# AN EXAMPLE OF $N$-BODY QUANTUM-MECHANICAL MODEL THAT ELUDES THE SECOND QUANTIZATION

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A quantum-mechanical system of $N$ spin-$^1/_2$ identical particles, where $\gamma$ matrices *anticommute* for different particles, is shown to escape the procedure of the second quantization (in the physical space). Instead, a quantization procedure in the Fock configurational space, called here the third quantization, is outlined. The respective particles are referred to as the non-Abelian Dirac particles. A system of $N$ such particles, tightly concentrated around its centre of mass, can *always* be described in the pointlike limit by the Dirac equation corresponding to a composite, reducible (for $N > 1$) representation of the Dirac algebra. According to a previous discussion by the author, such representations with $N = 1, 3, 5$ may be responsible for the puzzling phenomenon of three fermion generations, if an intrinsic exclusion principle is introduced.

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As is well known, the two-step quantization procedure is, especially for matter particles, a paradigm of quantum theory. It starts from the classical mechanics and leads *via* the quantum mechanics (the first quantization) to the quantum field theory (the second quantization). Though the first step of quantization leading to quantum mechanics may be formally avoided by starting from a properly constructed classical field theory and then passing directly to quantum field theory, in the case of matter particles this step is really crucial for our understanding of physical systems such as atoms, solids and nuclei.

In the present note we construct a mathematical example of a nonconventional quantum-mechanical system that cannot be second-quantized and so described by a quantum field.

To this end, let us consider a system of $N$ equal-mass identical particles described by their positions $\vec{r}_i$, momenta $\vec{p}_i$ and some new matrices

$\gamma_i^\mu$ defined through their anticommutation relation *viz.*, Dirac conventional ones:

$$\{\gamma_i^\mu, \gamma_i^\nu\} = 2g^{\mu\nu} \qquad \text{for } i = 1, 2, \ldots, N \tag{1}$$

and nonconventional:

$$\{\gamma_i^\mu, \gamma_j^\nu\} = 0 \qquad \text{for } i, j = 1, 2, \ldots, N \text{ with } i \neq j \tag{2}$$

(the latter are replacing the commutation relations $[\gamma_i^\mu, \gamma_j^\nu] = 0$ which appear in the Dirac conventional $N$-body problem). In order to coin a term for our particles obeying noncommutativity (2), we will call them the *non-Abelian Dirac particles* [1]. It is easy to show that the relations (1) and (2) can be minimally satisfied by $4^N \times 4^N$ matrices. For instance, in the case of $N = 3$ one can use the representation

$$\begin{aligned} \gamma_1^\mu &= \gamma^\mu \otimes 1 \otimes 1\,, \\ \gamma_2^\mu &= \gamma^5 \otimes i\gamma^5 \otimes i\gamma^5\gamma^\mu \otimes 1\,, \\ \gamma_3^\mu &= \gamma^5 \otimes \gamma^5 \otimes \gamma^\mu\,, \end{aligned} \tag{3}$$

where $\gamma^\mu$ and $1$ are the usual Dirac $4 \times 4$ matrices while $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ ($(\gamma^\mu) = (\beta, \beta\vec{\alpha})$, $\vec{\alpha} = \gamma^0\vec{\gamma} = \gamma^5\vec{\sigma}$). Thus, the dimension of minimal representation for our $\gamma_i^\mu$ does not change in comparison with the familiar $\gamma_i^\mu$ matrices in the Dirac conventional $N$-body problem.

Let us note that the relativistic dynamics for the system of non-Abelian Dirac particles may be based on the operator

$$\sum_{i=1}^{N} \gamma_i \cdot p_i \tag{4}$$

which is Lorentz-invariant, if the *physical* Lorentz group for such a system is generated by

$$J^{\mu\nu} = \sum_{j=1}^{N} \left( x_j^\mu p_j^\nu - x_j^\nu p_j^\mu + \tfrac{i}{4}[\gamma_j^\mu, \gamma_j^\nu] \right) = L^{\mu\nu} + \tfrac{1}{2}\sigma^{\mu\nu}\,, \tag{5}$$

where

$$\sigma^{\mu\nu} = \sum_{j=1}^{N} \frac{i}{2}[\gamma_j^\mu, \gamma_j^\nu] = \begin{cases} -\sum\limits_{j=1}^{N} i\alpha_j^k & \text{for } \mu = k\,, \nu = 0 \\ \sum\limits_{j=1}^{N} \epsilon^{klm}\sigma_j^m & \text{for } \mu = k\,, \nu = l \end{cases} \tag{6}$$

with $\vec{\alpha}_j = \gamma_j^0\vec{\gamma}_j = \gamma_j^5\vec{\sigma}_j$ and $\vec{\sigma}_j = \gamma_j^5\gamma_j^0\vec{\gamma}_j = \gamma_j^5\vec{\alpha}_j$. Here, the velocity matrices $\vec{\alpha}_j$ commute for different $j$. The same is true for the spin matrices $\vec{\sigma}_j$ as well as for the chirality matrices $\gamma_j^5 = i\gamma_j^0\gamma_j^1\gamma_j^2\gamma_j^3$. We can see that the non-Abelian Dirac particles forming our system carry simultaneously measurable spins$^1/_2$ and left-right chiralities.

In the one-time case, when $t_1 - \ldots = t_N \equiv t$ and $p_1^0 = \ldots = p_N^0 \equiv P^0/N$ with $P^0 = i\partial/\partial t$, one can write

$$\sum_{i=1}^{N} \gamma_i \cdot p_i = \frac{1}{\sqrt{N}} \Gamma^0 P^0 - \sum_{i=1}^{N} \vec{\gamma}_i \cdot \vec{p}_i = \frac{1}{\sqrt{N}} \Gamma^0 \Big( P^0 - \sqrt{N} \sum_{i=1}^{N} \Gamma^0 \vec{\gamma}_i \cdot \vec{p}_i \Big) , \quad (7)$$

where

$$\Gamma^0 = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_i^0 \quad (8)$$

satisfies together with $\gamma_i^k$ the anticommutation relations

$$\{\Gamma^0, \Gamma^0\} = 2 , \ \{\Gamma^0, \gamma_i^k\} = 0 , \ \{\gamma_i^k, \gamma_j^l\} = -2\delta^{kl}\delta_{ij} . \quad (9)$$

Thus, in the free case the one-time wave equation for our system can be assumed in the form

$$\Big(P^0 - \sqrt{N} \sum_{i=1}^{N} \vec{A}_i \cdot \vec{p}_i - BM\Big)\psi = 0 , \quad (10)$$

where $M = Nm$ is the rest mass of the system of $N$ equal-mass particles ($m_i \equiv m$) and

$$\vec{A}_i = \Gamma^0 \vec{\gamma}_i , \qquad B = \Gamma^0 \quad (11)$$

fulfil the anticommutation relations

$$\{A_i^k , A_j^l\} = 2\delta^{kl}\delta_{ij} , \quad \{A_i^k , B\} = 0 , \quad \{B , B\} = 2 . \quad (12)$$

Note that relations (12) can be minimally satisfied by less-dimensional matrices than the relations (1) and (2), for instance, in the case of $N = 3$ by $32 \times 32$ matrices.

The wave equation (10) implies the nonconventional second-order equation [2]

$$\Big(P^{02} - N \sum_{i=1}^{N} \vec{p}_i^{\,2} - M^2\Big)\psi = 0 \quad (13)$$

that, however, in the nonrelativistic limit goes over into the familiar $N$-body Schrödinger equation

$$\Big(P^0 - M - \sum_{i=1}^{N} \frac{\vec{p}_i^{\,2}}{2m}\Big)\psi_{\rm NR} = 0 , \quad (14)$$

since $(P^0 + M)_{\rm NR} \to 2M$. We can see that, inversely, a generalized $3N+1$-dimensional Dirac square-root procedure [3] leads from Eq. (13) to Eq. (10).

In the pointlike limit, when $\vec{r}_1 = \ldots = \vec{r}_N \equiv \vec{R}$ and $\vec{p}_1 = \ldots = \vec{p}_N \equiv \vec{P}/N$ with $\vec{P} = -i\partial/\partial\vec{R}$, one gets from Eq. (10) the Dirac equation for the motion of the centre of mass:

$$\left(P^0 - \vec{A}\cdot\vec{P} - BM\right)\psi_{\rm CM} = 0\,, \tag{15}$$

where

$$\vec{A} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\vec{A}_i = \Gamma^0\vec{\Gamma} \quad \text{or} \quad \vec{\Gamma} = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\vec{\gamma}_i \tag{16}$$

satisfies together with $B = \Gamma^0$ the Dirac anticommutation relations

$$\left\{A^k\,,\,A^l\right\} = 2\delta^{kl}\,,\quad \left\{A^k\,,\,B\right\} = 0\,,\quad \left\{B\,,\,B\right\} = 2 \tag{17}$$

or

$$\left\{\Gamma^\mu\,,\,\Gamma^\nu\right\} = 2g^{\mu\nu} \tag{18}$$

But, the Dirac equation (15) is not the familiar one, because it corresponds to a composite, reducible (for $N > 1$) representation of the Dirac algebra (18). It turns out that Eq. (15) realizes a generalization [4] both of the Dirac equation ($N = 1$) and the Kähler equation [5], the latter being equivalent [6] to Eq. (15) for $N = 2$. Note that, according to Eq. (15), the centre of mass carries *always* spin $^1/_2$ independently of the number $N = 1, 2, 3, \ldots$ of non-Abelian Dirac particles forming our system. The remaining $N-1$ spins $^1/_2$ are *hidden* in the internal structure of the system, since this structure becomes invisible in the pointlike limit (its remnant may be a correction to the rest mass $M = Nm$). Of course, the true total spin of the system remains half-integer or integer for $N$ odd or even, respectively, even though $N-1$ relative spins are hidden in the internal structure (unless the definition (5) of the physical Lorentz group including physical rotations were changed by excluding from it the relative spin degrees of freedom for the system of $N$ non-Abelian Dirac particles).

A nonconventional feature of the first-quantized Hamiltonian appearing in Eq. (10),

$$H = \sqrt{N}\sum_{i=1}^{N}\vec{A}_i\cdot\vec{p}_i + BNm\,, \tag{19}$$

is that its parts corresponding to different particles $i$ do not commute (even in the free case). This property causes that there *does not exist* a related second-quantized Hamiltonian. In fact, the Hamiltonian of the form

$$\int d^3\vec{r}\,\psi^+(\vec{r},t)(\vec{\alpha}\cdot\vec{p} + \beta m)\psi(\vec{r},t)\,, \tag{20}$$

with $\psi(\vec{r},t)$ being a quantized bispinor field, would imply that the matrices $\vec{\alpha}$ and $\beta$ resulting (in the Fock representation) for different particles of the field $\psi(\vec{r},t)$ should commute.

However, we would like to point out that our quantum-mechanical system of $N$ non-Abelian Dirac particles, though not second-quantizable (in the 3-dimensional physical space), can be "second-quantized" in the $3N$-dimensional configurational space. In fact, it is not difficult to see that in the free case such a "second-quantized" Hamiltonian (related to the first-quantized Hamiltonian (19)) can be written in the form

$$\int d^3\vec{r}_1 \dots d^3\vec{r}_N \psi^+(\vec{r}_1,\dots,\vec{r}_N,t)\big(\sqrt{N}\sum_{i=1}^{N}\vec{A}_i\cdot\vec{p}_i + BNm\big)\psi(\vec{r}_1,\dots,\vec{r}_N,t)\,, \tag{21}$$

where $\psi(\vec{r}_1,\dots,\vec{r}_N,t)$ is the *quantized* $N$-body wave function ("configurational quantum field") satisfying the wave equation (10) ("configurational quantum field equation"). For $N = 1,2,3,\dots$ the novel operators $\psi(\vec{r}_1,\dots,\vec{r}_N,t) \equiv \psi^{(N)}(\vec{r}_1,\dots,\vec{r}_N,t)$ form together a "Fock multicomponent quantum field". They define a novel step in the quantum theory that may be called the *third quantization*. In Eq. (21), the matrices $\vec{A}_i \equiv \vec{A}_i^{(N)}$ and $B \equiv B^{(N)}$ fulfil the anticommutation relations (12) and are assumed to *commute* for different $N$. The operators $\psi^{(N)}(\vec{r}_1,\dots,\vec{r}_N,t)$ are chosen to be quantized by means of *anticommutators* or *commutators* when $N$ is odd or even, respectively, and are assumed to *commute* for $N$'s differing by their parity. Of course, these anticommutators or commutators vanish for different odd or even $N$, respectively.

In both cases, the operators $\psi^{(N)}(\vec{r}_1,\dots,\vec{r}_N,t)$ and $\psi^{(N)+}(\vec{r}_1,\dots,\vec{r}_N,t)$ can be interpreted as, respectively, the annihilation and creation operators (of the system of $N$ non-Abelian particles as a whole) in the configuration $\vec{r}_1\alpha_1,\dots,\vec{r}_N\alpha_N$ at time $t$. Here, $\alpha_i$ with $i = 1,2,\dots,N$ are bispinor indices suppressed in $\psi^{(N)}(\vec{r}_1,\dots,\vec{r}_N,t)$ and $\psi^{(N)+}(\vec{r}_1,\dots,\vec{r}_N,t)$.

It should be noted that for the conventional (*i.e.*, Abelian) Dirac particles the procedure of the third quantization would run into trouble connected with *double* and, in general, *multiple* counting of particles. In fact, in this case there would be no way to distinguish a single $N$-configuration of particles from a conventional configuration of $N$ single particles (*i.e.*, form $N$ 1-configurations). In contrast, in the case of non-Abelian Dirac particles these two configurations are perfectly distinguishable since $\gamma$ matrices for different particles anticommute in the former configuration and commute in the latter (in general, $\gamma$ matrices commute for two particles from two different $N$-configurations, both with the same $N$ and with two different $N$).

Another nonconventional feature of the first-quantized Hamiltonian (19) is connected with the particle velocities

$$\dot{\vec{r}}_i \equiv \frac{1}{i}[\vec{r}_i, H] = \sqrt{N}\vec{A}_i\,, \tag{22}$$

whose distinguished components get the eigenvalues $\pm\sqrt{N}$ formally exceeding for $N > 1$ the velocity of light. Note that here $\dot{\vec{r}}_i$ are not given by the traditional velocity matrices $\vec{\alpha}_i = \gamma_i^0\vec{\gamma}_i$ but rather by $\sqrt{N}\Gamma^0\vec{\gamma}_i$, where $\sqrt{N}\Gamma^0 = \sum_{i=1}^{N}\gamma_i^0$. In contrast, for the velocity of the centre of mass

$$\dot{\vec{R}} \equiv \frac{1}{i}[\vec{R}, H] = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\vec{A}_i = \vec{A}\,, \tag{23}$$

any distinguished component has the Dirac conventional eigenvalues $\pm 1$. Notice that $\dot{\vec{r}}_i$ do not commute with each other for different $i$ and also with $\dot{\vec{R}}$. Thus, in their motion around the centre of mass moving with the velocity $\dot{\vec{R}}$, the non-Abelian Dirac particles have nonmeasurable velocities $\dot{\vec{r}}_i$, provided a distinguished component of $\dot{\vec{R}}$ is measured. Like a Dirac particle, the centre of mass performs here the zitterbewegung around a mean motion having the velocity $\vec{P}/\sqrt{\vec{P}^2 + M^2}$.

Finally, it is perhaps worthwhile to mention that the wave equation (15) for the motion of the centre of mass (valid in the pointlike limit) may be important for solving the puzzle of three lepton and quark generations [7]. This Dirac equation can be written in the covariant form

$$(\Gamma \cdot P - M)\psi_{\rm CM} = 0\,, \tag{24}$$

where

$$\Gamma^\mu = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_i^\mu \tag{25}$$

(*cf.* Eqs. (8) and (16)) with

$$\{\gamma_i^\mu, \gamma_j^\nu\} = 2g^{\mu\nu}\delta_{ij} \tag{26}$$

(*cf.* Eqs. (1) and (2)) define a sequence $N = 1, 2, 3, \ldots$ of composite, reducible (for $N > 1$) representations of the Dirac algebra (18). It has been recently shown [7] that for $N = 1, 3, 5$ they may correspond to three generations of fundamental fermions, if an *intrinsic exclusion principle* is introduced for those $N - 1$ bispinor indices of the wave function $\psi_{\rm CM}$ describing the hidden spins $^1/_2$ . This restricts, in fact, all possible states presented by $\psi_{CM}$ with various odd $N$ to three spin-$^1/_2$ states with $N = 1, 3, 5$.

In the Appendix we discuss in detail the simplest system of even number of non-Abelian Dirac particles *i.e.* that with $N = 2$, assuming an internal Coulombic attraction which enables us to find explicitly an exact solution.

## APPENDIX

### Two-body problem for non-Abelian Dirac particles

In the case of $N = 2$ non-Abelian Dirac particles interacting through a vector potential $V$ and a scalar potential $S$, the one-time wave equation (10) reads (for stationary states):

$$\left[E - V - \sqrt{2}(\vec{A}_1 \cdot \vec{p}_1 + \vec{A}_2 \cdot \vec{p}_2) - B(2m + S)\right]\psi = 0 . \qquad \text{(A.1)}$$

In the centre-of-mass variables this equation can be written as

$$\left(E - V - \vec{A} \cdot \vec{P} - 2\vec{\alpha} \cdot \vec{p} - B(2m + S)\right]\psi = 0 , \qquad \text{(A.2)}$$

where

$$\vec{P} = \vec{p}_1 + \vec{p}_2 , \quad \vec{p} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) \qquad \text{(A.3)}$$

are the momenta conjugate to the coordinates

$$\vec{R} = \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2) , \quad \vec{r} = \vec{r}_1 - \vec{r}_2 , \qquad \text{(A.4)}$$

while the matrices

$$\vec{A} = \tfrac{1}{\sqrt{2}}(\vec{A}_1 + \vec{A}_2) , \quad \vec{\alpha} = \tfrac{1}{\sqrt{2}}(\vec{A}_1 - \vec{A}_2 , \qquad \text{(A.5)}$$

and $B$ fulfil the anticommutation relations

$$\begin{aligned} &\{A^k , A^l\} = 2\delta^{kl} , \quad \{A^k , \alpha^l\} = 0 , \quad \{\alpha^k , \alpha^l\} = 2\delta^{kl} , \\ &\{A^k , B\} = 0 , \qquad \{\alpha^k , B\} = 0 , \quad \{B , B\} = 2 \end{aligned} \qquad \text{(A.6)}$$

following from Eq. (12) for $N = 2$. They can be minimally satisfied by $8 \times 8$ matrices: *e.g.* one can use the representation

$$\vec{A} = \vec{\sigma}_{\mathrm{P}} \otimes 1_{\mathrm{P}} \otimes \sigma^1_{\mathrm{P}} , \quad \vec{\alpha} = 1_{\mathrm{P}} \otimes \vec{\sigma}_{\mathrm{P}} \otimes \sigma^2_{\mathrm{P}} , \quad B = 1_{\mathrm{P}} \otimes 1_{\mathrm{P}} \otimes \sigma^3_{\mathrm{P}} , \qquad \text{(A.7)}$$

where $\sigma^k_{\mathrm{P}}$ and $1_{\mathrm{P}}$ are the usual Pauli $2 \times 2$ matrices.

In the case of internal interactions $V(\vec{r})$ and $S(\vec{r})$, the total momentum $\vec{P}$ is a constant of motion and so one can use the centre of mass frame, where $\vec{P} = 0$. Then, if $V$ and $S$ do not contain spin coupling, Eq. (12) reduces to the Dirac equation for internal motion

$$[E - V - 2\vec{\alpha}\cdot\vec{p} - B(2m + S)]\psi = 0\,, \tag{A.8}$$

where $\alpha^k$ and $B$ fulfil the usual Dirac anticommutation relations

$$\{\alpha^k\,, \alpha^l\} = 2\delta^{kl}\,, \quad \{\alpha^k\,, B\} = 0\,, \quad \{B\,, B\} = 2 \tag{A.9}$$

what is readily seen from Eq. (A.6). Thus, the two-body problem for non-Abelian Dirac particles is then *exactly* reduced to the usual Dirac one-body problem, what *never* happens in the case of two-body problem for conventional (*i.e.* Abelian) Dirac particles.

The angular momentum appearing in the reduced problem is

$$\vec{j} = \vec{L} + \tfrac{1}{2}\vec{\sigma} \tag{A.10}$$

with $\vec{L} = \vec{r}\times\vec{p}$ and $\vec{\sigma} = (-i\alpha^1\alpha^2\alpha^3)\vec{\alpha}$. Of course, the total angular momentum of our two-body system is

$$\vec{J} = \tfrac{1}{2}\vec{\Sigma} + \vec{j} = \vec{L} + \tfrac{1}{2}\vec{\Sigma} + \tfrac{1}{2}\vec{\sigma} \tag{A.11}$$

with $\vec{\Sigma} = (-iA^1A^2A^3)\vec{A}$, but in Eq. (A.8) there is *no coupling* of $\vec{\Sigma}$ and $\vec{j}$ if $V$ and $S$ do not contain spin coupling (note that $\vec{\sigma}_1 + \vec{\sigma}_2 = \vec{\Sigma} + \vec{\sigma}$ and $\vec{\sigma}_1 \cdot \vec{\sigma}_2 = \vec{\Sigma}\cdot\vec{\sigma}$). Thus, in the case of central interactions $V(r)$ and $S(r)$ leading to bound states, the energy levels corresponding to eigenvalues $J = j \mp {}^1\!/_2$ are degenerate. Here, $j = {}^1\!/_2\,, {}^3\!/_2\,, {}^5\!/_2\,, \ldots$, so for the ground state, where $j = {}^1\!/_2$, one gets $J = 0, 1$, which gives a degenerate quartet of bound states.

In particular, for the Coulombic attraction $V = -\alpha/r$ and $S \equiv 0$ the Dirac equation (A.8) leads to the Sommerfeld-type energy levels

$$E = 2m\left\{1 + \left[\frac{\alpha/2}{n_r + \sqrt{(j+\frac{1}{2})^2 - (\alpha/2)^2}}\right]^2\right\}^{-\frac{1}{2}}\,, \tag{A.12}$$

where $n_r = 0, 1, 2, \ldots$ and $j = {}^1\!/_2\,, {}^3\!/_2\,, {}^5\!/_2\,, \ldots$. Note that for the ground state ($n_r = 0\,, j = {}^1\!/_2$) the energy $E \to 0$ if the coupling constant $\alpha \to 2$ (from below). So, in this case, the quartet of bound states, mentioned before, becomes *massless.*

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