ON THE CONSTRUCTION OF HIGHER SPIN MULTIPLETS

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Two independent methods are employed to deduce the multiplet structure of a higher spin superspace. The first is based on the usual theta expansion. A general diagrammatic method is developed for working out the various components of the superfield. The case of the superfield with a vector spinor SUSY coordinate is worked out in detail. The second approach utilizes the Wigner method of induced representations. The two approaches are shown to give analogous results for the representations.

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1. Introduction

It is well known [1] that conventional SUSY is based on generators that belong to the spin half representation of the Lorentz group. However, consistent SUSY algebras with generators belonging to higher spin can be constructed [2, 3] provided the assumptions underlying the Haag, Łopuszański and Sohnius theorem [4] are relaxed. These algebras must also satisfy the constraints imposed by the Coleman Mandula theorem [5]. Explicit demonstration of this algebra has been shown to occur for the case of a vector spinor generator, $Q_{\mu\alpha}$, of the Lorentz group, where μ is a vector index and α is a spinor index. Here we show that the algebra also leads to a superspace [6, 7, 8] with vector spinor SUSY coordinates. Furthermore, we construct the superfield and discuss the particle content of the multiplet.

The particle content in a higher spin SUSY theory can be determined in at least two different ways. One is based on the supercoordinate expansion and does not require *a priori* knowledge of the algebra. The second method involves the Wigner method of induced representations [9, 10], and requires *a priori* knowledge of the SUSY algebra. In what follows we employ both approaches and determine the component fields for the $^1/_2 \leq j \leq ^3/_2$ superspace.

In Section 2 we work out the superspace formalism for the $(1, ^1/_2) + (^1/_2, 1)$ case explicitly. In Section 3, the theta variable expansion of the superfield is determined by the diagrammatic method. In Section 4 the Wigner method is applied to the algebra. It is shown that the two independent methods lead to the same particle content.

2. (1, 1/2)+(1/2, 1) superfield

Let $Q_{a\alpha}$ represent the $(1, ^1/_2)+(^1/_2, 1)$ fermionic, *i.e.*supersymmetric, generator; P_a, the four-dimensional translation generator and M_{ab}, the four-dimensional rotation generator. The algebra which has been shown to close under the Jacobi identities is as follows:

$$
\begin{aligned}
[M_{ab}, M_{cd}] &= -i\big(\eta_{ac}M_{bd} - \eta_{ad}M_{bc} + \eta_{bd}M_{ac} - \eta_{bc}M_{ad}\big)\,, && (2.1)\\
[M_{ab}, P_c] &= -i\big(\eta_{ac}P_b - \eta_{bc}P_a\big)\,, && (2.2)\\
[M_{ab}, Q_{c\alpha}] &= -\tfrac{i}{2}\big(\gamma_{ab}Q_c\big)_\alpha - i\big(\eta_{ac}Q_{b\alpha} - \eta_{bc}Q_{a\alpha}\big)\,, && (2.3)\\
[M_{ab}, \bar{Q}^\alpha_c] &= +\tfrac{i}{2}\big(\bar{Q}_c\gamma_{ab}\big)^\alpha - i\big(\eta_{ac}\bar{Q}^\alpha_b - \eta_{bc}\bar{Q}^\alpha_a\big)\,, && (2.4)\\
[P_a, P_b] &= 0\,, && (2.5)\\
[P_a, Q_{b\alpha}] &= 0 = [P_a, \bar{Q}^\alpha_b]\,, && (2.6)\\
\{Q_{a\alpha}, \bar{Q}^\beta_b\} &= a\Big[\eta_{ab}P^\beta_\alpha - \tfrac{1}{5}\big(\gamma_a P_b + \gamma_b P_a\big)^\beta_\alpha + \tfrac{3i}{5}\epsilon_{abcd}\big(\gamma^c\gamma_5\big)^\beta_\alpha P^d\Big]. && (2.7)
\end{aligned}
$$

This algebra contains negative norm states. The handling of the negative norm states will require the same techniques used in quantum electrodynamics and string theory *i.e.*either the Gupta–Bleurer method or the BRST operator method. However, in this article we do not attempt to construct a Lagrangian for the superfield. Therefore, we will present the detail of how to get rid of the negative norm states in a future publication. The $Q_{a\alpha}$ is assumed to be Majoranic, *i.e.*, $Q^C_{a\alpha} = \big(C\bar{Q}^{\mathrm{T}}_a\big)_\alpha = Q_{a\alpha}$. Furthermore, being a $(1, ^1/_2)+(^1/_2, 1)$ charge, it is transverse in spinor space, *i.e.*, obeys the identity $\gamma^a Q_a)_\alpha = 0$.

Associated with the irreducible vector-spinor charge $Q_{a\alpha}$ are the vector-spinor coordinates $\Theta_{a\alpha}$. The vector-spinor coordinates belong to the irreducible $(1, {}^1\!/_2)+({}^1\!/_2, 1)$ representation of the Lorentz group and are Majorana spinors,

$$\Theta_{a\alpha} = \left(C\bar{\Theta}_a^{\mathrm{T}}\right)_\alpha . \tag{2.8}$$

Since the Θ-coordinates are Grassmanian, they obey the following relations:

$$\left\{\Theta_{a\alpha}, \, \Theta_{b\beta}\right\} = 0 = \left\{\Theta_{a\alpha}, \, \bar{\Theta}_b^\beta\right\} . \tag{2.9}$$

The superspace is taken to consist of four bosonic coordinates x_a and twelve fermionic coordinates $\Theta_{a\alpha}$. These coordinates can collectively be referred to as z. Also, for ease of writing, we will suppress the spinor index on the fermionic coordinates Θ and ϵ. A superfield on this superspace is denoted by $\Phi(z)$ and is related to $\Phi(0)$ by exponentiation:

$$\begin{aligned}\Phi(z) = \Phi(x_a, \, \Theta_{a\alpha}) &= \exp(x^a P_a)\exp(\bar{\Theta}_a Q^a)\Phi(0,0) \\ &= \exp(x^a P_a + \bar{\Theta}^a Q_a)\Phi(0,0) ,\end{aligned} \tag{2.10}$$

where we have used the commutativity of P_a with $Q_{a\alpha}$ to arrive at equation (2.10).

Next, we introduce a supersymmetric transformation operator acting on $\Phi(z)$ which will take $\Phi(z)$ to $\Phi(z')$. One can write

$$\exp(z') = \exp\left(\bar{\epsilon}^a Q_a\right)\exp(z) , \tag{2.11}$$

where the $\epsilon_{a\alpha}$ is an anticommuting, Majorana spinor parameter. As with any $(1, {}^1\!/_2)+({}^1\!/_2, 1)$ irreducible spinor, the following general property, $\gamma^a\epsilon_a = 0 = \bar{\epsilon}^a\gamma_a$, holds.

Equation (2.11) can be simplified using the algebra given in Eqs (2.1) to (2.7). Using equation (2.11) and the identity

$$\exp(A)\exp(B) = \exp(A + B + \tfrac{1}{2}[A, B]) , \tag{2.12}$$

where A and B are arbitrary operators, one can show that

$$\exp\left(\bar{\epsilon}^a Q_a\right)\exp(z) = \exp\left(\bar{\epsilon}^a Q_a + x^a P_a + \bar{\Theta}^a Q_a + \tfrac{1}{2}\left[\bar{\epsilon}^a Q_a, \bar{\Theta}^b Q_b\right]\right) . \tag{2.13}$$

Furthermore, the commutator on the right-hand side can be worked out using equation (2.7); one obtains after some Dirac algebra:

$$\begin{aligned}\left[\bar{\epsilon}^a Q_a, \bar{\Theta}^b Q_b\right] &= \bar{\epsilon}^a\left\{Q_a, \bar{Q}_b\right\}\Theta^b \\ &= a\left[\bar{\epsilon}^a \not{P}\Theta_a + \tfrac{3}{5} i\epsilon_{abcd}\bar{\epsilon}^a\gamma^c\gamma_5\Theta^b P^d\right] ,\end{aligned} \tag{2.14}$$

where we have used the property that $\gamma^a\Theta_a = 0 = \bar{\epsilon}^a\gamma_a$.

The second term on the right-hand side of Eq. (2.14) can be further simplified by using the identity

$$\begin{aligned}\gamma_{abc} &\equiv i\epsilon_{abcd}\gamma^{d}\gamma_5 \\ &= \tfrac{1}{2}\{\gamma_{ab},\gamma_c\} = -\gamma_{bac} = +\gamma_{bca}\,. \end{aligned} \tag{2.15}$$

Again after some Dirac manipulation, a remarkable simplification results

$$\begin{aligned}\tfrac{3}{5}i\epsilon_{abcd}\bar{\epsilon}^{a}\gamma^{c}\gamma_5\Theta^{b}P^{d} &= \tfrac{-3}{10}iP^{d}\bar{\epsilon}^{a}\{\gamma_{ab},\gamma_d\}\Theta^{b} \\ &= \tfrac{3}{10}\left(P^{d}\bar{\epsilon}_{b}\gamma_{d}\Theta^{b} + P^{d}\bar{\epsilon}^{a}\gamma_{d}\Theta_{a}\right) \\ &= \tfrac{3}{5}\bar{\epsilon}^{a}\not{P}\Theta_{a}\,. \end{aligned} \tag{2.16}$$

By substituting the result of Eq. (2.16) back into Eq. (2.14) we obtain

$$\left[\bar{\epsilon}^{a}Q_{a},\bar{\Theta}^{b}Q_{b}\right] = \tfrac{8}{5}a\bar{\epsilon}^{b}\gamma_{a}\Theta_{b}P^{a}\,. \tag{2.17}$$

Therefore equation (2.13) now reads

$$\exp\left(\bar{\epsilon}^{a}Q_{a}\right)\exp(z) = \exp\left((x^{a} + \tfrac{1}{2}\bar{\epsilon}^{b}\gamma^{a}\Theta_{b})P_{a} + (\bar{\Theta}^{a} + \bar{\epsilon}^{a})Q_{a}\right), \tag{2.18}$$

where the coefficient "a" has been set equal to unity without loss of generality.

A supersymmetrically transformed superfield can thus be expressed as

$$\Phi(z') = \exp(z')\Phi(0,0) = \Phi\left(x^{a} + \tfrac{1}{2}\bar{\epsilon}^{b}\gamma^{a}\Theta_{b},\bar{\Theta}^{a} + \bar{\epsilon}^{a}\right), \tag{2.19}$$

where we have allowed equation (2.18) to operate on $\Phi(0,0)$. One sees that we have not only a translation in spinor space but also an ordinary space-time translation of the amount $\frac{1}{2}\bar{\epsilon}^{b}\gamma^{a}\Theta_{b}$. Following customary procedure we next expand (2.19) in a Taylor series about the point $(x^{a},\bar{\Theta}^{a})$ to obtain:

$$\Phi(z') = \Phi(z) + \bar{\epsilon}^{a}d_{a}\Phi(z) + \tfrac{1}{2}\bar{\epsilon}^{b}\not{\partial}\Theta_{b}\Phi(z)\,. \tag{2.20}$$

The spinor partial derivatives d_{ax} are defined as $d_{ax} \equiv \partial/\partial\bar{\Theta}^{ax}$.

The action of a supersymmetric variation with infinitesimal parameter $\epsilon_{a\alpha}$ can now be ascertained from equation (2.20). The result reads

$$\begin{aligned}\delta\left(\bar{\epsilon}Q_{a}\right)\Phi(z) &= \Phi(z') - \Phi(z) \\ &= \bar{\epsilon}^{a}d_{a} + \tfrac{1}{2}\bar{\epsilon}^{a}\gamma^{b}\Theta_{a}\partial_{b})\Phi(z)\,. \end{aligned} \tag{2.21}$$

A supersymmetric covariant spinor derivative can be found using the above (1, ½)+(½, 1) transformation law. It assumes the explicit form

$$D_{a\alpha} \equiv d_{a\alpha} - \tfrac{1}{2}\left(\not{\partial}\Theta_{a}\right)_{\alpha} + \tfrac{1}{4}\left(\gamma_{a}\Theta^{b}\right)_{\alpha}\partial_{b} \tag{2.22}$$

and one can easily demonstrate that the covariant derivative of a superfield transforms as the superfield itself with the help of a little Dirac algebra.

We can also show that the anticommutator of (2.22) gives

$$\{D_{a\alpha}, \bar{D}_{b\beta}\} = \not{\partial}_{\alpha\beta}\eta_{ab} - \tfrac{1}{4}\left(\gamma_a\partial_b + \gamma_b\partial_a\right)_{\alpha\beta}. \tag{2.23}$$

Note that the result is symmetric under simultaneous interchange of $(a \longleftrightarrow b)$ and $(\alpha \longleftrightarrow \beta)$. From (2.23) it follows that

$$\bar{D}_{a\alpha}D_{b\beta} - \begin{pmatrix} a \longleftrightarrow b \\ \alpha \longleftrightarrow \beta \end{pmatrix} = 0. \tag{2.24}$$

Therefore

$$\bar{D}_a\begin{pmatrix}\mathrm{L}\\ \mathrm{R}\end{pmatrix}D_b - (a \longleftrightarrow b) \equiv \bar{D}_a\left(\frac{1 \mp \gamma_5}{2}\right)D_b - (a \longleftrightarrow b) = 0. \tag{2.25}$$

We allow this identity to act on an arbitrary superfield:

$$\bar{D}_{\underline{[a}}\begin{pmatrix}\mathrm{L}\\ \mathrm{R}\end{pmatrix}D_{\underline{b]}}\Phi = 0. \tag{2.26}$$

From a study of electrodynamics, specifically how the vector potential follows from an antisymmetric fieldstrength, it should be clear that the general solution to this equation is

$$\Phi_{a\mathrm{L}} = \mathrm{L}\Phi_a \equiv \mathrm{L}D_a\Phi. \tag{2.27}$$

Again the electromagnetic analogue is the vanishing of the fieldstrength, an antisymmetric tensor, from which the general solution for the vector potential follows. We have demonstrated this for the left-handed superfields, the same can be demonstrated for the right-handed superfield. We see that instead of the ordinary left- and right-handed scalar chiral superfields in the $(^1/_2, 0)+(0, ^1/_2)$ case, we now have left- and right-handed *vector* chiral superfields for the $(1, ^1/_2)+(^1/_2, 1)$ case.

3. Expansion in the (1, 1/2) variable

In equation (2.8) we introduced the $(1, ^1/_2)+(^1/_2, 1)$ coordinate $\Theta_{a\alpha}$ associated with $Q_{a\alpha}$. Like $Q_{a\alpha}$ it is transverse in spinor space, *i.e.*, $\gamma^a\Theta_a = 0$ and represents twelve components. The Majoranic $\Theta_{a\alpha}$ can be written in SL(2, C) Weyl spinor formalism as

$$\Theta_{a\alpha} = \begin{pmatrix} \Theta_{(AB\dot{C})} \\ \Theta^{*(\dot{A}\dot{B}C)} \end{pmatrix}, \tag{3.1}$$

where $\Theta_{(AB\dot{C})}$ $\left(\Theta^{*(\dot{A}\dot{B}C)}\right)$ are the left- (right-)handed chiral projections of $\Theta_{a\alpha}$. In SU(2)×SU(2) notation these are the (1, ½)+(½, 1) components, respectively. They are transformed into each other by parity transformations.

The $\Theta_{a\alpha}$ are anticommuting coordinates as seen from equation (2.9); written out in SL(2,C) formalism we claim that

$$\left\{\Theta_{(AB\dot{C})},\, \Theta^{(DE\dot{F})}\right\} = 0\,, \qquad \left\{\Theta^{*(A\dot{B}\dot{C})},\, \Theta^{(DE\dot{F})}\right\} = 0\,, \tag{3.2a, b}$$

$$\left\{\Theta_{(AB\dot{C})},\, \Theta^{*}_{(D\dot{E}\dot{F})}\right\} = 0\,, \qquad \left\{\Theta^{*(A\dot{B}\dot{C})},\, \Theta^{*}_{(D\dot{E}\dot{F})}\right\} = 0\,, \tag{3.2c, d}$$

where the indices $A, B, \ldots = 1, 2$ and $\dot{A}, \dot{B}, \ldots = \dot{1}, \dot{2}$ are SL(2,C) Weyl indices.

In ordinary spin (½, 0)+(0, ½) SUSY an anticommuting Θ_α implies that a superfield $\Phi(x,\, \Theta)$ can be expanded in a finite power series in Θ. The same holds for our new superfield $\Phi(x,\, \Theta_a)$ where now the anticommuting $\Theta_{a\alpha}$ must be used. To keep things as transparent as possible we consider only a left-handed chiral multiplet where the expansion variable is $\Theta_{(AB\dot{C})} = (L\Theta_a)_\alpha$. This projection represents only six components.

To find which fields are represented in a (1, ½) supermultiplet we first have to find those bilinear, trilinear, quartic, ... terms in $(L\Theta_a)_\alpha = \Theta_{(AB\dot{C})}$ that are irreducible. We expect for $(\mathrm{L}\Theta_a)$, $(\mathrm{L}\Theta_a)^2$, $(\mathrm{L}\Theta_a)^3$, ... the following number of irreducible components. $(\mathrm{L}\Theta_a)^n$ gives $\binom{6}{n} = 6![n!(6-n)!]^{-1}$ irreducible components. For example, $(\mathrm{L}\Theta_a)^4$ implies 15 irreducible components fields. $(\mathrm{L}\Theta_a)^7$ and higher powers vanish because of the nilpotency of spinors. There is no way to successfully antisymmetrize a 6-component field seven times.

We now give the result leaving the proof for the subsequent discussion. The result is

$$\begin{aligned}
\Phi_{\mathrm{L}} = {} & \phi_1 + \Theta^{(AB\dot{C})}\varphi_{1(AB\dot{C})} + \Theta^{(AB\dot{C})}\Theta_{(AB\dot{C})}\phi_2 \\
& + \Theta^{(AB\dot{E})}\Theta^{(CD}{}_{\dot{E})}\phi_{2(ABCD)} + \Theta^{(AE\dot{C})}\Theta^{(B}{}_{E}{}^{\dot{D})}\phi_{2(AB\dot{C}\dot{D})} \\
& + \Theta^{(DF\dot{A})}\Theta_{(D}{}^{E\dot{B})}\Theta_{(EF}{}^{\dot{C})}\varphi_{2(\dot{A}\dot{B}\dot{C})} + \Theta^{(AF\dot{D})}\Theta^{(BE}{}_{\dot{D})}\Theta_{(EF}{}^{\dot{C})}\varphi_{2(AB\dot{C})} \\
& + \Theta^{(AB\dot{G})}\Theta^{(CF\dot{E})}\Theta^{(D}{}_{F\dot{G})}\varphi_{2(ABCD\dot{E})} + \Theta^{(AB\dot{E})}\Theta_{(B}{}^{C\dot{F})}\Theta_{(C}{}^{D}{}_{\dot{E})}\Theta_{(DA\dot{F})}\phi_3 \\
& + \Theta^{(AH\dot{E})}\Theta^{(BF}{}_{\dot{E})}\Theta^{(C}{}_{F}{}^{\dot{G})}\Theta^{(D}{}_{H\dot{G})}\phi_{3(ABCD)} \\
& + \Theta^{(AH\dot{E})}\Theta^{(BF}{}_{\dot{E})}\Theta_{(F}{}^{G\dot{C})}\Theta_{(GH}{}^{\dot{D})}\phi_{3(AB\dot{C}\dot{D})} \\
& + \Theta^{(AD\dot{H})}\Theta^{(BE\dot{I})}\Theta_{(D}{}^{F}{}_{\dot{I})}\Theta_{(F}{}^{G\dot{C})}\Theta_{(GE\dot{H})}\varphi_{3(AB\dot{C})} \\
& + \Theta^{(A}{}_{F}{}^{\dot{A})}\Theta^{(B}{}_{A}{}^{\dot{B})}\Theta^{(C}{}_{B}{}^{\dot{C})}\Theta^{(D}{}_{C\dot{A})}\Theta^{(E}{}_{D\dot{B})}\Theta^{(F}{}_{B\dot{C})}\phi_4\,.
\end{aligned} \tag{3.3}$$

Note that the superfield expansion (3.3) contains component fields with spin greater than two. However, this is not a worry since these higher spin component fields have mass dimensions greater than those of the spinor field ($M^{3/2}$). This follows from the fact that fermionic coordinate Θ is of mass dimension $M^{-1/2}$. In the above ϕ_1, ϕ_2, ϕ_3, ϕ_4 are spin (0,0) scalars; $\varphi_{1(AB\dot{C})}$ $\varphi_{2(AB\dot{C})}$ and $\varphi_{3(AB\dot{C})}$ are spin (1, $^1/_2$) spinors; $\phi_{2(ABCD)}$ $\phi_{3(ABCD)}$ are spin (2, 0) conformal tensors, $\phi_{2(AB\dot{C}\dot{D})}$ $\phi_{3(AB\dot{C}\dot{D})}$ are symmetric spin (1, 1) tensors; $\varphi_{2(\dot{A}\dot{B}\dot{C})}$ is a spin (0, $^3/_2$) spinor and, finally, $\varphi_{2(ABCD\dot{E})}$ is a spin (2, $^1/_2$) spinor. All tensors are designated by $\phi_{\dots}$ and are characterized by an even number of symmetrized SL(2, C) indices. All spinor have been designated by $\varphi_{\dots}$ and have an odd number of symmetrized Weyl indices. Even powers of $\Theta_{(AB\dot{C})}$ in (3.3) always have bosonic components associated with them; an odd number will lead to fermionic components. As in any supersymmetric theory the total number of fermionic components equals the total number of bosonic. In this case we have 32+32 giving a total of 64 components. The fields in expansion (3.3) can be given in the order of increasing Θ as they occur in (3.3). We have in SU(2)×SU(2) notation:

$$\begin{array}{l} \underset{1}{(0,0)}\,/\underset{6}{(1,\tfrac{1}{2})}\,/\underset{5+9+1}{(2,0)+(1,1)+(0,0)}\,/\underset{10+6+4}{(2,\tfrac{1}{2})+(1,\tfrac{1}{2})+(0,3)} \\ \qquad\qquad \underset{5+9+1}{(2,0)+(1,1)+(0,0)}\,/\underset{6}{(1,\tfrac{1}{2})}\,/\underset{1}{(0,0)}\,. \end{array} \tag{3.4}$$

The number of components associated with each field is specified below the field in (3.4). In general in SU(2)×SU(2) notation, the total number of components represented by (j_1, j_2) is given by (2j+1), where $|\, j_1 - j_2 \,| \leq j \leq (j_1 + j_2)$. With (3.3 and (3.4) we have reproduced the results of previous work by means of a new but equivalent method.

We now prove that the expansion given by (3.3) is unique. We start with the bilinear form in $(L\Theta_a)_\alpha$. Initially we have a total of 6×6 or 36 components for a bilinear form in $\Theta_{(AB\dot{C})}$. They are given in SU(2)×SU(2) formalism as:

$$(1, {}^1/_2) \times (1, {}^1/_2) = (2,1) + (2,0) + (1,1) + (1,0) + (0,1) + (0,0)\,. \tag{3.5}$$

The first term on the right-hand side corresponds to zero contractions in SL(2,C) indices between the Θ's the second to one dotted contraction, the third to one undotted contraction, the fourth to one dotted and one undotted contraction, *etc.* ... until no more contractions are possible.

Next we consider each of the products in turn. The first, (2,1), implies a symmetrized product in $(AB\dot{C}DE\dot{F})$ but

$$\begin{aligned} \Theta^{(AB\dot{C})}\Theta^{(DE\dot{F})}\phi^{(AB\dot{C}DE\dot{F})} &= -\Theta^{(DE\dot{F})}\Theta^{(AB\dot{C})}\phi_{(AB\dot{C}DE\dot{F})} \\ &= -\Theta^{(AB\dot{C})}\Theta^{(DE\dot{F})}\phi_{(DE\dot{F}AB\dot{C})}\,. \end{aligned} \tag{3.6}$$

In the first line we have used (3.2a); in the second we have relabeled indices. Clearly the vanishing result is because the indices are symmetrized for ϕ. (All irreducible fields in SL(2,C) notation are characterized by fully symmetrized indices; otherwise they are reducible.)

The second product, (2,0) implies one dotted contraction. We obtain that

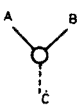
$$\begin{aligned} \Theta^{(AB\dot{C})}\Theta_{(A}{}^{E\dot{F})}\phi_{(BE\dot{C}\dot{F})} &= -\Theta_{(A}{}^{E\dot{F})}\Theta^{(AB\dot{C})}\phi_{(BE\dot{C}\dot{F})} \\ &= +\Theta^{(AE\dot{F})}\Theta_{(A}{}^{B\dot{C})}\phi_{(BE\dot{C}\dot{F})} \end{aligned} \qquad (3.7)$$

does not vanish. The third product on the right-hand side of (3.5) does not vanish for similar reasons. But the fourth gives

$$\begin{aligned} \Theta^{(AB\dot{C})}\Theta^{(D}{}_{B\dot{C})}\phi_{(AD)} &= -\Theta^{(D}{}_{B\dot{E})}\Theta^{(AB\dot{C})}\phi_{(AD)} \\ &= -\Theta^{(DB\dot{E})}\Theta^{(A}{}_{B\dot{C})}\phi_{(AD)} = 0\,. \end{aligned} \qquad (3.8)$$

Continuing in this fashion demonstrates that only the bilinear forms contained in (3.3) are nonvanishing. Because of the antisymmetry of the $\Theta_{(AB\dot{C})}$ upon interchange with another $\Theta_{(DE\dot{F})}$ we note that all zero and *even numbered* contractions must vanish.

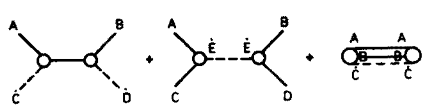
We now associate with every $\Theta_{(AB\dot{C})}$ a blob:

$\Theta_{(AB\dot{C})} \equiv$ 

(3.9)

where solid (dotted) lines represent undotted (dotted) indices. What kind of fully symmetrized Weyl field can be obtained if one contracts one $\Theta_{(AB\dot{C})}$ with another $\Theta_{(DE\dot{F})}$? The analysis above indicates that only an odd number of contractions between any two Θ's (or blobs) are variable. The result is:

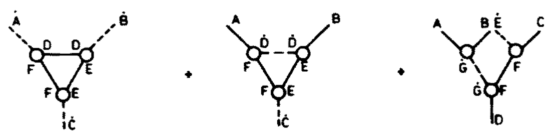
$(1, {}^1\!/\!_2) \times (1, {}^1\!/\!_2) =$



(3.10)

There are no others. At least one contraction is always necessary in order to antisymmetrize. Expansion (3.3) shows that these are exactly the represented bilinear terms in Θ. The fields associated with these bilinear forms are characterized by the exposed indices in (3.10): $(AB\dot{C}\dot{D})$, $(ABCD)$ and $(-)$.

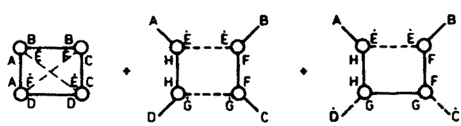
We continue the analysis using this diagrammatic method. Consider the trilinear forms in $(L\Theta_a)$. Antisymmetrizing gives $(1, {}^1\!/_2)\times(1, {}^1\!/_2)\times(1, {}^1\!/_2)=$



(3.11)

These are the only odd-numbered contractions which are possible between any two Θ's. Also all Θ's are necessarily contracted at least once in order to antisymmetrize. The exposed fully-symmetrized indices characterize the fields associated with these trilinear forms. These are given in Eq. (3.3) as the "coefficients" of the $(L\Theta_a)^3$.

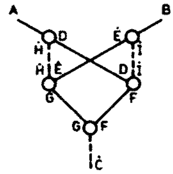
This line of developments can be continued. The following results must hold. First the quartic terms, give $(1, {}^1\!/_2)\times(1, {}^1\!/_2)\times(1, {}^1\!/_2)\times(1, {}^1\!/_2)=$



(3.12)

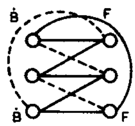
These are represented with their associated fields in the expansion given by equation (3.3). No two individual blobs or Θ's are contracted more than twice with each other. There are no other quartic terms possible.

For $(L\Theta_a)^5$ we obtain $(1, {}^1\!/_2)\times(1, {}^1\!/_2)\times(1, {}^1\!/_2)\times(1, {}^1\!/_2)\times(1, {}^1\!/_2)=$



(3.13)

This term is given in (3.3) along with the $(L\Theta_a)^6$ term. For the latter one obtains



(3.14)

This concludes the discussion of the expansion in $\Theta_{(AB\dot{C})}$.

4. The Wigner method

It is convenient to express the algebra (2.1) to (2.7) in Weyl SL(2, C) notation. Explicitly,

$$[M_{AB}, M_{CD}] = i(\epsilon_{CA}M_{BD} + \epsilon_{CB}M_{AD} + \epsilon_{DA}M_{CB} + \epsilon_{DB}M_{CA}), \tag{4.1}$$

$$[M_{AB}, M^*_{\dot{C}\dot{D}}] = 0, \tag{4.2}$$

$$[M^*_{\dot{A}\dot{B}}, M^*_{\dot{C}\dot{D}}] = i(\epsilon_{\dot{C}\dot{A}}M^*_{\dot{B}\dot{D}} + \epsilon_{\dot{C}\dot{B}}M^* + \epsilon_{\dot{D}\dot{A}}M^*_{\dot{B}\dot{C}} + \epsilon_{\dot{D}\dot{B}}M^*_{\dot{A}\dot{C}}), \tag{4.3}$$

$$[M_{AB}, P_{C\dot{C}}] = i(\epsilon_{CA}P_{B\dot{C}} + \epsilon_{CB}P_{A\dot{C}}), \tag{4.4}$$

$$[M^*_{\dot{A}\dot{B}}, P_{C\dot{C}}] = i(\epsilon_{\dot{C}\dot{A}}P_{C\dot{B}} + \epsilon_{\dot{C}\dot{B}}P_{C\dot{A}}), \tag{4.5}$$

$$[P_{A\dot{A}}, P_{B\dot{B}}] = 0, \tag{4.6}$$

$$[M_{(AB)}, Q_{C\dot{C}D)}] = i\big(\epsilon_{CA}Q_{(B\dot{C}D)} + \epsilon_{CB}Q_{(A\dot{C}D)} + \epsilon_{DA}Q_{(C\dot{C}B)} + \epsilon_{DB}Q_{(C\dot{C}A)}\big), \tag{4.7}$$

$$[M_{(AB)}, Q^*_{(C\dot{C}\dot{D})}] = i\big(\epsilon_{CA}Q^*_{(B\dot{C}\dot{D})} + \epsilon_{CB}Q^*_{(A\dot{C}\dot{D})}\big), \tag{4.8}$$

$$[M^*_{(\dot{A}\dot{B})}, Q_{(C\dot{C}\dot{D})}] = i\big(\epsilon_{\dot{C}\dot{A}}Q_{(\dot{B}CD)} + \epsilon_{\dot{C}\dot{B}}Q_{(\dot{A}CD)}\big), \tag{4.9}$$

$$[M^*_{(\dot{A}\dot{B})}, Q^*_{(C\dot{C}\dot{D})}] = i\big(\epsilon_{\dot{C}\dot{A}}Q^*_{(C\dot{B}\dot{D})} + \epsilon_{\dot{C}\dot{B}}Q^*_{(C\dot{A}\dot{D})} + \epsilon_{\dot{D}\dot{A}}Q^*_{(C\dot{B}\dot{C})} + \epsilon_{\dot{D}\dot{B}}Q^*_{(C\dot{A}\dot{C})}\big), \tag{4.10}$$

$$[P_{(A\dot{A})}, Q_{(C\dot{C}D)}] = 0 = [P_{(A\dot{A})}, Q^*_{(C\dot{C}\dot{D})}], \tag{4.11}$$

$$\{Q_{(A\dot{A}B)}, Q_{(C\dot{C}D)}\} = 0 = \{Q^*_{(A\dot{A}\dot{B})}, Q^*_{(C\dot{C}\dot{D})}\}, \tag{4.12}$$

$$\{Q_{(A\dot{A}B)}, Q^*_{(C\dot{C}\dot{D})}\} = -4/5\,(\epsilon_{AC}\epsilon_{\dot{A}\dot{C}}P_{(B\dot{D})} + \epsilon_{BC}\epsilon_{\dot{A}\dot{C}}P_{(A\dot{D})} + \epsilon_{AC}\epsilon_{\dot{A}\dot{D}}P_{(B\dot{C})} + \epsilon_{BC}\epsilon_{\dot{A}\dot{D}}P_{(A\dot{C})}, \tag{4.13}$$

where $A, B, \ldots = 1, 2$ and $\dot{A}, \dot{B}, \ldots = \dot{1}, \dot{2}$ denote Weyl SL(2, C) spinor indices. Furthermore, $M_{(AB)}$ and $M^*_{(\dot{A}\dot{B})}$ represent the irreducible (1,0)+(0,1) generators of the Lorentz group; $P_{(A\dot{A})}$, the ($^1/_2$, $^1/_2$) generator of translations and $Q_{(A\dot{A}B)}$ with $Q^*_{(A\dot{A}\dot{B})}$ the generators or charges associated with spin (1, $^1/_2$)+($^1/_2$, 1) SUSY. $\epsilon^{AB} = \epsilon_{AB}$ and $\epsilon^{\dot{A}\dot{B}} = \epsilon_{\dot{A}\dot{B}}$ are the SL(2, C) antisymmetric tensors used to raise and lower indices.

The independent generators are ($Q_{(11\dot{1})}$, $Q^*_{(11\dot{2})}$, $Q_{(12\dot{1})}$, $Q_{(12\dot{2})}$, $Q_{(22\dot{1})}$, $Q_{(22\dot{2})}$) and the set ($Q^*_{(1\dot{1}\dot{1})}$, $Q^*_{(2\dot{1}\dot{1})}$, $Q^*_{(1\dot{1}\dot{2})}$, $Q^*_{(2\dot{1}\dot{2})}$, $Q^*_{(1\dot{2}\dot{2})}$, $Q^*_{(2\dot{2}\dot{2})}$) and their nontrivial contributions (from equation (4.13)) are the following:

$$\{Q_{(11\dot{1})}, Q^*_{(2\dot{1}\dot{2})}\} = -{}^8/_5 P_{1\dot{1}}\,, \tag{4.14}$$

$$\{Q_{(12\dot{1})}, Q^*_{(1\dot{1}\dot{2})}\} = {}^4/_5 P_{1\dot{1}}\,, \tag{4.15}$$

$$\{Q_{(12\dot{1})}, Q^*_{(2\dot{2}\dot{2})}\} = -{}^8/_5 P_{2\dot{2}}\,, \tag{4.16}$$

$$\{Q_{(22\dot{1})}, Q^*_{(1\dot{2}\dot{2})}\} = {}^{16}/_5 P_{2\dot{2}}\,, \tag{4.17}$$

$$\{Q_{(11\dot{2})}, Q^*_{(2\dot{1}\dot{1})}\} = {}^{16}/_5 P_{1\dot{1}}\,, \tag{4.18}$$

$$\{Q_{(12\dot{2})}, Q^*_{(2\dot{1}\dot{2})}\} = {}^4/_5 P_{2\dot{2}}\,. \tag{4.19}$$

We only consider the massive case here. One can transform to the rest frame of the particle, *i.e.*, take $P_\mu = (m, \vec{0})$. Then $P_{A\dot{A}} \equiv (\sigma^\mu)_{A\dot{A}} P_\mu$ gives $P_{1\dot{1}} = P_{2\dot{2}} = m$ and $P_{1\dot{2}} = P_{2\dot{1}} = 0$. ($\sigma = (1, \vec{\sigma}) = (\sigma_\mu)_{A\dot{A}}$ are the Pauli spin matrices). Our equations (4.14) to (4.19) reduce to

$$\{Q_{(11\dot{1})}, Q^*_{(2\dot{1}\dot{2})}\} = -{}^8/_5 M\,, \tag{4.20}$$

$$\{Q_{(12\dot{1})}, Q^*_{(1\dot{1}\dot{2})}\} = {}^4/_5 M\,, \tag{4.21}$$

$$\{Q_{(12\dot{1})}, Q^*_{(2\dot{2}\dot{2})}\} = -{}^8/_5 M\,, \tag{4.22}$$

$$\{Q_{(22\dot{1})}, Q^*_{(1\dot{2}\dot{2})}\} = {}^{16}/_5 M\,, \tag{4.23}$$

$$\{Q_{(11\dot{2})}, Q^*_{(2\dot{1}\dot{1})}\} = {}^{16}/_5 M\,, \tag{4.24}$$

$$\{Q_{(12\dot{2})}, Q^*_{(2\dot{1}\dot{2})}\} = {}^4/_5 M\,. \tag{4.25}$$

The particle spectrum is given by

$$\begin{aligned}
&|0\rangle\,,\\
&Q_{(AB\dot{C})}|0\rangle\,,\\
&Q_{(AB\dot{C})}Q_{(DE\dot{F})}|0\rangle\,,\\
&Q_{(AB\dot{C})}Q_{(DE\dot{F})}Q_{(GH\dot{I})}|0\rangle\,,\\
&Q_{(AB\dot{C})}Q_{(DE\dot{F})}Q_{(GH\dot{I})}Q_{(JK\dot{L})}|0\rangle\,,\\
&Q_{(AB\dot{C})}Q_{(DE\dot{F})}Q_{(GH\dot{I})}Q_{(JK\dot{L})}Q_{(MN\dot{O})}|0\rangle\,,\\
&Q_{(AB\dot{C})}Q_{(DE\dot{F})}Q_{(GH\dot{I})}Q_{(JK\dot{L})}Q_{(MN\dot{O})}Q_{(PQ\dot{R})}|0\rangle\,,
\end{aligned} \tag{4.26}$$

where $|0\rangle$ represents the vacuum. The series terminates because of the anticommuting nature of $Q_{(AB\dot{C})}$. One obtains 64 components, half bosonic and half fermionic. The $Q_{(AB\dot{C})}|0\rangle$ generates a $(1, 1/2)$ particle which has six components; the $Q_{(AB\dot{C})}Q_{(DE\dot{F})}|0\rangle$ generates a 6×6 antisymmetric number of component states or $(6)(5)/2 = 15$. Three Q's acting on $|0\rangle$ give a $6 \times 6 \times 6$ antisymmetric (upon interchange of the Q's) number of particle states, rendering $(6)(5)(4)/((3)(2)) = 20$ components. Continuing in this fashion gives us the total number of components, namely, $1+6+15+20+15+6+1$ particle states, corresponding to $|0\rangle$, $Q|0\rangle$, $QQ|0\rangle$, *etc.*

All the representations except the first two in equation (4.26) are reducible. Their decomposition is given as follows:

$$Q_{(AB\dot{C})}Q_{(DE\dot{F})}|0\rangle = (2,0) + (1,1) + (0,0)\,,$$

$$\begin{aligned} Q_{(AB\dot{C})}Q_{(DE\dot{F})}Q_{(GH\dot{I})}|0\rangle &= (2, 1/2) + (1, 1/2) + (0, 3/2)\,, \\ Q_{(AB\dot{C})}Q_{(DE\dot{F})}Q_{(GH\dot{I})}Q_{(JK\dot{L})}|0\rangle &= (2,1) + (1,1) + (0,0)\,, \\ Q_{(AB\dot{C})}Q_{(DE\dot{F})}Q_{(GH\dot{I})}Q_{(JK\dot{L})}Q_{(MN\dot{O})}|0\rangle &= (1, 1/2)\,, \\ Q_{(AB\dot{C})}Q_{(DE\dot{F})}Q_{(GH\dot{I})}Q_{(JK\dot{L})}Q_{(MN\dot{O})}Q_{(PQ\dot{R})}|0\rangle &= (0,0)\,. \end{aligned} \tag{4.27}$$

We can identify the right-hand side of the above decomposition as the particle content of the theory. These exactly match the component fields in the theta expansion of the superfield, equation (3.3).

5. Conclusion

Two independent methods have been used to determine the particle content of a higher spin superspace, the theta expansion method and the Wigner method of induced representations. Both methods are shown to give a similar multiplet as worked out explicitly for the spin $(1, 1/2) + (1/2, 1)$ SUSY. In all sixty-four components interact, half bosonic and half fermionic. A greater than spin two particle is generated; being an auxiliary field, however, it cannot propagate.

The former method does not require an explicit knowledge of the algebra and it should be clear how to generalize the diagrammatic method to any higher spin superfield. Perhaps the new fermionic symmetry will prove interesting in ordinary gravity as regards attempts of renormalization. For grand unified theories the particle content may play a fruitful role.

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