DIFFUSIVE DYNAMICS OF A DILUTE GAS OF INTERACTING SINE–GORDON SOLITONS*

F. MARCHESONI

Dipartimento di Fisica dell'Universita'

AND

Istituto Nazionale di Fisica Nucleare
I-06100 Perugia, Italy

(Received October 1, 1991)

The diffusive properties of the soliton gas borne by a damped, unbiased sine–Gordon theory coupled with a dissipative heat-bath are reviewed in the dilute gas approximation. It is shown that, contrary to the biased case, no anomalous diffusion occurs at time longer than the soliton lifetime. The corrections to the effective diffusion constant due to the interaction of a single soliton with a gas of both breathers (*or* phonons) and (anti)solitons at equilibrium are calculated analytically.

PACS numbers: 12.40. -y

The perturbed sine–Gordon (SG) equation (in dimensionless units)

$$\phi_{tt} - \phi_{zz} + \sin\phi = -\gamma\phi_t - F + \zeta(z, t) \tag{1}$$

has been proposed [1] to describe a variety of diffusive processes in condensed matter. In Eq. (1) the unperturbed SG equation is coupled to a heat-bath at temperature T through a damping term $-\gamma\phi_t$, with γ a constant, and a gaussian noise with zero mean and correlation function

$$\langle\zeta(z, t)\zeta(z', t')\rangle = 2\frac{\gamma}{\beta}\delta(t - t')\delta(z - z') \tag{2}$$

* Presented at the IV Symposium on Statistical Physics, Zakopane, Poland, September 19–29, 1991.

($\beta \equiv 1/kT$). The constant force F is an external physical bias that breaks the $\phi \to -\phi$ symmetry of the SG potential $V[\phi] = 1 - \cos\phi$, but preserves its multistable nature for $F < 1$.

In the absence of external bias ($F = 0$), Eq. (1) provides an efficient thermalization mechanism for the unperturbed SG theory [2]

$$H[\phi] = \tfrac{1}{2}(\phi_t^2 + \phi_z^2) + V[\phi] \tag{3}$$

independently of the value of the damping constant. The unperturbed SG equation bears both extended and localized solutions. The extended solutions are phonons with continuum non-negative energy spectrum. Localized solutions can be well approximated as an appropriate superposition of solitons, ϕ_K, and antisolitons $\phi_{\bar{K}}$, in the limit when the separation among their centers is very large compared with their size (*dilute gas approximation)*. For reader's convenience, we write explicitly the single soliton (antisoliton) solution (*mod 2π*)

$$\phi_{K(\bar{K})}(z, u) = 4tg^{-1} \exp[\pm\, ch\alpha(z - x(t))]\,. \tag{4}$$

Here, $x(t) = z_0 + ut$, denotes the centre of mass of the solution (4), which moves with constant rapidity α, $ch\alpha \equiv 1/\sqrt{1-u^2}$. The energy of $\phi_{K(\bar{K})}$ is given by $E(u) = E_0\ ch\alpha$, where $E_0 = \int H[\phi_{K(\bar{K})}(z, 0)]dz = 8$ is called soliton rest mass. The soliton and antisoliton solutions (4) carry opposite topological charge and, therefore, may only be created by the pair. The equilibrium soliton (antisoliton) density n_0 reads

$$n_0(T) = \left(\tfrac{2}{\pi}\right)^{1/2} (\beta E_0)^{1/2} e^{-\beta E_0}\,, \tag{5}$$

where the global topological charge has been set to zero for simplicity [3]. The dilute gas approximation is thus legitimate in the low temperature limit $\beta E_0 \ll 1$, only, when $n_0^{-1} \gg 1$.

$\phi_{K(\bar{K})}$ are stable under the perturbation in Eq. (1) for any value of γ and F, a part from a rigid translation, against which they are in neutral equilibrium [3]. It has been shown [2] that in the unbiased case a *single* soliton (antisoliton) undergoes brownian motion, whereas the external bias pulls ϕ_K and $\phi_{\bar{K}}$ in opposite directions according to the Langevin equation.

$$\dot{p} = -\gamma p \pm 2\pi F + E_0 \eta(t)\,, \tag{6}$$

$\eta(t)$ is a zero-mean valued gaussian noise with correlation function $\langle \eta(t)\,\eta(0)\rangle = 2D_K\delta(t)$, $D_K \equiv \gamma/\beta E_0$ and p is the relevant momentum, $p = uE(u)$.

In the following we assume that the system is *damped*, *i.e.*$\gamma > 1$. Such a limitation allows two major simplifications:

(i) Eq. (6) can be treated in the non-relativistic limit, that is

$$\dot{u} = -\gamma u \pm \tfrac{2\pi F}{E_0} + \eta(t)\,, \tag{7}$$

whence the mean, $u_F = \pm 2\pi F/\gamma E_0$, and the variance of the $\phi_{K(\bar{K})}$ speed, $\langle (u - u_F)^2 \rangle = (\beta E_0)^{-1}$. The single soliton (antisoliton) mean square displacement is given by the well-known Einstein law

$$\langle \Delta x^2(t) \rangle = 2D_K t + u_F^2 t^2\,. \tag{8}$$

The diffusion law (8) only applies for short times when the effects due to the soliton-(anti)soliton interactions may be disregarded;

(ii) the soliton-antisoliton collisions are always *destructive* [4]. This suggests to estimate the soliton (antisoliton) lifetime τ_F by simply requiring that $\langle \Delta x^2(\tau_F) \rangle = n_0^{-2}$, whence [5]

$$\tau_F(F) = \left[D_K n_0^2 \left(1 + \sqrt{1 + \left(\frac{u_F}{D_K n_0} \right)^2} \right) \right]^{-1}\,. \tag{9}$$

Note that this estimate of τ_F has been obtained by neglecting the pair nucleation process which becomes dominant [5] at $2\pi\beta F \gg 1$. Such a further limitation has no bearing on the results of the present work.

In this paper we are concerned with the diffusive dynamics of a dilute gas of *interacting* damped SG solitons and antisolitons at thermal equilibrium. Our derivation of τ_F, (9), clearly implies that the diffusion law (8) is valid only for $t \ll \tau_F$. Since we are to consider time scales much longer than the soliton lifetime, a hydrodynamical picture of the diffusive process is more appropriate [6]. Let $n_\pm(x, t)$ be the local density of solitons and antisolitons, respectively. The topological charge density $\rho(x, t) = n_+(x, t) - n_-(x, t)$ is locally conserved, *i.e.*

$$\frac{\partial \rho}{\partial t}(x, t) + \frac{\partial j}{\partial x}(x, t) = 0\,, \tag{10}$$

where $j(x, t) = -D_K \frac{\partial \rho}{\partial x}(x, t) + j_F(x, t)$, and $j_F(x, t) \equiv u_F[n_+(x, t) + n_-(x, t)]$ is the driven topological charge current. Without loss of generality we restrict ourselves to the case of zero global topological charge, $\langle \rho \rangle = n_+ - n_- = 0$ (*i.e.*$n_\pm = n_0$). Here $n_\pm \equiv \langle n_\pm(x, t) \rangle$ are the uniform densities of the soliton (antisoliton) taken over a SG string of arbitrary length $L (L \gg n_0^{-1})$.

Krug and Spohn [7] have shown that for finite F it suffices to expand $j_F(x, t)$ in Eq. (10) around its mean value $\langle j_F \rangle = 2u_F/\tau_F$ up to second order in the topological charge *fluctuation* $\epsilon(x, t) \equiv \rho(x, t) - \langle\rho\rangle$ to reduce Eq. (10) to a Burgers equation

$$\frac{\partial\epsilon}{\partial t} = -\left(\frac{u_F}{2\tau_F}\right)^{1/2} \frac{\partial\epsilon^2}{\partial x} + D_K \frac{\partial^2\epsilon}{\partial x^2} \tag{11}$$

with random initial conditions. The asymptotic (anti)soliton mean square displacement associated with Eq. (11) follows a general scaling argument [7], *i.e.*$\langle\Delta x^2(t)\rangle = At^{4/3}$, with the constant $A(\beta, \gamma, F)$ given in implicit form [7]. The anomalous diffusion of the topological charge thus derived, is the consequence of the *nonlinear dispersive* term on the r.h.s. of Eq. (11) and must vanish for $F = 0$, *i.e.*when the $\phi \to -\phi$ symmetry of the problem is restored.

We develop now a hydrodynamical formalism for the limit of *vanishingly small* values of F, $2\pi\beta F \ll n_0$, where diffusive many-body effects dominate over the dispersive action of the external bias [4, 5]. We anticipate that no anomalous diffusion is predicted. The ordinary diffusion law (8) is still valid provided that D_K is replaced by an effective diffusion constant for the soliton gas.

Corrections to the ordinary diffusion regime (8) are due to the momentum exchange between the single soliton, whose direct coupling with the heat-bath is described by Eq. (7), and the other excited modes of the thermalized SG string $\phi(x, t)$. We distinguish three sources of interaction: the phonon, the breather and the soliton (antisoliton) gas, respectively.

(a) interaction with the phonon gas. This problem has been addressed in Refs [8, 9]. The effect due to the coupling of a single soliton with the phonon bath at thermal equilibrium is well accounted for by a small T-dependent correction of the damping constant. At low temperature, such contributions are expected to show up at the second order T, only [9].

(b) interaction with the breather gas. It has been guessed by several authors [10] that the breather and the phonon gas picture are equivalent in that they lead to the same statistical mechanical description of the SG theory. If such an *Ansatz* is to hold, the effective damping constant of the single soliton due to its interaction with the breather gas should coincide with the result of Ref. [9] mentioned in *(a)*. We have verified that such a prediction is correct, indeed, at least at the leading order in T. For the sake of brevity we limit ourselves to outlining our statistical mechanical analysis of the soliton-breather interaction, the details of which are left for a forthcoming publication.

We have calculated, first, the breather uniform density, n_B, as the ratio of the canonical partition function, Z_B, for the field $\phi(x, t)$ to bear one breather solution,

$$\phi_B(z,\,t) = 4tg^{-1}\left[tg\theta\frac{\cos[\cos\theta ch\alpha(t-uz)+\psi]}{ch[\sin\theta ch\alpha(z-ut)]}\right], \tag{12}$$

to the canonical partition function in the absence of breather modes, Z_0, integrated over the relevant breather parameter domains, *i.e.*$\alpha \in [-\infty,\,+\infty]$, $\psi \in [0,\,2\pi]$ and $\theta \in [0,\,\pi/2]$. Here, our notation is as in Ref. (10). On expanding the relevant $\phi(x,\,t)$ configurations on the complete set of phonon modes [3] and integrating over the phonon density of states in the presence of the breather [10] $\frac{1}{2\pi}\frac{\partial\delta_B}{\partial k}$ with $\delta_B(k) = 4tg^{-1}\left[\sin\theta ch\alpha/\left(k-u\sqrt{1+k^2}\right)\right] + o(\sin^2\theta)$, we obtain

$$n_B = \int n(\alpha,\,\theta)d\alpha\;d\theta\,, \tag{13a}$$

with

$$n(\alpha,\,\theta) = \frac{(\beta E_0)^2}{\pi}\sin\theta\,ch\alpha(ch\alpha+\sin\theta)^2 e^{-2\beta E_0 ch\alpha\sin\theta}\,. \tag{13b}$$

At low temperature we can expand $n(\alpha,\,\theta)$ in the integral (13a) in powers of θ to obtain

$$n_B = \frac{1}{4\pi}\int d\,sh\alpha + o(T^2)\,. \tag{14}$$

When properly regularized, the T-independent term on the r.h.s. of Eq. (14) differs from the phonon density by a factor 1/2, since for $\theta \to 0$ each breather is made up of two phonon modes with wave-vector $|k| = sh\alpha$. Most notably, on accounting for the next-to-leading term in $\delta_B(k)$ [10], we have shown that the finite temperature corrections to n_B cancel out up to the second order in T, at least.

The correction to the effective damping constant due to the breather-soliton interaction has been calculated explicitly by working out Eq. (10) of Ref. [9] in the breather gas picture. Now the leading term of the force acting upon the soliton centre of mass is given by the first term on the r.h.s. of Eq. (7) of Ref. [9]. At low temperature, we are allowed to take the limit of $\phi_B(z,\,t)$ and the breather density (13b) for $\theta \to 0$, whence the following analytical expression for the breather contribution to the effective damping constant

$$\gamma_B(T) = \gamma + \frac{I}{\beta^2 E_0}\,, \tag{15}$$

with $\frac{I}{3} = \frac{2\pi^2}{E_0^4} = 4.8\cdot10^{-3}$. Correspondingly, the effective diffusion constant is $D_B(T) = D_K + I/\beta^3E_0^2$. In Ref. [9] the multiplicative factor I was computed numerically, $\frac{I}{3} \simeq 4.6 \cdot 10^{-3}$. (Note that a misprint occurred in Eq. (7) and (20) of Ref. [9].) Such a remarkable agreement *proves the equivalence* between the phonon and the breather gas description of the diffusive dynamics of a single soliton interacting with a heat-bath at low temperature. Finally, the characteristic time scale for the soliton-breather (or phonon) interactions to affect the single soliton dissipative dynamics [9], $\tau_B \simeq (2\beta/3\pi I)^{1/2}$, turns out to be *much shorter than the soliton lifetime* τ_F. This justifies our hydrodynamical formalism for treating the soliton (antisoliton) gas at thermal equilibrium.

(c) interaction with the soliton (antisoliton) gas. Such a diffusive mechanism sets in at times much longer than $\tau_0 \equiv \tau_F(0) = (D_K n_0^2)^{-1}$, only. Let us start considering a probe soliton (antisoliton) located in $X(t)$. The effective potential which describes its interaction with the soliton (antisoliton) gas at equilibrium with density $\rho(x, t)$ is

$$V_\rho(X, t) = \int_{-\infty}^{\infty} \rho(x, t) V_\pm(x - X) dx\,. \tag{16}$$

$V_\pm$ represents the interaction between two solitons (4) with the same (+) or opposite topological charge (−) placed at the relative distance R from one another. For $R \gg 1$ [3]

$$V_\pm \simeq \pm 2E_0 e^{-2|R|}\,. \tag{17}$$

At low temperature $\rho(x, t)$ is expected to vary sensibly on a spatial scale of the order of n_0^{-1} (with $n_0 \ll 1$), so that $V_\rho(X, t)$ can be approximated by

$$V_\rho(X, t) \simeq \pm 2E_0 \rho(X, t)\,. \tag{18}$$

The probe soliton (antisoliton) experiences an effective force

$$F_\rho(X, t) = \mp 2E_0 \frac{\partial \rho}{\partial x}(X, t) \tag{19}$$

which, in the damped regime $\gamma_B(T)\tau_0 > 1$, drives it with the instantaneous speed

$$\pm u[\rho] = \mp \frac{2}{\gamma_B} \frac{\partial \rho}{\partial x}(X, t) \tag{20}$$

obtained from Eq. (7) after replacing $2\pi F$ with F_ρ.

Local conservation of the topological charge leads to the equation of continuity (10), where now

$$j(x, t) = -D[\rho] \frac{\partial \rho}{\partial x}(x, t) + u[\rho] |\rho(x, t)|\,. \tag{21}$$

As shown in *(a)* and *(b)*, at low temperature $D[\rho] \simeq D_B(T)$ and the equation of continuity (21) can be rewritten in its final form

$$\frac{\partial \rho}{\partial t}(x, t) = \frac{\partial^2}{\partial x^2}\left[D_B + \frac{1}{\gamma_B}|\rho(x, t)|\right]\rho(x, t). \tag{22}$$

The diffusion law for a soliton (antisoliton) gas at thermal equilibrium, can be obtained by assuming random initial conditions. If we replace $\rho(x, t)$ with $\langle\rho\rangle + \epsilon(x, t)$ and assume for simplicity $\langle\rho\rangle = 0$, as we did for the biased case, Eq. (11), we see immediately that $\lim_{t\to\infty}\langle\Delta x^2(t)\rangle = 2D_S t$ with

$$D_S(T) = D_B(T) + \frac{\langle|\rho|\rangle}{2\gamma_B}. \tag{23}$$

In the dilute gas approximation with $n_\pm = n_0$, $\langle|\rho|\rangle = 2n_0$, whence $D_S = D_B + n_0/\gamma_B$.

In conclusion, Eqs (15) and (23) prove that in the absence of external bias the diffusive dynamics of a dilute gas of solitons (antisolitons) at low temperature obeys the ordinary Einstein law. In order to appreciate the relevance of our predictions, we remark that our results apply to models with weak thermalization mechanism [8, 9], too. For instance, if we assume suitable *noisy boundary conditions* for the SG string (like in the low impurity density model [11]), the same equilibrium densities for the soliton (antisoliton) and the breather gas are achieved as in Eqs (5) and (13), respectively. In this case, however, the breather and the soliton (antisoliton) contributions to the effective dissipation constant are the *leading terms*, being $D_K = 0$ by assumption. Verification of such predictions lies within the reach of certain simulation algorithms available at present [12] and, in fact, preliminary evidence of deviations from the single-soliton diffusion law (8) has been reported recently [13].

Work supported in part by the Istituto Nazionale di Fisica della Materia, Sezione di Perugia. The author wishes to thank Prof. B.I. Halperin for his kind hospitality at the Harvard Physics Laboratories.

REFERENCES

[1] for a review see *e.g.* Y.S. Kivshar, B.A. Malomed, *Rev. Mod. Phys.* **61**, 763 (1990).

[2] F. Marchesoni, *Phys. Lett.* **115A**, 29 (1986).

[3] R. Rajaraman, *Solitons and Instantons*, North Holland, Amsterdam 1982, chapter 7.

[4] F. Marchesoni, *Phys. Rev.* **B34**, 6536 (1986).
[5] P. Hänggi, F. Marchesoni, P. Sodano, *Phys. Rev. Lett.* **60**, 2563 (1988).
[6] C.H. Bennett, M. Büttiker, R. Landauer, H. Thomas, *J. Stat. Phys.* **24**, 419 (1981).
[7] J. Krug, H. Spohn, *Europhys. Lett.* **8**, 219 (1989).
[8] F. Marchesoni, C.R. Willis, *Phys. Rev.* **A36**, 4559 (1987) and references therein.
[9] F. Marchesoni, C.R. Willis, *Europhys. Lett.* **12**, 491 (1990).
[10] K. Sasaki, *Phys. Rev.* **B33**, 2214 (1986) and references therein.
[11] F. Marchesoni, *Europhys. Lett.* **8**, 83 (1989).
[12] N. Theodorakopoulos, E.W. Weller, *Phys. Rev.* **B38**, 2749 (1988); A.I. Bokharev, Ph. de Forcrand, *Phys. Rev. Lett.* **63**, 2337 (1989); P. Biller, F. Petruccione, *Phys. Rev.* **B41**, 2139 (1990).
[13] Ph. de Forcrand, private communication.