LONG-LIVED PERIODIC TRANSIENTS

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A critical behavior before saddle-node bifurcation is examined. The existence of long-lasting periodic transients near not-yet-born periodic orbits is shown. Influence of external small perturbation is discussed. Properly chosen perturbation may stop escaping from neighborhood of these orbits. In such dynamical system we can store well-determined amount of energy for reasonably long time and by switching off external perturbation we can release this energy.

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Unstable periodic orbits play a fundamental role in the analysis of chaotic dynamical systems [1,2]. It is well known that strange hyperbolic attractors or repellers can be regarded as a closure of the set of all unstable periodic orbits. Such orbits offer a clear insight into the hierarchal organization of these complicated fractal sets. Knowing the number and the stability of unstable cycles one can determine the whole spectrum of dynamical invariants, like generalized dimensions, entropies, and Lyapunov exponents. For example, by simply counting the number of unstable m-cycles it is possible to estimate the topological entropy K_0

$$K_0 = \lim_{m\to\infty} \frac{1}{m} \ln M(m), \tag{1}$$

where $M(m)$ stands for the number of all unstable orbits with period m. Fortunately, the dynamical characteristics seem to converge fast enough with the increasing length of period and therefore K_0 can be obtained with quite good accuracy from relatively short cycles [3].

Varying continuously the control parameter λ of a given dynamical system we can observe how different periodic orbits are born, evolve, and eventually die. Many unstable orbits are a reminder of stable ones which lost earlier their stability as λ was changed from the previous value to the current one. Sometimes, when λ passes over a critical value λ^*, a phenomenon called the boundary crisis can occur and chaotic transient is observed [4,5,6]. In the boundary crisis case the basin of attraction of chaotic attractor is destroyed, the basin boundary is rendered and trajectories escape the region coinciding with the basin just before the crisis. The stability of attractor is lost and instead of an attracting invariant set we have a repulsing one, called repeller. The escape time of a particular trajectory depends strongly on the choice of initial point, but starting with sufficiently many different points from a given region, one can construct the appropriate statistics and calculate the averaged lifetime which depends only on system parameter λ. The distribution of lifetimes of particular transient trajectories is given by

$$L(n) \sim \mathrm{e}^{-\kappa n}, \tag{2}$$

where $L(n)$ is the number of trajectories not yet escaped from the given neighborhood of the repeller after n iterations. Coefficient κ is called the escape rate from the given region and is connected with the mean lifetime $\langle n \rangle$ as $\kappa = \langle n \rangle^{-1}$. This quantity is related to other dynamical invariants which, as stated above, can be extracted from the knowledge of unstable orbits [4].

In this paper we are interested in another range of the control parameter λ, in which stable periodic orbits start to exist via saddle-node bifurcation when λ is passing through a critical value λ_0 from below [7], see Fig. 1. In some sense the control parameter λ measures the 'age' of cycle: all unstable orbits are 'old', because they had enough 'time' to lose their stability if they were not initially born as unstable ones. Here, we investigate very 'young' stable cycles which have just been born or even in their 'prenatal' life, when λ is close to but less than λ_0. We call these not-yet-born orbits as the precursors [8] of stable periodic orbits (periodic precursors). In what follows, we limit our considerations to the two-dimensional dissipative smooth maps, although extension to higher-dimensional systems is straightforward.

Let us consider a dissipative 2-d map $\boldsymbol{T}_\lambda$

$$\boldsymbol{x}_{n+1} = \boldsymbol{T}_\lambda \boldsymbol{x}_n, \tag{3}$$

where $\boldsymbol{T}_\lambda$ is a smooth map and λ stands for a control parameter of the map. Let us assume that at $\lambda = \lambda_0^{(m)}$ a stable m-cycle is born via a saddle-node bifurcation. One then finds an $\boldsymbol{x}^*$ for which

$$\boldsymbol{x}^* = \boldsymbol{T}^m_{\lambda_0^{(m)}} \boldsymbol{x}^*. \tag{4}$$

For simplicity let us choose $m = 1$ and denote $\lambda_0 \equiv \lambda_0^{(m)}$ (considerations for $m > 1$ are nearly the same and numerical results for $m = 2$ and $m = 3$

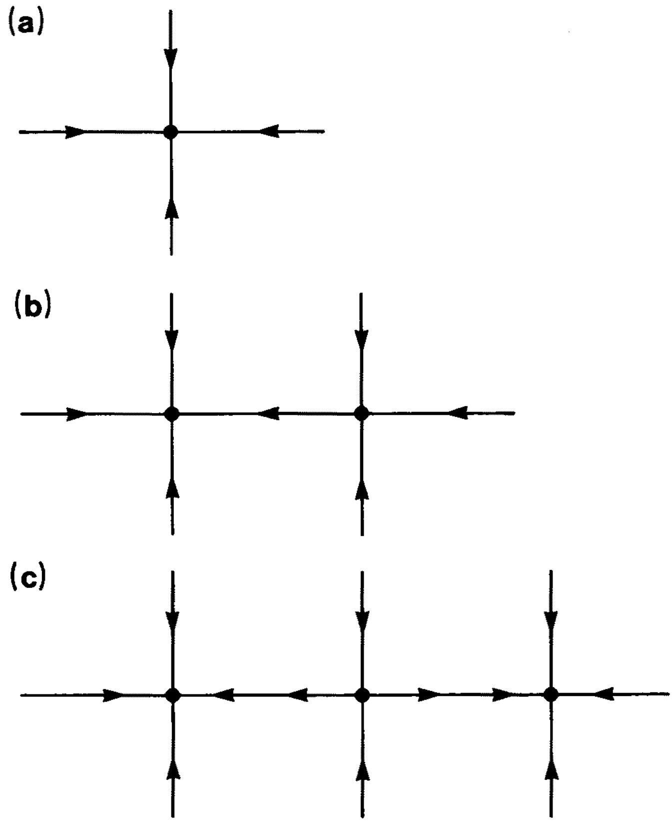

Fig. 1. Schematic phase portraits for control parameter near the saddle-node bifurcation where a pair of stable and unstable cycle is born. Additionally, there exist an attractor in the whole discussed here range of λ: a — $\lambda < \lambda_0$; b — $\lambda = \lambda_0$; c — $\lambda > \lambda_0$.

will be given). For $\lambda < \lambda_0$ Eq. (4) has no real solution in the neighborhood of $\boldsymbol{x}^*(\lambda_0)$ and one can observe a systematic drift of successive images $\boldsymbol{x}_n$ of point $\boldsymbol{x}_0 = \boldsymbol{x}^*(\lambda_0)$, see Fig. 2. However, if λ is close to λ_0 one can make the approximation $\boldsymbol{x}_n \approx \boldsymbol{x}^* + \delta\boldsymbol{x}_n$, where the time evolution of vector length $||\delta\boldsymbol{x}_n||$ is linear in a suitable range of n

$$||\delta\boldsymbol{x}_n|| \approx (1 + n\alpha(\lambda)). \tag{5}$$

The drift velocity $\alpha(\lambda)$ can be expanded up to the first order

$$\alpha(\lambda) \approx (\lambda_0 - \lambda), \tag{6}$$

with the condition $\alpha(\lambda_0) = 0$.

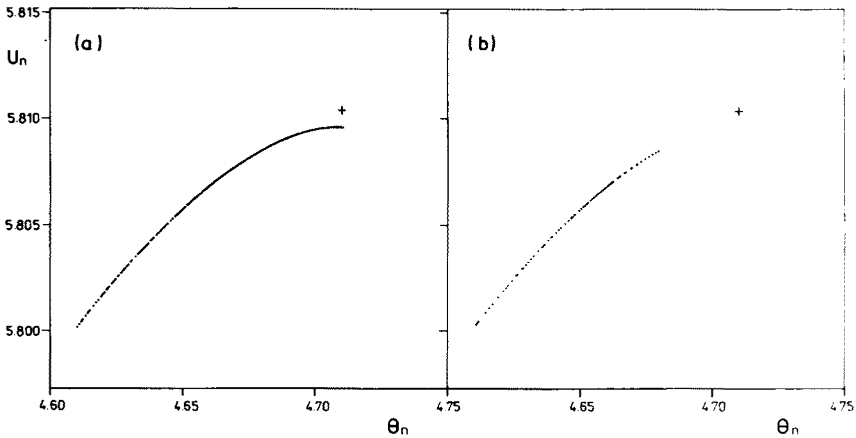

Fig. 2. Drift of nearly periodic evolution near precursor of period one (marked by cross) for $\lambda = \lambda_0 - \Delta\lambda$. Here, the map $\boldsymbol{T}_\lambda$ is given by Eqs (7), (8) and $\boldsymbol{x}_n = (u_n, \theta_n)$. 60×60 initial points are iterated n_1 times and if after next n_2 iterations their images fall into the given neighborhood of precursor they are plotted. The value of parameters: $\Delta\lambda = 2 \times 10^{-4}, n_1 = 10$, and a — $n_2 = 30$; b — $n_2 = 60$.

Thus, in the subcritical region of λ, just before the saddle-node bifurcation, the situation is as follows. We have an exponentially fast attraction along the contracting direction and a slow linear drift along the other. These two different time scales cause a very surprising effect. Trajectories starting from a given neighborhood of $\boldsymbol{x}^*$ are initially attracted to this point which will soon become a normal attractor as λ passes λ_0, then they spend a certain time in the close vicinity of this point, and finally they all leave this region and never return to it. We can define the lifetime of a single transient trajectory as the largest number N such that $\|\delta\boldsymbol{x}_N\| < \varrho$, where $\varrho > 0$ is a small constant. The value of N depends both on the initial point and on the value of the control parameter. For λ closer to λ_0 one can observe longer and longer periodic transient, see Eq. (6). A drift velocity α measures how fast trajectories leave the region around $\boldsymbol{x}^*$. Thus, α may be regarded as a counterpart of the escape rate κ for chaotic transient.

Below we demonstrate this type of behavior on the example of the bouncing ball model [9]. In this system a ball is jumping in constant gravitational field on periodically vibrating surface $h_s(\theta)$. The dynamics of the ball can be reproduced by iterating the following two-dimensional dissipative map

$$v_{n+1} = k(2\tau_n - v_n) + (1+k)\dot{h}_s(\theta_{n+1}), \qquad \theta_{n+1} = \theta_n + \tau_n, \qquad (7)$$

where $\dot{h}_s$ is the velocity of surface and τ_n denotes the time interval between two successive impacts satisfying the relation

$$-\tau_n^2 + v_n \tau_n + h_s(\theta_n) = h_s(\theta_n + \tau_n). \tag{8}$$

Here, the dimensionless variables θ_n and v_n denote the phase and the velocity with which the ball starts to fly just after the n-th impact, k stands for the coefficient of restitution and is kept constant ($k = 0.86$), while λ is proportional to the amplitude of surface vibration and is used as a control parameter. Trajectories generated by the introduced above map are usually observed on $[u_n, \theta_n]$ or $[\tau_n, \theta_n]$ plane, where $u_n = v_n - \dot{h}_s(\theta_n)$, because these variables are accessible in the real mechanical experiments [9]. For $h_s(\theta) = \lambda/(1+k)\cos\theta$, which is used in the current study, the map can be reduced to the standard Zaslavsky map if we neglect the changes in the position of the vibrating surface at the moment of impact, *i.e.*, $h_s(\theta_n) = 0$ but $\dot{h}_s(\theta) \neq 0$. In this model there are many different coexisting attractors and the structure of their basins may be very complicated [10]. Fig. 3 gives

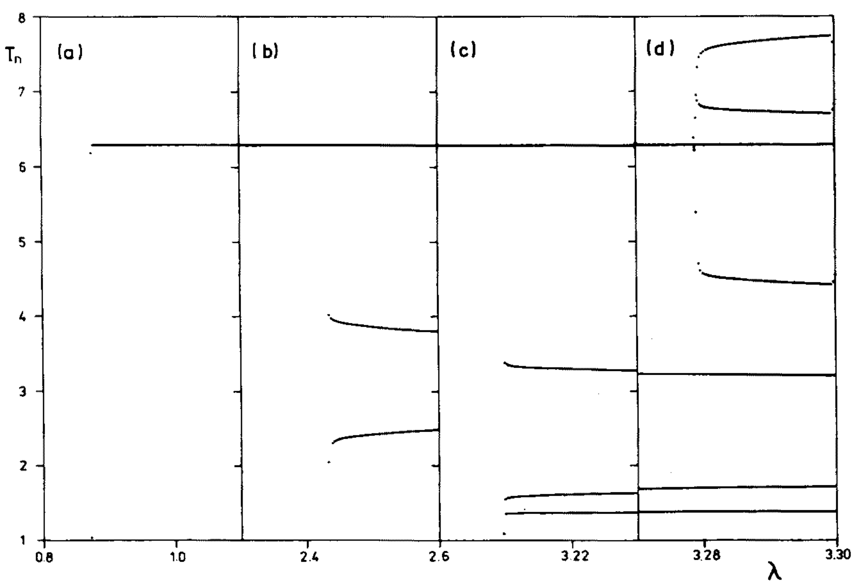

Fig. 3. A part of bifurcation diagram τ_n *vs* λ where stable cycles are born: a — one-period; b — two-period; c — three-period; d — another three-period. In the whole range of λ exists also attractor defined by $\tau_n = 0$ ('mute' mode [9]).

the bifurcation diagram in the range of the control parameter λ where the lowest stable 1-cycle, 2-cycle, and two 3-cycles are born. For period-one we know the value of $\lambda_0^{(1)}$ explicitly as well as the position of the periodic point $\boldsymbol{x}^* = (\theta^*, \tau^*)$

$$\lambda_0^{(1)} = (1-k)2\pi, \tag{9}$$

and

$$\tau^* = 2\pi, \qquad \sin\theta^* = -\frac{\lambda_0^{(1)}}{\lambda}\,. \tag{10}$$

Thus, period one denotes the simple sequence of regular jumps on the vibrating surface with constant period τ^*. There are also possible other fixed points of period one, for which

$$\lambda_0^{(1,j)} = j(1-k)2\pi, \tag{11}$$

and

$$\tau^* = j2\pi, \qquad \sin\theta_j^* = -\frac{\lambda_0^{(1,j)}}{\lambda}\,, \tag{12}$$

where $j = 2, 3, 4, \ldots, j_{\mathrm{max}}$, and for a given λ and k the highest allowed excitation of period-one is determined by

$$j_{\mathrm{max}} = [\lambda/\lambda_0^{(1,1)}]_{\mathrm{int}}\,. \tag{13}$$

For the other m-cycles only numerically determined values of $\lambda_0^{(m)}$ are available. Let us mention here that dynamics of the bouncing ball seems to be much more complicated than that generated by the logistic map. In the latter case the number of cycles with period one is independent of the control parameter and is equal to two. Similarly, the number of other cycles is bounded from above by 2^m for all value of the control parameter, while such an upper bound is not known to exist in the bouncing ball case.

In our model the numerical calculations are performed in the following way. We determined with a given accuracy the value of $\lambda_0^{(m)}$. Then we choose λ close to but still greater than $\lambda_0^{(m)}$ and we determine the basin of attraction for the m-cycle. Next, we slightly decrease λ below the critical value and we iterate forward the starting points placed in the region coinciding with the basin of attraction for $\lambda > \lambda_0^{(m)}$. First n_1 iterations are discarded, and if after the next n_2 iterations we find trajectories inside a small ball of radius ϵ around the points of stable cycle, *i.e.*, the cycle which starts to exist and be stable at $\lambda = \lambda_0^{(m)}$, we plot such starting points on the $[u_n, \theta_n]$ plane. The results shown in Figs 4,5 confirm the existence of long-lived periodic transients. Depending on the starting point and the value of λ one can see even hundreds of repetitions of nearly the same periodic points, Fig. 5d. These regular sequences are always ended by a sudden transition to another period, on which the system stays forever. Of course, periodic precursors are not the real stable orbits and therefore the 'basins' of attraction for $\lambda < \lambda_0^{(m)}$ cannot be regarded as the usual ones.One should

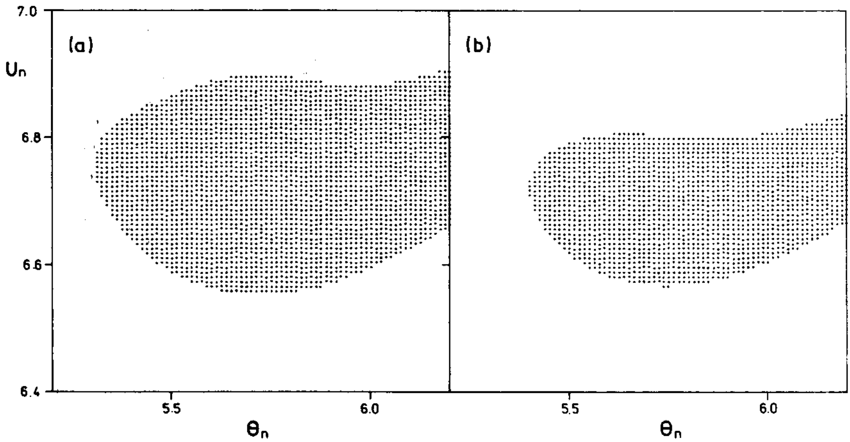

Fig. 4. A part of 'basin' of attraction of three-cycle on $[u, \theta]$ plane. 80×80 starting points are iterated n_1 times and if after it the distance to any point of cycle is less than ϵ they are iterated n_2 times further. The starting point is plotted if its image after $n_1 + n_2$ iterations is still within the ball of radius ϵ centred at any point of three-cycle. The value of parameters: $\lambda = 3.2819, n_1 = 25, \epsilon = 0.1$ and a — $n_2 = 10$; b — $n_2 = 100$.

rather think of the 'basin' of attraction for periodic precursor as of a set of all initial points which tend to precursor and stay in its close neighborhood during at least n_2 iterations. Let us notice that with n_2 larger and larger the volume of basin is not shrinking uniformly, see Fig. 4, and some parts of basin disappear very quickly (for small n_2) while others are more persistent.

In order to check numerically that the periodic transient is caused by two different time scales along the attracting and non-attracting direction, we verify Eq. (5) for the lowest 1-cycle where (θ^*, τ^*) and $\lambda_0^{(1)}$ is given analytically by Eqs (9), (10). Indeed, the results shown in Fig. 6 confirm this explanation. On the vertical axis the quantity $||\delta \boldsymbol{x}_n||/||\delta \boldsymbol{x}_0|| - 1$ versus the number of iterations n is plotted ($\delta \boldsymbol{x}_0$ is the vector of small initial deviation). After a short nonlinear part of evolution, which corresponds to the initial attraction in the stable direction, one can see the linear dependence in a large range of n. It corresponds to a temporary locking on the periodic precursor. The ending part of the evolution is again nonlinear and it describes the escape from the neighborhood of $\boldsymbol{x}^*$. According to Eqs (5), (6) one can see that the slope of linear part of evolution depends strongly on the value of control parameter λ. Thus, one can observe longer periodic transient for λ closer to bifurcation point $\lambda_0^{(1)}$, see Fig. 6.

Described above behavior near a saddle-node bifurcation seems to be very similar to the critical behavior near tangent bifurcation where intermit-

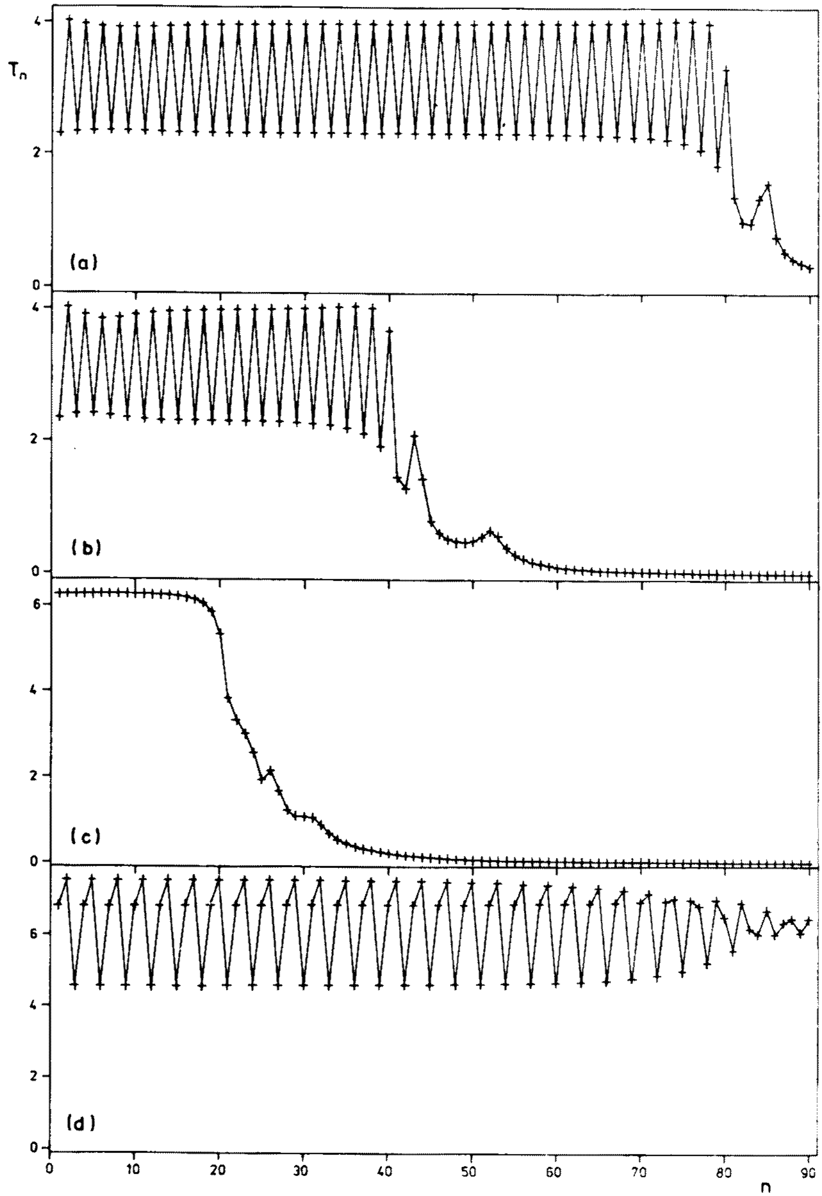

Fig. 5. Examples of transient trajectory initiated inside the basin of attraction of different periodic precursors. The variable τ_n *versus* the number of iteration n is plotted: a — $\lambda = 2.44, u_1 = 3.5, \theta_1 = 2.8$; b — $\lambda = 2.44, u_1 = 3.54, \theta_1 = 2.8$; c — $\lambda = \lambda_0 - 0.001, u_1 = 5.95, \theta_1 = 4.6$; d — $\lambda = 3.2819, u_1 = 7.605, \theta_1 = 5.6715$. In the last case we discarded 380 initial iterations during which the trajectory was locked on precursor of period three. The final attractor is $\tau_n = 0$ for a — c, and $\tau_n = 2\pi$ for d).

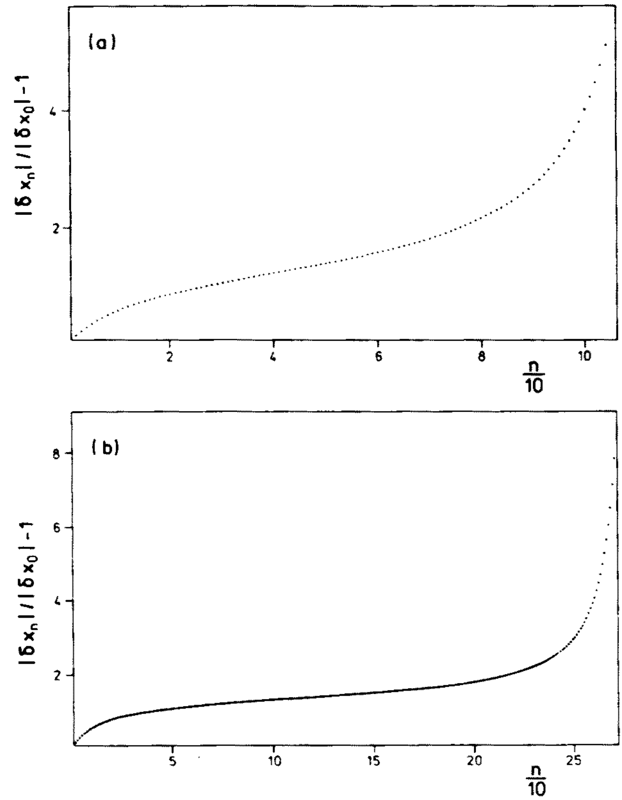

Fig. 6. Growth of the length of deviation vector $\delta \boldsymbol{x}_n = \boldsymbol{x}^*(\lambda_0) - \boldsymbol{x}_n$. Here $\boldsymbol{x}^*$ is the lowest one-cycle, $||\delta \boldsymbol{x}_0|| = 0.01$ and $\lambda = \lambda_0 - \Delta\lambda$: a — $\Delta\lambda = 10^{-4}$; b — $\Delta\lambda = 2 \times 10^{-5}$. On the vertical axis the quantity $||\delta \boldsymbol{x}_n||/||\delta \boldsymbol{x}_0|| - 1$ is plotted, see Eq. (5). Notice the increase in the range of n for b).

tency occurs. In both cases one can observe a long lasting, nearly periodic evolution, which is suddenly interrupted. However, there is an essential difference between these two types of behavior. In the intermittency case we observe alternating chaotic and laminar sequences. After irregular evolution the system is again reinjected to the regular one and so on. On the contrary, the critical behavior described above near a saddle-node bifurcation has no such vivifying mechanism. This is caused by the fact that the final attractor, on which a transient trajectory finally settles down, is a very simple one (a single point characterized by $\tau = 0$ or $\tau = 2\pi$, see Fig. 4). If this final attractor were a fractal one and if it could penetrate the neighborhood of periodic precursor (its 'basin' of attraction), then observed behavior would have an intermittent character. Therefore, because of the difference in critical behavior near the saddle-node and the tangent bifurcation, we use the name 'periodic transient' for the first phenomenon, as it shows better analogy with chaotic transient: trajectories have only one and abrupt transition from one kind of motion to another, but in contrast to chaotic transient the observed motion is now periodic.

Up to now we discussed the behavior of dynamical system without any external perturbation. However, when control parameter is close to critical value the presence of perturbation has an essential influence on time evolution of dynamical system. Periodic transient can be viewed as a slow systematic drift near periodic precursor. Thus, applying a proper small external perturbation acting against this drift we can delay or even completely stop escape from vicinity of periodic precursor. Instead of Eq. (3) we have now

$$\boldsymbol{x}_{n+1} = \boldsymbol{T}_\lambda \boldsymbol{x}_n + \boldsymbol{p}_n, \tag{14}$$

where $\boldsymbol{p}_n$ is small external perturbation. In order to get a full compensation of drift and observe stationary behavior of system this perturbation should fulfill following condition: if for a given n $\|\boldsymbol{x}^* - \boldsymbol{x}_n\| > \varrho_0$, where $\varrho_0 > 0$ is a small constant, then $\boldsymbol{p}_n$ is such that $\|\boldsymbol{x}^* - \boldsymbol{x}_{n+1}\| < \varrho_0$. This condition is very general and nearly periodic stationary evolution can be sustained by any specific perturbation. In particular, perturbing signal may be periodic as well as aperiodic. Thus, even small but properly chosen perturbation can change qualitatively the global character of evolution — from transient to permanent nearly periodic oscillation. Transient behavior may occur if the above mentioned condition for $\boldsymbol{p}_n$ is not fulfilled at few successive moments $n, n+1, \ldots$. In this case external perturbation can be effectively directed downward or upward to the drift and in comparison with a free nonperturbated drift observed escape from neighborhood of precursor may be accelerated or delayed, respectively.

Numerical investigation of the bouncing ball model with added small perturbation confirm all discussed here kinds of behavior. We choose parameter λ close to λ_0, where the lowest stable cycle one starts to exist. Next, we begin to iterate initial point from the 'basin' of attraction of corresponding precursor and we stop this process before transient trajectory starts to leave fairly the close neighborhood of precursor ($n < 15$ in Fig. 5c). The last iterated point is chosen as a point of reference $\boldsymbol{x}_r = (\theta_r, \tau_r)$. At this moment we switch on an external perturbation and start to iterate Eq. (14). As a specific realization of perturbation we choose the following one: $\boldsymbol{p}_n = \beta(\boldsymbol{x}_r - \boldsymbol{x}_{n-j})\delta_{nl}$, where δ_{nl} is the Kronecker symbol, $j \geq 0$ and β stands for scale factor ($0.8 < |\beta| < 1.2$). Successive nonzero kicks occur for $n = l_1, l_2, \ldots$ and by proper choice of l_i we can deal with periodic or aperiodic perturbation. Typical mean amplitude of $\|\boldsymbol{p}_n\|$ does not exceed 2×10^{-3}. When transient trajectory starts to leave neighborhood of precursor and $\|\boldsymbol{p}_n\|$ becomes larger than a certain number ($\sim 5 \times 10^{-2}$) we stop external perturbation and continue iterations of original map (3). Choosing different values of parameters β and j we can reproduce all described earlier kinds of behavior. Generated in this way pictures are quite similar to this one in Fig. 5c, the only difference concerns the range of n where rapid transition takes place from higher energy state ($\tau_n = 2\pi$) to the ground state ($\tau_n = 0$). Now, in contrast to free drift shown in Fig. 5c, we can keep the region of transition under control and by proper choice of parameters β and j we can shift this region to the right or left, *i.e.*, we can make periodic transient arbitrary long or shorter comparing to the unperturbated one.

Although described phenomenon is demonstrated on peculiar mechanical model, similar behavior is expected in any dynamical system for control parameter close to critical value, where configuration in phase space looks like in Fig. 1. Such system with properly selected perturbation has a very interesting property. It can serve as a storage of energy which can be released on request. Because the initial and final state have unambiguously determined energy, transition between periodic precursor and periodic attractor is connected with well defined amount of energy. In order to get this energy it suffers to switch off external perturbation. Of course, in order to sustain nearly periodic oscillation we must pump into the system an extra amount of energy. Its total quantity depends on our requirement, *i.e.*, how long we need to prevent the natural escape from vicinity of periodic precursor. In particular, this energy can be arbitrary large for infinitely long time. It should be noticed however that drift near precursor is very slow and averaged amplitude of perturbing kicks p_n is few orders of magnitude smaller comparing to available transition energy. Thus, during the time scale of practical interest we do not need to pay high price for keeping dynamical system ready to emit energy.

It should be stressed that dynamical system has this useful property only for properly chosen control parameter and initial conditions. Any other transient behavior, for example chaotic transient, or intermittency does not have these features. Although in the case of intermittency we can apply similar external perturbation and we can sustain nearly periodic oscillation as long as we wish but transition energy between two states is not well determined due to chaotic character of one of the states (counterpart of permanent attractor in the case of periodic transient). Therefore only periodic transient with added proper perturbation can be used as a storage of strictly determined quantity of energy which can be released on request.

Application of properly selected external perturbation is very similar to the recently proposed procedure known as the controlling chaos [11]. This procedure is based on the standard concept of chaotic attractor as a set of unstable periodic cycles. Assuming our knowledge (based only on experimental observation) about position and stability of a given cycle as well as our ability to change the control parameter in a small interval around its given value, we can force dynamical system to leave chaotic evolution and follow this particular cycle. This procedure can be applied to permanent as well as transient chaos [12]. In the last case the controlling is even easier because a time in which control is achieved is shorter comparing to permanent chaos. Described here periodic transient is the simplest case and we can control it not only by one specific perturbation but by a very vast class of small external perturbations. This is due to the fact that permanent and transient chaos are very sensitive simultaneously to a value and a direction of the perturbing kicks while for periodic transient the dependence on a perturbation amplitude is much weaker.

The last remark concerns the relation of periodic precursors to the 'old' unstable orbits. From purely theoretical point of view the existence of precursors should not be dangerous. However, in practical applications when one must deal with experimental noisy time-series or wants to extract the dynamical invariants from the numerically determined periodic unstable orbits, some problems may arise [3]. In both cases one can overestimate the number of 'real' unstable orbits. In order to estimate the entropy K_0 of the strange set, only 'old' unstable orbits should be counted and therefore all periodic precursors, if by chance recorded or numerically found, should be discarded . This situation can be especially dangerous for systems which have many saddle-node bifurcations at values which are not well separated on the axis of control parameter λ. Then, the different subintervals $\Delta\lambda_0^{(m)}$, in which the corresponding not-yet-born orbits exist, may overlap and for a given fixed value of control λ many such orbits can simultaneously have their own 'basins' of attraction and finite, nonzero lifetimes (see Eqs (11), (13) when $k \to 1$). This may lead to essential errors in estimating the topological entropy. Similar phenomenon was recently found in another high dimensional dissipative system (laser system), where something like 'drifting repeller' was observed [13]. Normal repeller can be viewed as a set of all unstable periodic orbits. Thus, periodic precursor and periodic transient may be a prototype of more complicated behavior.

Summarizing, we can conclude that a counterpart of chaotic transient arising after boundary crisis is the critical behavior just before a saddle-node bifurcation. Both transients can have very long lifetimes. The existence of periodic transients and periodic precursors may probably be easier to overlook because they behave initially like normal stable cycles and seem to be much less interesting than chaotic transients and repellers. However, if the population of coexisting precursors is high, then their existence is of great importance. Periodic transient may be arbitrarily prolonged by properly selected external perturbation. Mean amplitude of perturbing kicks is few orders of magnitude smaller comparing to transition energy between periodic precursor and periodic attractor. Therefore in such system we can store precisely determined amount of energy for reasonable long time and by switching off perturbation we can release this energy.

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