\rangle = G^{2n}z_0 + \left[G\left(\nu_1 x_1 + \nu_2 x_2\right) + \left(\nu_2 x_1 + \nu_1 x_2\right)\right] B\frac{1-G^{2n}}{1-G^2}\,, \tag{3.14a}$$

$$\langle z_{2n+1}\rangle = G^{2n+1}z_0 + B\left\{\langle x_{2n}\rangle + G\left(G\langle x_{2n}\rangle + \langle x_{2n+1}\rangle\right)\frac{1-G^{2n}}{1-G^2}\right\}, \tag{3.14b}$$

$$\begin{aligned}\left\langle z_{2n}^2\right\rangle &= \left(G^{2n}z_0\right)^2 + 2G^{2n}z_0\left(\langle z_{2n}\rangle - G^{2n}z_0\right)\\ &+ B^2\left(\frac{1-G^{2n}}{1-G^2}\right)^2\left(G^2\left\langle x_{2n}^2\right\rangle + 2G\langle x_1x_2\rangle + \left\langle x_{2n+1}^2\right\rangle\right),\end{aligned} \tag{3.14c}$$

$$\begin{aligned}\left\langle z_{2n+1}^2\right\rangle &= \left(G^{2n+1}z_0\right)^2 + 2G^{2n+1}z_0\left(\langle z_{2n+1}\rangle - G^{2n}z_0\right)\\ &\times B^2\left\{\left(\frac{1-G^{2n+2}}{1-G^2}\right)^2\left\langle x_{2n}^2\right\rangle + 2G\frac{(1-G^{2n+2})(1-G^{2n})}{(1-G^2)^2}\langle x_1x_2\rangle\right.\\ &\left. + G^2\left(\frac{1-G^{2n}}{1-G^2}\right)^2\left\langle x_{2n+1}^2\right\rangle\right\},\end{aligned} \tag{3.14d}$$

$$\langle z_{2n}x_{2m}\rangle = G^{2n}z_0\langle x_{2m}\rangle + B\frac{1-G^{2n}}{1-G^2}\left(G\left\langle x_{2n}^2\right\rangle + \langle x_1x_2\rangle\right), \tag{3.14e}$$

$$\langle z_{2n}x_{2m+1}\rangle = G^{2n}z_0\langle x_{2m+1}\rangle + B\frac{1-G^{2n}}{1-G^2}\left(G\langle x_1x_2\rangle + \left\langle x_{2n+1}^2\right\rangle\right), \tag{3.14f}$$

and

$$\sum_{l=0}^{2n-1} G^{2n-1-l}x_l = \sum_{j=0}^{n-1}(G^2)^{n-j-1}(Gx_{2j} + x_{2j+1})\,, \tag{3.15a}$$

$$\sum_{l=0}^{2n} G^{2n-l}x_l = \sum_{j=0}^{n}(G^2)^{n-j}x_{2j} + G\sum_{j=0}^{n-1}G^{2(n-j-1)}x_{2j+1}\,. \tag{3.15b}$$

Similarly, we obtain

$$\begin{aligned}\left\langle z_{2n+1} x_{2m+1}\right\rangle &= G^{2n+1} z_0 \left\langle x_{2n+1}\right\rangle \\ &+ \frac{B}{(1-G^2)}\left\{\left(1-G^{2n+2}\right)\left\langle x_1 x_2\right\rangle + G\left(1-G^{2n}\right)\left\langle x_{2n+1}\right\rangle^2\right\},\end{aligned} \tag{3.16a}$$

$$\begin{aligned}\left\langle z_{2n+1} x_{2m}\right\rangle &= G^{2n+1} z_0 \left\langle x_{2m}\right\rangle \\ &+ \frac{B}{(1-G^2)}\left\{\left(1-G^{2n+2}\right)\left\langle x_{2m}^2\right\rangle + G\left(1-G^{2n}\right)\left\langle x_1 x_2\right\rangle\right\},\end{aligned} \tag{3.16b}$$

$$\begin{aligned}\left\langle z_{2n} z_{2m}\right\rangle &= G^{2(n+m)} z_0^2 \\ &+ B\left(G\left\langle x_{2n}\right\rangle + \left\langle x_{2n-1}\right\rangle\right)\left[G^{2n} + G^{2m}\frac{\left(1-G^{2n}\right)}{\left(1-G^2\right)}\right] z_0 \\ &+ \frac{B}{\left(1-G^2\right)^2}\left(1-G^{2n}\right)\left(1-G^{2m}\right)\left(G^2\left\langle x_{2n}^2\right\rangle + 2G\left\langle x_1 x_2\right\rangle + \left\langle x_{2n+1}^2\right\rangle\right),\end{aligned} \tag{3.16c}$$

$$\begin{aligned}\left\langle z_{2n+1} z_{2m+1}\right\rangle &= G^{2(n+m+1)} z_0^2 \\ &+ \frac{B}{(1-G^2)}\left(\left(1-G^{2n+2}\right)\left\langle x_{2n}\right\rangle + \left(1-G^{2n}\right)\left\langle x_{2n+1}\right\rangle\right) z_0 \\ &+ \frac{B^2}{\left(1-G^2\right)^2}\left\{\left(1-G^{2n+2}\right)\left(1-G^{2m+2}\right)\left\langle x_{2n}^2\right\rangle\right. \\ &+ G\left[\left(1-G^{2n+2}\right)\left(1-G^{2m}\right) + \left(1-G^{2m+2}\right)\left(1-G^{2n}\right)\right]\left\langle x_1 x_2\right\rangle \\ &\left. + G^2\left(1-G^{2n}\right)\left(1-G^{2m}\right)\left\langle x_{2n+1}^2\right\rangle\right\},\end{aligned} \tag{3.16d}$$

$$\begin{aligned}\left\langle z_{2n} z_{2m+1}\right\rangle &= G^{2n+2m+1} z_0^2 \\ &+ \frac{z_0 B}{(1-G^2)}\left\{\left(G^{2n}\left(1-G^{2m+2}\right) + G^{2m+2}\left(1-G^{2n}\right)\right)\left\langle x_{2m}\right\rangle\right. \\ &\left. + G\left(G^{2n}\left(1-G^{2m}\right) + G^{2m}\left(1-G^{2n}\right)\right)\left\langle x_{2m+1}\right\rangle\right\} \\ &+ B^2\frac{\left(1-G^{2n}\right)}{\left(1-G^2\right)^2}\left\{G\left(1-G^{2m+2}\right)\left\langle x_{2n}^2\right\rangle + \left(1-G^{2m}\right)\left\langle x_{2n+1}^2\right\rangle\right. \\ &\left. + \left(1+G^2-2G^{2m+2}\right)\left\langle x_1 x_2\right\rangle\right\}.\end{aligned} \tag{3.16e}$$

If the "noise" χ is zero-centered (*c.f.* (2.18) and (2.19)), the above formulae lead to the following long time behaviour:

$$\langle z_n \rangle \to 0, \tag{3.17a}$$

$$\langle z_{2n} z_{2m} \rangle \to \langle z_{2n}^2 \rangle \to \left(\frac{BD}{1+G} \right)^2, \tag{3.17b}$$

$$\langle z_{2n+1} z_{2m+1} \rangle \to \langle z_{2n+1}^2 \rangle \to \left(\frac{BD}{1+G} \right)^2, \tag{3.17c}$$

$$\langle z_{2n} z_{2m+1} \rangle \to \left(\frac{BD}{1+G} \right)^2 \frac{G}{1-G}. \tag{3.17d}$$

Both $\langle z_{2n} \tilde{x}_{2m} \rangle$ and $\langle z_{2n+1} \tilde{x}_{2m+1} \rangle$ approach the same value:

$$\langle z_{2n} \tilde{x}_{2m} \rangle \to -\frac{B}{1+G} D^2, \tag{3.17e}$$

and so do $\langle z_{2n} \tilde{x}_{2m+1} \rangle$ and $\langle z_{2n+1} \tilde{x}_{2m} \rangle$:

$$\langle z_{2n} \tilde{x}_{2m+1} \rangle \to +\frac{B}{1+G} D^2, \tag{3.17f}$$

where

$$D^2 = \langle \tilde{x}_{2n}^2 \rangle = \nu (1-\nu)(x_1 - x_2)^2 = \sigma_{\tilde{x}}^2. \tag{3.18}$$

Hence, by using directly (3.7), for even $m = 2m'$, such that $\mathrm{Ent}(\tau/\Delta) = 2m'$ and in the limit $t \to \infty$ one gets the following expression for the correlation function:

$$\begin{aligned} \langle z(t) z(t+\tau) \rangle \to \Bigg\{ & \left(\frac{1-G}{1+G} \right)^2 \exp\{-\gamma \Theta_1\} \exp\{-\gamma \Theta_2\} \\ & - \frac{1-G}{1+G} \Big[\exp\{-\gamma \Theta_1\} (1 - \exp\{-\gamma \Theta_2\}) \\ & + \exp\{-\gamma \Theta_2\} (1 - \exp\{\gamma \Theta_1\}) \Big] \\ & + (1 - \exp\{-\gamma \Theta_1\})(1 - \exp\{-\gamma \Theta_2\}) \Bigg\} \left(\frac{\lambda D}{\gamma} \right)^2. \end{aligned} \tag{3.19a}$$

For m odd, $m = 2m' + 1$, such that $\mathrm{Ent}(\tau/\Delta) = 2m' + 1$ the long-time limit yields:

$$\langle z(t)z(t+\tau)\rangle \to \Bigg\{ \left(\frac{1-G}{1+G}\right)^2 \frac{G}{1-G} \exp\{-\gamma\Theta_1\}\exp\{-\gamma\Theta_2\} + \frac{1-G}{1+G}\Big[\exp\{-\gamma\Theta_1\}(1-\exp\{-\gamma\Theta_2\}) + \exp\{-\gamma\Theta_2\}(1-\exp\{\gamma\Theta_1\})\Big] - (1-\exp\{-\gamma\Theta_1\})(1-\exp\{-\gamma\Theta_2\})\Bigg\} \left(\frac{\lambda D}{\gamma}\right)^2 . \tag{3.19b}$$

To summarize this Section, let us briefly discuss behaviour of the quantities of interest in the limit of "continuous-time perturbation processes", *i.e.* for $\Delta \to 0$. A natural time scale of the process (3.1) is $\tau_0 = \gamma^{-1}$ which characterizes relaxation of $z(t)$. Further, we will analyze limit of $\Delta \to 0$ with a physically obvious condition $\gamma\Delta \ll 1$, so that

$$n = \mathrm{Ent}\,(t/\Delta) \to \infty\,, \qquad n\Delta \sim t\,, \tag{3.20}$$

and t remains finite. Following these conditions one has:

$$\Theta = t - n\Delta \longrightarrow 0\,, \qquad (\gamma\Theta \ll 1)\,, \tag{3.21a}$$

$$G^n = \exp\{-\gamma n\Delta\} \longrightarrow \exp\{-\gamma t\}\,, \tag{3.21b}$$

$$G^m \longrightarrow \exp\{-\gamma\tau\}\,, \tag{3.21c}$$

$$G = \exp\{-\gamma\Delta\} \longrightarrow 1 + O(\gamma\Delta)\,. \tag{3.21d}$$

From (3.5)–(3.9) we get now:

$$\langle z(t)\rangle \xrightarrow[\Delta\to 0]{} \exp\{-\gamma t\}\, z_0 \xrightarrow[t\to\infty]{} 0\,, \tag{3.22a}$$

$$\left\langle z(t)^2\right\rangle \xrightarrow[\Delta\to 0]{} (\exp\{-\gamma t\}\, z_0)^2 + \tfrac{1}{8}\left(\frac{\lambda}{\gamma}\right)^2 (1-\exp\{-2\gamma t\}) \xrightarrow[t\to\infty]{} \tfrac{1}{8}\left(\frac{\lambda}{\gamma}\right)^2, \tag{3.22b}$$

$$\langle z(t)z(t+\tau)\rangle \xrightarrow[\Delta\to 0]{} (\exp\{-\gamma t\}\, z_0)^2 + \tfrac{1}{8}\left(\frac{\lambda}{\gamma}\right)^2 \exp\{-\gamma\tau\}(1-\exp\{-2\gamma t\}) \xrightarrow[t\to\infty]{} \tfrac{1}{8}\left(\frac{\lambda}{\gamma}\right)^2 \exp\{-\gamma\tau\}\,. \tag{3.22c}$$

Let us note also, that by taking the limit $\Delta \to 0$ from r.h.s of Eqs (3.19a) and (3.19b) one arrives at:

$$\langle zz(\tau)\rangle_\infty^0 = \lim_{\Delta\to 0}\lim_{t\to\infty}\langle z(t)z(t+\tau)\rangle\,, \tag{3.23a}$$

$$\langle zz(\tau)\rangle_\infty^0 = \begin{cases} \lambda^2(\Delta-\Theta_1)(\Delta-\Theta_2)\,D^2 \to 0\,, & \text{if } \dfrac{\tau}{\Delta} = \text{even}\,,\\ -(\lambda/\gamma)^2\,D^2\,, & \text{if } \dfrac{\tau}{\Delta} = \text{odd}\,, \end{cases} \tag{3.23b}$$

wheras the same operation performed in the reverse order (by calculating first the limit $\Delta \to 0$ and then taking $t \to \infty$) yields:

$$\begin{aligned}\langle z(t)z(t+\tau)\rangle &\to \exp\{-\gamma(2t+\tau)\}\,z_0^2\\ &\quad + \left(\frac{\lambda}{\gamma}\right)^2 (1-\exp\{-\gamma t\})\,(1-\exp\{-2\gamma\tau\})\,\langle(\tilde{x}_{2n}+\tilde{x}_{2n+1})^2\rangle\\ &\quad + O(\gamma\Delta)\\ &= \exp\{-\gamma(2t+\tau)\}\,z_0^2 + O(\gamma\Delta) \xrightarrow[t\to\infty]{} 0\,. \end{aligned} \tag{3.24}$$

As it stands, the perturbation process χ, does not possess single trajectories mimicking typical noise-like stochastic process. (In principle, we can think of idealization of a noise by using ensembles of trajectories with all possible distributions of initial points). One can, however, compare the long-time limit properties of a "nearly ergodic" two-cycle to its analogon in the theory of continuous time stochastic processes, *i.e.* to a symmetric dichotomic noise. Let a random variable $F(t)$ be a dichotomous (not necessarily Markov) process, alternately taking on the values α_1, $-\alpha_2$. The times, that $F(t)$ retains the value α_i are governed by the distribution ψ_i. If $F(t)$ is supposed to be a Markov process, then these distributions are exponential [11], $\psi_i = \beta_i \exp\{-\beta_i t\}$ and β_i^{-1} are the average residence times in the states α_i (they are just the average times between the switches from α_1 to α_2: $\alpha_2\beta_1 = \alpha_1\beta_2$). Now, let us consider the case of Brownian motion governed by an ordinary Langevin equation:

$$\dot{z}(t) = -\gamma z(t) + F(t) \tag{3.25}$$

with the above mentioned assumptions for $F(t)$. Moreover, if $F(t)$ is a centered dichotomous noise (with $\alpha_1 = -\alpha_2$ and $\beta_1 = \beta_2 = \beta/2$), its average becomes trivially zero and a correlation function of $F(t)$ is given by:

$$\langle F(t)F(t+\tau)\rangle = \alpha^2 \exp\{-\beta\,|\tau|\}\ . \tag{3.26}$$

The formal solution of (3.25), assuming $z(t=0)=0$ is:

$$z(t) = \exp\{-\gamma t\} \int_0^t \exp\{\gamma t'\}\, F(t')\, dt'\,, \tag{3.27}$$

from which one can easily calculate the long-time limit properties of the moments of $z(t)$. In particular, we obtain:

$$\langle z(t)\rangle \xrightarrow[t\to\infty]{} 0\,, \tag{3.27a}$$

$$\langle z^2(t)\rangle \xrightarrow[t\to\infty]{} \frac{\alpha^2}{\gamma(\beta+\gamma)}\,, \tag{3.27b}$$

$$\langle z(t)z(t+\tau)\rangle \xrightarrow[t\to\infty]{} \frac{\alpha^2\exp\{-\gamma\tau\}}{\gamma(\beta+\gamma)}\,. \tag{3.27c}$$

By comparison with appropriate averages $\langle \quad \rangle_\infty^0$ estimated for the dynamic process (3.1) driven by a "two-cycle" generator, we see that, contrary to a real noise situation, additive perturbation by a "two-cycle" produces correlations which decay in the long-time limit. Dispersion of the trajectory becomes constant in this limit with the ratio:

$$\frac{\sigma^2_{\text{noise}}}{\sigma^2_{\text{two-cycle}}} = \frac{8\gamma^2}{\gamma(1+\gamma)}\,, \tag{3.28}$$

where we have put $\beta = 1$ and $\alpha \equiv \lambda$.

A similar analysis of the process in the region of a fully developed chaos brings

$$\frac{\sigma^2_{\text{noise}}}{\sigma^2_{\text{chaos}}} = 4\gamma\,, \tag{3.29}$$

with a properly exponentially correlated[7] $\langle z(t)z(t+\tau)\rangle_\infty^0$.

---

[7] The Ornstein-Uhlenbeck process is a stationary Gaussian process with exponentially decreasing correlation function:

$$\langle z(t)z(t+\tau)\rangle = (\lambda^2/2\gamma)\exp\{-\gamma\tau\}\ .$$

Obviously, even in the continuous time limit, regular perturbations generated by the logistic mapping in the domain of multi-cycles do not reproduce fully characteristic features of its stochastic analogons. This issue is a subject of further studies in the subsequent Section.

4. Perturbations induced by shot-chaos process

As a natural generalization of (3.1), we analyze a linear process affected by an additive χ modelled by a "shot-chaos":

$$\dot{z}(t) = -\gamma z(t) + \xi(t)\,, \tag{4.1a}$$

$$\xi(t) = \Lambda \sum_n \tilde{x}_n \delta(t - n\Delta)\,, \tag{4.1b}$$

where $\tilde{x}$ is generated by a zero-centered logistic map (2.18), (2.19).

Between the pulses, integration of (4.1) yields

$$z(t) = \exp\{-\gamma(t - n\Delta)\}\, z_n\,, \tag{4.2a}$$

and

$$z_n \equiv z(n\Delta + 0) = z(n\Delta - 0) + \Lambda \tilde{x}_n\,, \tag{4.2b}$$

so that effectively,

$$z_{n+1} = \exp\{-\gamma\Delta\}\, z_n + \Lambda \tilde{x}_n\,, \tag{4.3}$$

or

$$z_n = G^n z_0 + \Lambda \sum_{l=0}^{n-1} G^{n-l-1} \tilde{x}_l\,, \tag{4.4}$$

where, as previously, $G = \exp\{-\gamma\Delta\}$. Ensemble averages over trajectories produce now:

$$\langle z(t)\rangle = \exp\{-\gamma(t - n\Delta)\}\, \langle z_n\rangle\,, \tag{4.5a}$$

$$\langle z^k(t)\rangle = \exp\{-\gamma(t - n\Delta)\}\, \langle z_n^k\rangle\,, \tag{4.5b}$$

$$\langle z(t) z(t+\tau)\rangle = \exp\{-\gamma(t - n\Delta)\} \exp\{-\gamma(t - n\Delta - m\delta)\}\, \langle z_n z_{n+m}\rangle\,, \tag{4.5c}$$

where $n = \mathrm{Ent}\,(t/\Delta)$, $m = \mathrm{Ent}\,((t+\tau)/\Delta)$.

For a fully developed chaos and for $\langle \tilde{x} \rangle = 0$ one gets:

$$\langle z_n \rangle = G^n z_0 \,, \tag{4.6a}$$

$$\langle z_n^2 \rangle = (G^n z_0)^2 + \Lambda^2 \frac{1 - G^{2n}}{1 - G^2} \langle \tilde{x}^2 \rangle \,, \tag{4.6b}$$

$$\langle z_n z_{n+m} \rangle = G^{2n+m} z_0^2 + \Lambda^2 G^m \frac{1 - G^{2n}}{1 - G^2} \langle \tilde{x}^2 \rangle \,, \tag{4.6c}$$

which in the long-time limit $(t \to \infty)$ tend to

$$\langle z_n \rangle \longrightarrow 0 \,, \tag{4.7a}$$

$$\langle z_n^2 \rangle \longrightarrow \frac{\Lambda^2}{8\,(1 - G^2)} \,, \tag{4.7b}$$

$$\langle z_n z_{n+m} \rangle \longrightarrow \frac{\Lambda^2 G^m}{8\,(1 - G^2)} \,. \tag{4.7c}$$

In the case of $\tilde{x}_n$ driven by a two-cycle, analogously calculated averages turn to be:

$$\begin{aligned} \langle z_{2n} z_{2n+2m} \rangle &= G^{4n+2m} z_0^2 \\ &+ \frac{\Lambda^2}{(1 - G^2)^2} \left(1 - G^{2n}\right) \left(1 - G^{2(n+m)}\right) \left\langle \left(G\tilde{x}_{2j} + \tilde{x}_{2j+1}\right)^2 \right\rangle \,, \end{aligned} \tag{4.8a}$$

$$\begin{aligned} \langle z_{2n} z_{2n+2m+1} \rangle &= G^{4n+2m+1} z_0^2 \\ &+ \Lambda^2 \frac{(1 - G^{2n})}{(1 - G^2)^2} \Big[\left(1 - G^{2n+2m+2}\right) \langle \tilde{x}_{2i} \left(G\tilde{x}_{2j} + \tilde{x}_{2j+1}\right) \rangle \\ &+ \left(1 - G^{2n+2m}\right) G \, \langle \tilde{x}_{2i+1} \left(G\tilde{x}_{2j} + \tilde{x}_{2j+1}\right) \rangle \Big] \,, \end{aligned} \tag{4.8b}$$

$$\begin{aligned} \langle z_{2n+1} z_{2n+2m} \rangle &= G^{4n+2m+1} z_0^2 \\ &+ \Lambda^2 \frac{(1 - G^{2(n+m)})}{(1 - G^2)^2} \Big[\left(1 - G^{2n+2}\right) \langle \tilde{x}_{2i} \left(G\tilde{x}_{2j} + \tilde{x}_{2j+1}\right) \rangle \\ &+ \left(1 - G^{2n+2m}\right) G \, \langle \tilde{x}_{2i+1} \left(G\tilde{x}_{2j} + \tilde{x}_{2j+1}\right) \rangle \Big] \,, \end{aligned} \tag{4.8c}$$

$$\begin{aligned}\langle z_{2n+1} z_{2n+2m+1}\rangle &= G^{4n+2m+2} z_0^2 \\ &+ \frac{\Lambda^2}{(1-G^2)^2}\left(1-G^{2n+2}\right)\left(1-G^{2n+2m+2}\right)\langle \tilde{x}_{2i}^2\rangle \\ &+ \frac{\Lambda^2}{(1-G^2)^2}\Big[\left(1-G^{2n+2}\right)\left(1-G^{2n+2m}\right) \\ &\qquad + \left(1-G^{2n}\right)\left(1-G^{2n+2m+2}\right)\Big] G\,\langle \tilde{x}_{2i}\tilde{x}_{2j+1}\rangle \\ &+ \frac{\Lambda^2}{(1-G^2)^2}\left(1-G^{2n}\right)\left(1-G^{2n+2m}\right) G^2\,\langle \tilde{x}_{2i+1}^2\rangle\,. \end{aligned} \tag{4.8d}$$

Asymptotically, for $t \to \infty$ one gets then:

$$\langle z_{2n} z_{2n+2m}\rangle \longrightarrow \left(\frac{\Lambda}{1-G^2}\right)^2 \left\langle \left(G\tilde{x}_{2j} + \tilde{x}_{2j+1}\right)^2 \right\rangle, \tag{4.9a}$$

$$\begin{aligned}\langle z_{2n} z_{2n+2m+1}\rangle &\longrightarrow \langle z_{2n+1} z_{2n+2m}\rangle \\ &\longrightarrow \left(\frac{\Lambda}{1-G^2}\right)^2 \langle (G\tilde{x}_{2j} + \tilde{x}_{2j+1})(\tilde{x}_{2j} + G\tilde{x}_{2j+1})\rangle\,, \end{aligned} \tag{4.9b}$$

$$\langle z_{2n+1} z_{2n+2m+1}\rangle \longrightarrow \left(\frac{\Lambda}{1-G^2}\right)^2 \left\langle \left(G\tilde{x}_{2j+1} + \tilde{x}_{2i}\right)^2 \right\rangle. \tag{4.9c}$$

Some comments need to be said also concerning the proper scaling of the process when Δ is supposed to approach zero. In fact, one can rederive the same type of asymptotic behaviour of the averages $\langle \quad \rangle_\infty^0$ as in the last Section (*c.f.* formulae (3.22a)–(3.22c)) provided the following scaling is used[8]:

$$\frac{\Lambda^2}{1-G^2} = \left(\frac{\lambda}{\gamma}\right)^2. \tag{4.10}$$

In other words, taking the limit $\Delta \to 0$ requires now to take $\Lambda \to 0$ in such a way that (Λ^2/Δ) remains constant.

$$\Lambda^2/\Delta = \text{const} = 2\lambda^2/\gamma\,. \tag{4.11}$$

[8] It is so, because $\dfrac{\Lambda^2}{1-G^2} \longrightarrow \dfrac{\Lambda^2}{2\gamma\Delta}$ in this limit.

This constraint to be satisfied[9], we scale the intensity of the ξ process:

$$\xi(t) = \lambda\sqrt{\frac{2\Delta}{\gamma}}\sum_n \delta(t - n\Delta) = \Lambda_0\sqrt{\Delta}\sum_n \delta(t - n\Delta)\,. \tag{4.12}$$

Keeping in mind that for a "two-cycle" $\tilde{x}_{2j}+\tilde{x}_{2j+1} \equiv 0$ for $t \to \infty$, we get the following expression for $\Delta \to 0$ limit of the correlation function $\langle z_{2n} z_{2m}\rangle$:

$$\begin{aligned}\langle z_{2n} z_{2m}\rangle^0_\infty &\longrightarrow \left(\frac{\Lambda_0\sqrt{\Delta}}{2\gamma\Delta}\right)^2 \left\{\left\langle[(1-\gamma\Delta)\tilde{x}_{2j} + \tilde{x}_{2j+1}]^2\right\rangle + O(\Delta^2)\right\} \\ &= -\frac{\Lambda_0^2}{2\gamma}\langle\tilde{x}_{2n}(\tilde{x}_{2n} + \tilde{x}_{2n+1})\rangle + O(\Delta^2) = 0\,. \end{aligned} \tag{4.13}$$

By inspection, the same result is obtained for other averages. In conclusion, our conjecture is that rapid shot-like oscillations imposed on a deterministic process will not change (asymptotically) properties of its moments. One can expect also that a "slight mixture" of any multi-cycle will decay sufficiently fast in time to be unobservable experimentally, provided its period N satisfies

$$N\Delta \ll \tau_0 \equiv \frac{1}{\gamma}\,, \tag{4.14}$$

with τ_0 representing the deterministic relaxation time.

5. Evolution equations for the probability distribution function of a variable perturbed by a parametric shot noise

Our aim is now to derive evolution equation for the probability density function describing time-dependent statistical properties of the system driven by a noise generator of the type discussed earlier. Let us start with a general one-dimensional kinetic equation with a parametric "noise":

$$\dot{x} = f(x) + \gamma g(x)\xi(t)\,, \tag{5.1}$$

[9] The procedure is a common manner of dealing with the continuous realization of discrete in time "noises", *c.f.* [11].

where

$$\xi(t) = \lambda \sum_n w_n \delta(t - n\Delta),$$

$$w_{n+1} = 2w_n^2 - 1,$$

$$w \in [-1,+1], \quad \langle w \rangle = \overline{w} = 0. \tag{5.2}$$

Hence, ξ satisfies conditions of a "white shot noise" [21] with a distribution of peaks given by

$$p(w) = \frac{1}{\pi\sqrt{1-w^2}}, \qquad \int_{-1}^{+1} p(w)\, dw = 1.$$

Following the procedure of Hänggi, with ξ interpreted in the Stratonovich sense [22] Eq (5.1) is equivalent to the Master equation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[f(x) - \lambda\gamma \langle w \rangle g(x)\right] P(x,t) + \lambda \left\{ \int \exp\left[-w\frac{\partial}{\partial x}\gamma g(x)\right] p(w)\, dw - 1 \right\} P(x,t), \tag{5.3}$$

where $P(x,t)$ stands for the probability that a variable $x(t)$ takes value x at time t. From the definition of a cylindric function [23] we get

$$\frac{1}{\pi}\int_{-1}^{+1} \frac{\exp\{\pm zt\}}{\sqrt{1-t^2}} = I_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{z}{2}\right)^{2k}. \tag{5.4}$$

So that the final form of (5.3) is

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x} f(x) P(x,t) + \lambda \sum_{k=1}^{\infty} \frac{(-1)^k}{(k!)^2}\left[-\tfrac{1}{2}\frac{\partial}{\partial x}\gamma g(x)\right]^{2k} P(x,t), \tag{5.5}$$

which approximated up to the order γ^2 yields a standard Fokker-Planck equation.

As it has been argued earlier, in the case of multi-cycles a single trajectory does not represent a particular "noise-realization". An ensemble of these trajectories starting from all possible initial points apparently resembles characteristics of the white shot noise and for an M-cycle is described by a distribution

$$p(\tilde{w}) = \sum_j^M \nu_j \delta(\tilde{w} - \tilde{w}_j), \qquad \sum_j^M \nu_j = 1\,. \tag{5.6}$$

Equation (5.1) with the noise whose stationary properties are expressed by $p(\tilde{w})$ of the form (5.6) can be easily transformed to:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x} f(x) P(x,t) + \lambda \sum_{k=1}^{\infty} \frac{1}{k!} \langle \tilde{w}^k \rangle \left[-\frac{\partial}{\partial x} \gamma g(x) \right]^k P(x,t)\,. \tag{5.7}$$

For a particular choice of a linear system driven by a multiplicative noise, $g(x) = ax$ represented by a two-cycle generator and by assuming $\tilde{w}_1 = -\alpha$, $\tilde{w}_2 = +\alpha$, $\nu_1 = \nu$, $\nu_2 = 1 - \nu$ r.h.s of (5.7) can be further transformed to produce effectively

$$\begin{aligned} \frac{\partial P(x,t)}{\partial t} &= -\frac{\partial}{\partial x} f(x) P(x,t) - \lambda P(x,t) \\ &+ \lambda \nu \mathrm{e}^{a\alpha} P\left(\mathrm{e}^{a\alpha} x, t\right) + \lambda (1-\nu) \mathrm{e}^{-a\alpha} P\left(\mathrm{e}^{-a\alpha} x, t\right)\,. \end{aligned} \tag{5.8}$$

In the case of a purely periodic forcing ($f(x) \equiv 0$) the process can be compared to a two-state telegraphic process with the Poisson distribution of waiting times in a given state. By an analogy, let us think of the system whose evolution is governed by a propagator $\exp\{\tau A\}$ or $\exp\{-\tau A\}$ for some random time τ, then reversing, with the Poisson distribution of the reversals. It has been shown [24] that for a general form of the operator A (we accept here $A = ax$, $\alpha_{1,2} = \pm 1$) probability density of observing x at time t provided given sign of the operator A

$$P_{\pm}(x,t) \equiv \left\{ \text{Prob } x \text{ at time } t \,\middle|\, \text{evolution governed by } \pm A \right\} \tag{5.9}$$

satisfies the equation

$$\frac{\partial P_{\pm}(x,t)}{\partial t} = \mp \nu P_{+}(x,t) \pm (1-\nu) P_{-}(x,t) \pm A P_{\pm}(x,t)\,. \tag{5.10}$$

Let us assume for simplicity $\nu = 1/2$. The sum of P_+ and P_- satisfies then a telegrapher equation:

$$\frac{\partial^2 P(x,t)}{\partial t^2} + \frac{\partial P(x,t)}{\partial t} - A^2 P(x,t) = 0\,. \tag{5.11}$$

As it stands, both of the Eqs (5.8) and (5.10) have different operators describing time-evolution of the probability mass and, *a priori* they do not need to give the similar asymptotic behaviour for $P(x,t)$ in the long-time limit. It seems thus challenging to investigate closely [25] differences in the dynamic properties (kinetic rates, mean first passage times, relaxation dynamics) of the system driven by forcing leading to Eqs (5.8) and (5.10), respectively. This problem will be the subject of separate studies.

6. Final remarks

In this paper we have concentrated our attention on direct calculations of statistical properties of a kinetic process perturbed by a chaos-generator. Our intention was to clarify existence of limits allowing to envision the long-time properties of such systems by use of the concept of ergodicity. In particular, it has been demonstrated that a perturbing mapping which bifurcates to an M-cycle does not possess semi-ergodic properties (they are restored however in the limit case, when the dynamics of the perturbations converge to an ordinary diffusion-like stochastic process). On the other hand, a mixture of M-cycle in the dynamics of the system does not produce any observable effects in the long time scale (properly scaled correlation functions of the M-cycle vanish in this limit).

In addition to giving the background of the formalism we have derived evolution equations for the probability density function describing time-dependent properties of the ensemble of perturbed trajectories. The utility of them will be discussed in a future analysis [25] where dynamic properties of the systems perturbed by regular disturbances will be brought up and interrelated with the long time behaviour of the systems driven by continuous-time noises.

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REFERENCES

[1] H. Mori, *Prog. Theor. Phys.* **33**, 423 (1965); M. Tokuyama, H. Mori, *Prog. Theor. Phys.* **55**, 411 (1976).
[2] I. Prigogine, C. George, F. Henin, L. Rosenfeld, *Chemica Scripta* **4**, 5 (1973); B. Misra, I. Prigogine, M. Courbage, *Physica* **98A**, 1 (1979).
[3] A. Fuliński, W. Kramarczyk, *Physica* **39**, 575 (1968); A. Fuliński, *Physica* **92A**, 198 (1978); R. Zwanzig, *J. Chem. Phys.* **33**, 1338 (1960).
[4] K. Tomita, *Prog. Theor. Phys. Supplement* **79**, 1 (1979).
[5] P. Collet, J.-P. Eckman, *Iterated Maps on the Interval as Dynamical Systems*, Birkäuser, Boston 1981.

[6] H.G. Schuster, *Deterministic Chaos*, Physik-Verlag, Weinheim 1984.
[7] I.P. Cornfield, S.V. Fomin, Ya.G. Sinai, *Ergodic Theory*, Springer, Berlin 1982.
[8] *Chaos, Noise and Fractals*, eds E.R. Pike, L.A. Lugiato, Malvern Physical Series, Bristol 1987; A.J. Lichtenberg, M.A. Liebermann, *Regular and Stochastic Motion*, Springer, Berlin 1983.
[9] J.-P. Eckman, *Rev. Mod. Phys.* **53**, 643 (1981).
[10] J. Crutchfield, J.D. Farmer, B. Huberman, *Phys. Rev.* **92**, 42 (1982); G. Meyer-Kress, H. Haken, *J. Stat. Phys.* **26**, 149 (1981); A. Fuliński, *Phys. Lett.* **126A**, 84 (1987); T. Kapitaniak, *Chaos in Systems with Noise*, World Scientific, New York 1988.
[11] W. Horsthemke, R. Lefever, *Noise Induced Transitions*, Springer, Berlin 1984.
[12] P. Grassberger, I. Procaccia, *Physica* **9D**, 189 (1983); P. Grassberger, I. Procaccia, *Phys. Rev. Lett.* **50**, 346 (1983); J.D. Farmer, *Physica* **4D**, 366 (1982); J. Guckenheimer, G. Buzyna, *Phys. Rev. Lett.* **51**, 1348 (1983).
[13] E. Gudowska-Nowak, A.Kleczkowski, G.O. Williams, *J. Stat. Phys.* **54**, 539 (1989); A.R. Bulsara, E.W. Jacobs, W.C. Schieve, *Phys. Rev.* **42A**, 4614 (1990).
[14] C. Beck, G. Roepstorff, *Physica* **145A**, 1 (1987).
[15] D. Mayer, G. Roepstorff, *J. Stat. Phys.* **31**, 309 (1983).
[16] M. Wang, G. Uhlenbeck, in: *Selected Papers on Noise and Stochastic Processes*, ed. N. Wax, Dover, New York 1954.
[17] C. Beck, Higher Order Correlation Functions of Chaotic Dynamical Systems — a Graph Theoretical Approach, preprint of the Inst. für Theoretische Physik, Aachen 1990.
[18] C.G. Grosjean, *J. Math. Phys.* **28**, 1266 (1987); S. Katsura, W. Fukuda, *Physica* **130A**, 597 (1985); G. Györgyi, P. Szepfalusy, *J. Stat. Phys.* **34**, 437 (1984).
[19] B. Dorizzi, B. Grammaticos, M. Le Berre, Y. Pomeau, E. Ressayre, A Tallet, *Phys. Rev.* **35A**, 328 (1987).
[20] M. Le Berre, E. Ressayre, A. Tallet, Y. Pomeau, *Phys. Rev.* **41A**, 6635 (1990).
[21] C. Van Den Broeck, *J. Stat. Phys.* **31**, 467 (1983).
[22] P. Hänggi, *Z. Phys.* **B36**, 271 (1980).
[23] I.S. Gradshteyn, I.M. Ryzhik, *Tables of Integrals, Series and Products*, Academic, New York 1980.
[24] B. Gaveau, T. Jacobson. M. Kac, L.S. Schulman, *Phys. Rev. Lett.* **53**, 419 (1984).
[25] A. Fuliński, E. Gudowska-Nowak, Relaxation Properties of a System Perturbed by a Chaos-Generator, to be published.