THE ELEMENTARY METHOD IN PAIRING ENERGY I. THE LIKE PARTICLES

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The elementary method in pairing energy calculations has been presented: *(i)* for like-nucleons in the $j-j$ coupling; *(ii)* for like nucleons in the $l-s$ coupling; *(iii)* for bosons on the degenerated l-level. The simple explanation how and, more involved why the elementary method works has also been given.

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1. Introduction

The pairing interaction is, by definition, the interaction between pairs of nucleons which are coupled to zero angular momenta either $J=0$ in $j-j$ coupling or$L=0$ in $L-S$ coupling. A number of uncoupled nucleons in a given nuclear state are not active and they give no direct contribution to the pairing energy; however, they are responsible for the well known "blocking effect". A number of those nucleons is the seniority number v introduced in the old paper of Racah [1] for electrons in atoms and then by Flowers [2] for nucleons in nuclei. Since the introduction of the pairing interaction to nuclear theory in the early 1950, it has become one of the main ingredients of the nuclear interaction and still it has its fundamental meaning in the modern nuclear theory. The pairing interaction by comparison with the radial delta interaction has the short range character and it pushes strongly down one of the nuclear energy level with the total $J=0$ (for even nuclei). Hence, the pairing interaction acts toward the spherical symmetry of nuclei and in many cases it competes with the long range forces simulated by the quadrupole–quadrupole interaction. The actual behaviour of nuclei is

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just the result of the interplay between these two interactions. The picture is, of course, only an approximation because many other components of the nuclear forces, both of phenomenological and microscopical characters, enter in detailed considerations of the ground and excited states of nuclei.

There are, generally speaking two methods to deal with the pairing interaction. The first one had come from the theory of superconductivity [3] and had been adopted very soon in nuclear physics [4, 5]. The method is based on the quasi-particle approximation to the exact solution but it can be applied to the complicated system of many nucleons on many single particle levels with a relatively simple numerical program. The second method, an exact one, is based on the symmetry connected with the pairing interaction [6, 7]. The method gives exact results in the theory of the pairing interaction but there is a technical difficulty in applying them to many single particle nuclear levels not because of matrix element calculations but because of a very large numerical matrix program.

Since my first involvement (with Lord Flowers) in the pairing interaction some 25 years ago [8] I have been and still am astonished that some of the very sophisticated results in pairing energies can be immediately obtained by the very simple reasoning. At last I have decided to look more carefully on such an elementary simplicity and the results of my considerations are presented here.

In this paper I deal with a system of like particles, say neutrons only, both in j–j and L–S couplings. There is also considered a boson system which has become, since the Interacting Boson Model introduction to nuclear theory [9], a very important and modern nuclear problem. The next paper to be prepared, will be devoted to a much more complicated problem of a system with neutrons and protons including the full isospin formalism. We also plan to discuss the matrix element calculations using the elementary method in the pairing interaction. The off–diagonal matrix elements are necessary to consider the many single–particle level configurations.

2. The system of like–nucleons (neutrons) on the j–level

Suppose we have a system of n-neutrons on the $2j+1$ degenerated single particle level $j(n \leq 2j+1)$ out of which v-neutrons are not paired and $n-v$ neutrons are coupled in pairs with $J = 0$ each. There are $\frac{n-v}{2}$ such pairs ($n-v$ has to be an even number). Let us distribute the neutrons according to the pictorial scheme (Fig. 1). The black circles represent nucleons, the white holes, and each state $(m, \ -m)$ enters the picture with different $m > 0$ because we deal with fermions. There are three "boxes" in the picture. The first one represents the two-particle state $(m, \ -m)$ fully occupied and taken n_1 times. They are considered as paired neutrons. The second box is the box

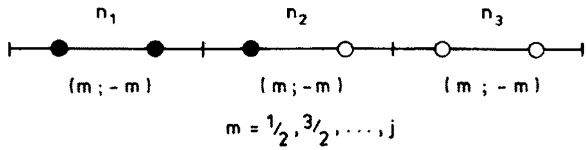

Fig. 1. Schematic picture presenting the specific configuration of like-particles (neutrons) on the single particle j-level.

repeated n_2 times with only unpaired neutrons distributed over the different $m > 0$ single particle states which prevents them from pair coupling to $J = 0$. The third box repeated n_3 times represents the rest of the unoccupied states allowed on the single-particle j-level. Before further explanation, just to attract the reader's attention we apply to the pictorial scheme the pairing Hamiltonian which can be constructed by the pair creation $a_m^+ a_{-m}^+$ and pair annihilation $a_{-m} a_m$ operators. The pairing Hamiltonian does nothing else but annihilates the pair $(m, -m)$ on each of the place (n_1 possibilities) and creates the pair $(m, -m)$ on every possible place ($1+n_3$ possibilities). There are altogether

$$n_1(1 + n_3) \tag{1}$$

such pair annihilation and pair creation actions and each single action gives an unit contribution to the pairing energy and hence

$$E_{\text{pair}} = n_1(1 + n_3), \tag{2}$$

where we assume the pairing strength $G = 1$. From Fig. 1 we get

$$\begin{aligned} n_1 + n_2 + n_3 &= \frac{2j+1}{2} \equiv \Omega \\ 2n_1 + n_2 &= n \\ n_2 &= v, \end{aligned} \tag{3}$$

where Ω is so called pair degeneracy, n is the number of neutrons and v is the seniority number. Taking a reverse transformation

$$\begin{aligned} n_1 &= \frac{n-v}{2} \\ n_2 &= v \\ n_3 &= \Omega - \frac{n}{2} - \frac{v}{2}, \end{aligned} \tag{4}$$

and introducing (4) to (2) we get

$$\begin{aligned} E_{\text{pair}} &= \frac{n-v}{4}(2\Omega + 2 - n - v) \\ &= \frac{n-v}{4}(2j + 3 - n - v). \end{aligned} \tag{5}$$

This is an exact formula for the pairing energy which could be also obtained in rather sophisticated considerations [4, 6].

There can be several objections to such a simplified treatment of the pairing energy. At first, Fig. 1 cannot represent a real physical state. It represents rather one of the configurations of a very complicated linear combination of similar terms. Secondly, even if, for some good reason, we are right to consider only a single configuration, then the H_{pair}, as we explained above, changes that particular configuration because it has moved one pair of particles from the block n_1 to the block n_3. Such an action could be rather connected with the non-diagonal matrix elements of H_{pair} than with its energy. Next, the particular configuration in the linear combination of many configurations forming a physical state has a proper weight factor coming from the complicated couplings of many particles to the total angular momentum. Hence it is not possible to consider a change of one configuration into another without taking weight factors into account. The same argument holds for the pairing Hamiltonian which also has many terms.

Let us at first explain the last two points. In the configuration presented in Fig. 1 there are two separate particle parts. One part, n_1, involves paired particles and the other one, n_2, — unpaired ones. The unpaired particles have, by definition, the zero paired energy and hence, their structure has no influence on the pairing energy or, in other words, the structure of unpaired particles forms a scalar with respect to the pairing Hamiltonian. Then the structure of unpaired particles can be taken out of the action of the pairing Hamiltonian. However, there is a visible blocking effect of those particles because they block the n_2-part of the single particle j-states and the pairing Hamiltonian can act only in the space represented by n_1 and n_3. The blocking effect is demonstrated in the energy formula (5) by the seniority v. Let us consider the configuration n_1 of neutron pairs coupled to $J = 0$ each. In the language of creation a^+ fermion operators and with the usual Clebsch–Gordan coupling coefficients a coupled pair reads

$$\begin{aligned} \left(a_j^+ a_j^+\right)^{J=0} &= \sum_m (jm\, j-m|00) a_{jm}^+ a_{j-m}^+ \\ &= \sum_m \frac{(-1)^{j-m}}{\sqrt{2j+1}} a_{jm}^+ a_{j-m}^+ \\ &= \frac{2}{\sqrt{2j+1}} \sum_{m>0} (-1)^{j-m} a_{jm}^+ a_{j-m}^+ \end{aligned} \tag{6}$$

The last expression is obtained with the help of the common fermion anticommutation relations

$$\begin{aligned} \left\{a_{jm}\,,a^+_{jm'}\right\} &\equiv a_{jm}a^+_{jm'} + a^+_{jm'}\,,a_{jm} = \delta_{mm'} \\ \left\{a^+_{jm'}\,,a^+_{jm'}\right\} &= 0 \\ \left\{a_{jm}\,,a_{jm'}\right\} &= 0 \end{aligned} \tag{7}$$

For further convenience we take the two particle coupled creation operator with a factor $\frac{\sqrt{2j+1}}{2}$

$$\frac{\sqrt{2j+1}}{2}(a^+_j a^+_j)^{J=0} = \sum_{m>0}(-1)^{j-m}a^+_{jm}a^+_{j-m} \equiv S_+\,. \tag{8}$$

It is now seen that each $(m, -m)$ configuration appears in (8) with the weight factor $(-1)^{j-m}$ equals ± 1. The same, however, phase factor is present in the proper term of the H_{pair}, and hence each of the overall weight factor brings the same unit contribution to the pairing energy. The pairing Hamiltonian is constructed in the form

$$\begin{aligned} H_{\text{pair}} &= -G\left\{\frac{\sqrt{2j+1}}{2}(a^+_j a^+_j)^{J=0}\right\}\left\{\frac{\sqrt{2j+1}}{2}(a_j a_j)^{J=0}\right\} \\ &= -G\hat{S}_+\hat{S}_-\,, \end{aligned} \tag{9}$$

where

$$\hat{S}_- = (\hat{S}_+)^+ = \sum_{m>0}(-1)^{j-m}a_{j-m}a_{jm}\,, \tag{10}$$

and $-G$ is the strength of the pairing interaction.

In our considerations the value of $-G$ is nonrelevant and we simply take $-G = 1$ and then

$$H_{\text{pair}} = \hat{S}_+\hat{S}_-\,. \tag{11}$$

The next problem is to answer the question why it is possible to consider only one of the many nucleon configurations. Suppose the physical eigenstate is formed by a complicated linear combination of the configurations, Fig. 1. Suppose that the Hamiltonian acts on that state. The result should be also the same complicated linear combination but with a common factor equal to the energy E. Hence, if we consider any single configuration of the state and if we are able to answer the question how many times that particular configuration is repeated after the action of the Hamiltonian on the full eigenstate, we will get the eigenenergy as a repetition factor. That is the basis upon which we can consider the single configuration instead of the complicated linear combination of configurations. The pairing Hamiltonian (and any other two-body interaction) could change at most the state of two particles. Hence, if we pick up any particular configuration, say that in Fig. 1 then we must look only on configurations which differ by the position at most of one pair of particle. In our case, those are the configurations with one pair of particles taken from the block n_1 to the block n_3. The proper part of H_{pair} will take back that pair from the block n_3 to the block n_1. There are $n_1 \times n_3$ such possibilities and that is also a number of repetition of a given configuration. But the H_{pair} can also annihilate the pair in the block n_1 and create it on the same place. A number of repetitions of such an action is equal to n_1. Because each of the configurations comes into the physical state with the same weight factor, the pairing energy is equal to the total number of repetitions of a given configuration

$$E = n_1 \times n_3 + n_1 = n_1(1 + n_3)$$

as obtained before (2). The language we used to get (5) differs slightly from the real action of the H_{pair} on the eigenstate. Namely, in elementary calculations of the energy (5) we said that the H_{pair} annihilated a pair in the n_1 configuration and created it in the n_3 configuration while in fact there is an opposite action. But a number of repetitions of the action and its opposite is the same and that statement ends the verification of the elementary method in pairing energy of the system with like-nucleons on the j-level.

We have given here the detailed description of the elementary method in pairing energy calculations for the simplest system of one kind of nucleons on the j-level. The other systems to be considered are more or much more complicated but the main line of verification of the elementary methods also holds but will not necessarily be given in detail. However, the rules of the elementary method in the calculation of the pairing energy are still extremely simple and calculations based on those rules are of the same simplicity as for the neutrons on the j-level.

3. The system of like-nucleons (neutrons) on the l-level

In light nuclei for which the angular momentum of a nucleon is not necessarily coupled with its spin to total j, the better basis is provided by the states with the separate coupling of angular momenta to total L and spins to total S. Hence, we should consider a configuration of neutrons on the single particle l-level. The number of single particle states on the l-level is equal to $2(2l+1)$ because of a 2-fold spin degeneracy. The pair degeneracy Ω in this case is equal to

$$\Omega = \frac{2(2l+1)}{2} = 2l + 1 \tag{12}$$

Two neutrons coupled to a zero angular momentum, $L = 0$, are symmetric in the L-space, hence, they must be antisymmetric in the spin-space and then $S = 0$ follows. For the system of n neutrons with the seniority number v (the seniority v means now the number of unpaired neutrons in the L-space) we consider the configuration similar to $j - j$ coupling.

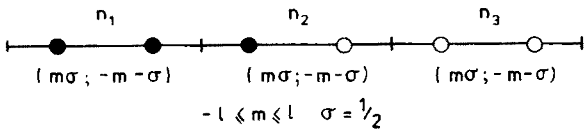

Fig. 2. Schematic picture presenting the specific configuration of like-particles (neutrons) on the single particle l-level.

In Fig. 2 σ stands for the third spin component of a neutron but an angular momentum quantum number l and spin quantum number $s = 1/2$are understood without being written. The remarks concerning the m-numbers and n_i numbers are the same as for Fig. 1. Pairing Hamiltonian acts in a quite similar way as in the j-case: it annihilates a pair of neutrons on the block n_1 (n_1 possibilities) and creates a pair on each empty space ($1+n_3$ possibilities). The pairing energy is then

$$E_{\text{pair}} = n_1(1 + n_3) \tag{13}$$

From Fig. 2 we get the same relations of n_1 , n_2, and n_3 with physical quantum numbers as in (3)

$$\begin{aligned} n_1 + n_2 + n_3 &= \Omega \\ 2n_1 + n_2 &= n \\ n_2 &= v \end{aligned} \tag{14}$$

and hence

$$E_{\text{pair}} = \frac{n-v}{4}(2\Omega + 2 - n - v) = \frac{n-v}{4}(4l + 4 - n - v) \tag{15}$$

The shape of the formula is exactly the same as in the $j - j$ coupling (5) in terms of a pair degeneracy Ω. If Ω is replaced by j or l, two formulas differ slightly. The verification of the elementary method in the $L - S$ coupling is quite the same as in the $j - j$ case.

4. The system of bosons on the l-boson level

Bosons have become very popular in nuclear theory since the construction of the Interacting Boson Model (IBM) by Arima and Iachello [9]. A boson in the model represents a properly correlated pair of shell model nucleons. Similarly to the nucleon shell model there has been considered the boson shell model. Although bosons represent pairs of nucleons, they are usually treated in the IBM as ideal bosons fulfilling the common boson commutation relations

$$[b_m , b^+_{m'}] = \delta_{mm'} \qquad [b^+_m , b^+_{m'}] = [b_m , b_{m'}] = 0 \tag{16}$$

Together with the IBM there has been introduced the pairing interaction among bosons defined similarly as the interaction acting between two bosons coupled to the total angular momentum $L = 0$. In our treatment of the boson pairing energy we will consider bosons as ideal and we neglect their nuclear structure.

Let us consider the n-boson system with the seniority number v on the single particle level of a given angular momentum l (l — necessarily an integer). The boson eigenstate of the pairing Hamiltonian is also a very complicated linear combination of many boson configurations but the reasoning of the elementary method is, in the main lines, the same as for nucleons. The difference in constructing the full eigenstate or one of its configurations as compared to the nucleon system is the possibility of putting as many bosons as one wishes on the single particle state. Sometimes it simplifies the problem, but sometimes it makes the problem more difficult. In our elementary method as well as in the group theory consideration it is rather a complication.

Let us now choose the boson configuration presented in Fig. 3 where all bosons are put in the state $(m, -m) = (l, -l)$. The schematic configuration of bosons in Fig. 3 is also divided into three parts. The first box represents all the bosons out of which $n - v$ are paired and v are unpaired with $m = -1$. The second box represents $(l - 1)$ empty pair states $(m, -m)$ for $m = 1, 2, \ldots, l - 1$. The third empty box is a single one for $m = 0$.

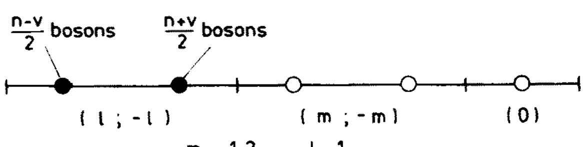

Fig. 3. Schematic picture presenting the specific configuration of like-particles (bosons) on the single particle l-level (l-integer).

In the elementary method the question is: how many times the pairing Hamiltonian will repeat the specific configuration in Fig. 3. There are altogether $\frac{n-v}{2} \times \frac{n+v}{2}$ possibilities of forming pairs in the $(l, -l)$ states, and the repetition factor from the first box is just

$$\frac{n-v}{2} \times \frac{n+v}{2}.$$

But each pair from $\frac{n-v}{2}$ pairs can be also taken either to the state $(m, -m)$ in $l-1$ different ways or to the state $m = 0$ with a weight factor $1/2$which will be explained later. Hence, the total repetition factor equal to the pairing energy reads

$$\begin{aligned} E_{\text{pair}} &= \frac{n-v}{2} \times \frac{n+v}{2} + \frac{n-v}{2} \times (l-1) + \frac{n-v}{2} \times \tfrac{1}{2} \\ &= \frac{n-v}{2}\left(\frac{n+v}{2} + l - 1 + \tfrac{1}{2}\right) \\ &= \frac{n-v}{4}(2l - 1 + n + v), \end{aligned} \tag{17}$$

and that is the exact formula for the boson pairing energy. The pair degeneracy has no meaning in the boson system because there may be on the level l as many boson pairs as one wants to take.

The explanation why the elementary method works properly in the boson system is rather different from that in the fermion case and hence, it will be presented here in detail. The simple configuration in Fig. 3 can be exactly constructed with the boson creation operator b^+

$$|c_1\rangle \equiv (b_l^+)^{(n-v)/2}(b_{-l}^+)^{(n+v)/2}|0\rangle = (b_{-l}^+)^v(b_l^+ b_{-l}^+)^{(n-v)/2}|0\rangle. \tag{18}$$

The nice feature of the configuration (18) comes from the fact that unpaired bosons are coupled to a good angular momentum, namely

$$L = vl; \qquad M = -vl. \tag{19}$$

The coupling of unpaired bosons to a given total L does not influence the pairing energy and hence, we can consider the simplest coupling (18)-(19). For a such coupling we can also write the exact eigenstate $|f\rangle$ with the help of the quasi-spin boson operator $\hat{Q}^+$

$$|f\rangle = (b_{-l}^+)^v (\hat{Q}^+)^{(n-v)/2}|0\rangle, \tag{20}$$

$$\hat{Q}_+ \equiv \frac{\sqrt{2l+1}}{2}(b^+b^+)^{L=0} = \tfrac{1}{2}\sum_{m=-l}^{l}(-1)^{l-m}b_m^+b_{-m}^+$$
$$= \sum_{m>0}(-1)^{l-m}b_m^+b_{-m}^+ + \tfrac{1}{2}(-1)^l b_0^+b_0^+ . \tag{21}$$

For further convenience we write the vector $|f\rangle$ in the form

$$|f\rangle = (b_{-l}^+)^v\left\{b_l^+b_{-l}^+ + \sum_{m=1}^{l-1}(-1)^{l-m}b_m^+b_{-m}^+ + \tfrac{1}{2}(-1)^l b_0^+b_0^+\right\}^{(n-v)/2}|0\rangle . \tag{22}$$

The pairing Hamiltonian is constructed analogously to the fermion case as

$$H_{\text{pair}} = \hat{Q}_+\hat{Q}_- , \tag{23}$$

where

$$\hat{Q}_- = (\hat{Q}_+)^+ = \sum_{m>0}(-1)^{l-m}b_{-m}b_m + \tfrac{1}{2}(-1)^l b_0b_0 .$$

The main question in the explanation of the elementary method is: how many times the specific configuration (18) will be repeated after the action of the H_{pair} (23) on the full state (22)? To discuss the answer we need only such a part of the full state (22) which differs from the specific configuration (18) at most by the position of one pair. That part is

$$|f\rangle_{\text{part}} = (b_{-l}^+)^v\left\{(b_l^+b_{-l}^+)^{(n-v)/2} + \frac{n-v}{2}(b_l^+b_{-l}^+)^{[(n-v)/2]-1}\sum_{m=1}^{l-1}(-1)^{l-m}b_m^+b_{-m}^+ + \frac{n-v}{2}\times\tfrac{1}{2}(-1)^l(b_l^+b_{-l}^+)^{[(n-v)/2]-1}b_0^+b_0^+\right\}|0\rangle . \tag{24}$$

Let us analyse each of the three parts of (24). The first part is just our chosen configuration $|c_1\rangle$ (18). The proper part of the H_{pair} (23) to repeat this configuration is

$$b_l^+b_{-l}^+b_{-l}b_l ,$$

and

$$b_{-l}b_l|c_1\rangle = b_{-l}b_l(b_l^+)^{(n-v)/2}b_{-l}^{(n+v)/2}|0\rangle = \frac{n-v}{2}\times\frac{n+v}{2}(b_l^+)^{[(n-v)/2]-1}(b_{-l}^+)^{[(n+v)/2]-1}|0\rangle$$

and hence

$$b_l^+ b_{-l}^+ b_{-l} b_l |c_1\rangle = \frac{n-v}{2} \times \frac{n+v}{2} |c_1\rangle \tag{25}$$

showing that the repetition factor coming from $|c_1\rangle$ is

$$\frac{n-v}{2} \times \frac{n+v}{2}.$$

Let us now pick up from the second part of (24) *i.e.* from

$$|c_2\rangle \equiv \left(b_{-l}^+\right)^v \times \frac{n-v}{2} \left(b_l^+ b_{-l}^+\right)^{[(n-v)/2]-1} \sum_{m=1}^{l-1} (-1)^{l-m} b_m^+ b_{-m}^+ |0\rangle$$

one term with a fixed m_0

$$|c_2\rangle_{m_0} = \left(b_{-l}^+\right)^v \times \frac{n-v}{2} \left(b_l^+ b_{-l}^+\right)^{[(n-v)/2]-1} (-1)^{l-m_0} b_{m_0}^+ b_{-m_0}^+ |0\rangle \tag{26}$$

The proper part of H_{pair} to act on (26) is

$$b_l^+ b_{-l}^+ (-1)^{l-m_0} b_{-m_0} b_{m_0}$$

and hence

$$b_l^+ b_{-l}^+ (-1)^{l-m_0} b_{-m_0} b_{m_0} |c_2\rangle_{m_0} = \frac{n-v}{2} |c_1\rangle\,. \tag{27}$$

There are $(l-1)$ configurations of the type (26) and the repetition factor from the second part, $|c_2\rangle$, of (24) is

$$\frac{n-v}{2}(l-1)\,. \tag{28}$$

The last part of (24), $|c_3\rangle$, is

$$|c_3\rangle = \left(b_{-l}^+\right)^v \times \frac{n-v}{2} \tfrac{1}{2} (-1)^l b_0^+ b_0^+ \left(b_l^+ b_{-l}^+\right)^{[(n-v)/2]-1} |0\rangle$$

and the proper part of H_{pair}, to act on $|c_3\rangle$, is

$$b_l^+ b_{-l}^+ \times \tfrac{1}{2} (-1)^l b_0 b_0$$

and then

$$b_l^+ b_{-l}^+ \times \tfrac{1}{2} (-1)^l b_0 b_0 |c_3\rangle = \frac{n-v}{2} \times \tfrac{1}{2} |c_1\rangle\,, \tag{29}$$

where we have got the weight factor $1/2$which is a product of three factors $1/2 \times 1/2 \times 2$ coming from the calculation (29).

In the formulas (28) and (29) the factor $\frac{n-v}{2}$ is connected rather with creation of a pair and the factor $(l-1)$ in (28) has come from a pair annihilation because the H_{pair} changes the configuration (24) to the specific configuration (18) while in our "elementary language" we assumed an opposite transformation. In the first contribution to E_{pair} given by (25) the repetition factor has come entirely from the pair annihilation and that is the reason that we did not divide this term, in our elementary method, into two parts. Taking the three contributions (25), (28), (29) we get the formula (17).

Although, the group theory treatment in the boson pairing energy is much more complicated than for the fermions mostly because of the uncommon non-compact group SU(1,1) [10] involved, the elementary method in both cases is astonishingly similar. In the next paper we will apply the elementary method to the case of protons and neutrons including the isospin formalism and we will show that the pairing energy treatment is also very simple in spite of the fact that the symmetry groups are still much more complicated, namely there are orthogonal groups in five- and eight-dimensional abstract spaces.

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