VARIOUS FORMS OF RADIAL EQUATIONS FOR THE DIRAC TWO-BODY PROBLEM*

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The first- and second-order radial equations, derived from the two-body Dirac equation (called also the Breit equation), are described. A new Klein–Gordon-type form of the second-order radial equations is reported on. This is convenient for numerical calculations and may be also useful for a general discussion of the order of energy levels is quarkonia. Beside the usual perturbative case of weak Coulombic potential, the less familiar case of strong Coulombic potential is briefly discussed.

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In this paper, we return to the two-body Dirac equation and its first- and second-order radial equations, the latter arising when redundant radial components of its wave function are eliminated from the former. We report on new, Klein–Gordon-type, second-order radial equations derived from the two-body Dirac equation. These have a convenient form for numerical calculations and may be also useful for a general discussion of the order of energy levels in quarkonia, following the method developed in the case of Schrödinger equation [1] and Klein–Gordon equation [2].

The relativistic wave equations for a system of two spin-$^1/_2$particles such as a leptonium or quarkonium, interacting through a vector potential V, resists an exact analytic treatment [3] because the Sommerfeld polynomial method of solving the second-order differential equations [4] does not work in this case. For the Coulombic potential $V = -\alpha/r$ it is caused by a singularity at $r = -\alpha/E$ that, though regular, appears in the analytic extension of the respective radial equations in addition to the familiar regular singularity at $r = 0$ and irregular singularity at $r = \infty$.

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The two-body Dirac equation (called also the Breit relativistic equation) [5]

$$[E - V(\vec{r}) - (\vec{\alpha}_1 - \vec{\alpha}_2)\cdot\vec{p} - (\beta_1 + \beta_2)(m + \tfrac{1}{2}S(\vec{r}))]\psi(\vec{r}) = 0\,, \qquad (1)$$

where $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, $\vec{p} \equiv \vec{p}_1 = -\vec{p}_2$ and $m \equiv m_1 = m_2$, offers the simplest relativistic wave equation for a system of two spin-$^1/_2$particles with equal masses, interacting through a vector potential $V(\vec{r})$ and a scalar potential $S(\vec{r})$ in the centre-of-mass frame. Although, in contrast to the much more complicated Salpeter equation [6], it does not include the hole theory, one can argue that in the case of static potentials V and S it is pretty well applicable (deviations from the hole theory being manifested only when the nonstatic corrections depending on $\vec{\alpha}_i$ are added to the static potentials and treated beyond the lowest-order perturbative calculation [5,3]).

In the case of central potentials $V(r)$ and $S(r)$, when using the multipole method of eliminating the angular variables [7], one can split the Eq. (1) into the following three independent subsets of radial equations [8]:

(i) subset ^1j_j with total parity $\eta(-1)^j$,

$$\begin{aligned}
\tfrac{1}{2}(E-V)\phi &= (m+\tfrac{1}{2}S)\phi^0,\\
\tfrac{1}{2}(E-V)\phi^0 + i(\tfrac{d}{dr}+\tfrac{2}{r})\phi_E + i\tfrac{\sqrt{j(j+1)}}{r}\phi_L &= (m+\tfrac{1}{2}S)\phi,\\
\tfrac{1}{2}(E-V)\phi_E + i\tfrac{d}{dr}\phi^0 &= 0,\\
\tfrac{1}{2}(E-V)\phi_L - i\tfrac{\sqrt{j(j+1)}}{r}\phi^0 &= 0,
\end{aligned} \qquad (2)$$

(ii) subset $^3(j \mp 1)_j$ with total parity $\eta(-1)^{j+1}$,

$$\begin{aligned}
\tfrac{1}{2}(E-V)\chi_E + i\tfrac{d}{dr}\chi^0 &= (m+\tfrac{1}{2}S)\chi^0_E,\\
\tfrac{1}{2}(E-V)\chi_L - i\tfrac{\sqrt{j(j+1)}}{r}\chi^0 &= (m+\tfrac{1}{2}S)\chi^0_L,\\
\tfrac{1}{2}(E-V)\chi^0_E + \tfrac{\sqrt{j(j+1}}{r}\phi^0_M &= (m+\tfrac{1}{2}S)\chi_E,\\
\tfrac{1}{2}(E-V)\chi^0_L - (\tfrac{d}{dr}+\tfrac{1}{r})\phi^0_M &= (m+\tfrac{1}{2}S)\chi_L,\\
\tfrac{1}{2}(E-V)\chi &= 0,\\
\tfrac{1}{2}(E-V)\chi^0 + i(\tfrac{d}{dr}+\tfrac{2}{r})\chi_E + i\tfrac{\sqrt{j(j+1)}}{r}\chi_L &= 0,\\
\tfrac{1}{2}(E-V)\phi_M &= 0,\\
\tfrac{1}{2}(E-V)\phi^0_M + \tfrac{\sqrt{j(j+1)}}{r}\chi^0_E + (\tfrac{d}{dr}+\tfrac{1}{r})\chi^0_L &= 0,
\end{aligned} \qquad (3)$$

(iii) subset ${}^3j_j$ with total parity $\eta(-1)^j$ $(j > 0)$,

$$
\begin{aligned}
&\tfrac{1}{2}(E-V)\chi_M &&= (m+\tfrac{1}{2}S)\chi^0_M,\\
&\tfrac{1}{2}(E-V)\chi^0_M + \tfrac{\sqrt{j(j+1)}}{r}\phi^0_E + \left(\tfrac{d}{dr}+\tfrac{1}{r}\right)\phi^0_L &&= (m+\tfrac{1}{2}S)\chi_M,\\
&\tfrac{1}{2}(E-V)\phi^0_E + \tfrac{\sqrt{j(j+1)}}{r}\chi^0_M &&= 0,\\
&\tfrac{1}{2}(E-V)\phi^0_L - \left(\tfrac{d}{dr}+\tfrac{1}{r}\right)\chi^0_M &&= 0.
\end{aligned}
\tag{4}
$$

The spectroscopic signature ${}^{2s+1}l_j$ of the subsets *(i)*, *(ii)* and *(iii)* refers to the "large-large" wave-function components involved. The total parity is $\eta\beta_1\beta_2(-1)^l$ with the eigenvalues $\beta_1\beta_2 = \pm 1$. For a fermion–antifermion system $\eta = -1$.

Denote by $\psi^{\gamma_1^5\gamma_2^5}_{\beta_1\beta_2}({}^{2s+1}l_j)$ the radial components of wave function ψ corresponding to the states ${}^{2s+1}l_j$ (where $s = 0, 1$, and $l = j,\ j \mp 1$), additionally labelled by the eigenvalues $\beta_1\beta_2 = \pm 1$ and $\gamma_1^5\gamma_2^5 = \pm 1$ (of course, the matrices $\beta_1\beta_2$ and $\gamma_1^5\gamma_2^5$ commute). Then, one gets

$$
\begin{aligned}
&\left.\begin{matrix}\phi\\ \phi^0\end{matrix}\right\} = \psi^{\mp}_{+}({}^1j_j),\\
&\left.\begin{matrix}\phi_E\\ \phi^0_E\end{matrix}\right\} = \sqrt{\frac{j}{2j+1}}\psi^{\pm}_{-}({}^3(j-1)_j) + \sqrt{\frac{j+1}{2j+1}}\psi^{\pm}_{-}({}^3(j+1)_j),\\
&\left.\begin{matrix}\phi_L\\ \phi^0_L\end{matrix}\right\} = -\sqrt{\frac{j+1}{2j+1}}\psi^{\pm}_{-}({}^3(j-1)_j) + \sqrt{\frac{j}{2j+1}}\psi^{\pm}_{-}({}^3(j+1)_j),\\
&\left.\begin{matrix}\phi_M\\ \phi^0_M\end{matrix}\right\} = \psi^{\pm}_{-}({}^3j_j)
\end{aligned}
\tag{5}
$$

and

$$
\begin{aligned}
&\left.\begin{matrix}\chi\\ \chi^0\end{matrix}\right\} = \psi^{\mp}_{-}({}^1j_j),\\
&\left.\begin{matrix}\chi_E\\ \chi^0_E\end{matrix}\right\} = \sqrt{\frac{j}{2j+1}}\psi^{\pm}_{+}({}^3(j-1)_j) + \sqrt{\frac{j+1}{2j+1}}\psi^{\pm}_{+}({}^3(j+1)_j),\\
&\left.\begin{matrix}\chi_L\\ \chi^0_L\end{matrix}\right\} = -\sqrt{\frac{j+1}{2j+1}}\psi^{\pm}_{+}({}^3(j-1)_j) + \sqrt{\frac{j}{2j+1}}\psi^{\pm}_{+}({}^3(j+1)_j),\\
&\left.\begin{matrix}\chi_M\\ \chi^0_M\end{matrix}\right\} = \psi^{\pm}_{+}({}^3j_j).
\end{aligned}
\tag{6}
$$

Note from Eqs (3) that $\phi_M \equiv 0$ and $\chi \equiv 0$ (in the case of $m_1 = m_2$). One can see from Eqs (5) and (6) that the "large–large" components (superposed with the "small–small" components) are contained in ϕ and ϕ^0 as well as χ_E, χ_L, χ_M and χ_E^0, χ_L^0, χ_M^0 (which correspond to $\beta_1\beta_2 = +1$). Other ϕ's and χ's (corresponding to $\beta_1\beta_2 = -1$) are superpositions of the "large–small" and "small–large" components. The labels E, L and M of $s = 1$ components stand here for "electric", "longitudinal" and "magnetic" to mark some analogy with the multipole expansion of the electromagnetic field.

Eliminating from the subset (2) all components but ϕ^0 one obtains the equation

$$\left[\frac{1}{4}(E-V)^2 + \frac{1}{r}\frac{d^2}{dr^2}r - \frac{j(j+1)}{r^2} - \left(m+\tfrac{1}{2}S\right)^2 + \frac{1}{E-V}\frac{dV}{dr}\frac{d}{dr}\right]\phi^0 = 0\,, \quad (7)$$

where $s = 0$ and $l = j$. Similarly, from the subset (4) one gets the equation

$$\left[\frac{1}{4}(E-V)^2 + \frac{1}{r}\frac{d^2}{dr^2}r - \frac{j(j+1)}{r^2} - (m+\tfrac{1}{2}S)^2 + \frac{1}{E-V}\frac{dV}{dr}\left(\frac{d}{dr}+\frac{1}{r}\right)\right]\chi_M^0 = 0\,, \quad (8)$$

where $s = 1$ and $l = j > 0$. From the more complicated subset (3) one deduces in the simplest case of $j = 0$ (where $\chi_L \equiv 0$, $\chi_L^0 \equiv 0$ and $\phi_M^0 \equiv 0$, while χ_E, $\chi_E^0 = \psi_{+}^{\pm}(^3P_0)$) the equation

$$\left[\frac{1}{4}(E-V)^2 + \frac{1}{r}\frac{d^2}{dr^2}r - \frac{2}{r^2} - (m+\tfrac{1}{2}S)^2 + \frac{1}{E-V}\left(\frac{d}{dr}+\frac{2}{r}\right)\right]\chi_E = 0\,, \quad (9)$$

where $s = 1$ and $l = j + 1 = 1$. In the general case, eliminating from the subset (3) the components χ_E^0 and χ_L by the algebraic equations contained in Eq. (3), one obtains the system of four first-order radial equations:

$$\begin{aligned}
&\left[\tfrac{1}{4}(E-V)^2 - (m+\tfrac{1}{2}S^2\right]\chi_E + \tfrac{1}{2}(E-V)\frac{d}{dr}i\chi^0 \\
&\qquad = -(m+\tfrac{1}{2}S)\frac{\sqrt{j(j+1)}}{r}\phi_M^0\,, \\
&\left[\tfrac{1}{4}(E-V)^2 - (m+\tfrac{1}{2}S)^2\right]\chi_L^0 - \tfrac{1}{2}(E-V)\left(\frac{d}{dr}+\frac{1}{r}\right)\phi_M^0 \\
&\qquad = (m+\tfrac{1}{2}S)\frac{\sqrt{j(j+1)}}{r}i\chi^0\,, \\
&\left[\tfrac{1}{4}(E-V)^2 - \frac{j(j+1)}{r^2}\right]i\chi^0 - \tfrac{1}{2}(E-V)\left(\frac{d}{dr}+\frac{2}{r}\right)\chi_E \\
&\qquad = (m+\tfrac{1}{2}S)\frac{\sqrt{j(j+1)}}{r}\chi_L^0\,,
\end{aligned}$$

$$\left[\tfrac{1}{4}(E-V)^2 - \frac{j(j+1)}{r^2}\right]\phi^0_M + \tfrac{1}{2}(E-V)\left(\frac{d}{dr}+\frac{1}{r}\right)\chi^0_L$$
$$= -(m+\tfrac{1}{2}S)\frac{\sqrt{j(j+1)}}{r}\chi_E\,, \tag{10}$$

where $s=1$ and $l=j\mp 1$ for "large–large" components. In the case of $j=0$ this system gives Eq. (9) by further eliminating χ^0.

All radial components of $\psi(\vec{r})$ discussed above are so normalized as radial components of the (two-body) Dirac wave function should be. Therefore, when divided by $\sqrt{E-V(r)}$, they are normalized with the weight $E-V(r)$, like radial components of a (two-body) Klein–Gordon wave function. For such truncated functions, Eqs (7), (8) and (9) lead to the following new radial equations:

for ${}^1j_j$ states

$$\left[\frac{1}{4}(E-V)^2 + \frac{d^2}{dr^2} - \frac{j(j+1)}{r^2} - (m+\tfrac{1}{2}S)^2 - \frac{3}{4}\left(\frac{\frac{dV}{dr}}{E-V}\right)^2\right.$$
$$\left. - \frac{1}{2}\frac{\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}}{E-V}\right]\frac{r\phi^0}{\sqrt{E-V}} = 0\,, \tag{11}$$

for ${}^3j_j$ states ($j>0$)

$$\left[\frac{1}{4}(E-V)^2 + \frac{d^2}{dr^2} - \frac{j(j+1)}{r^2} - (m+\tfrac{1}{2}S)^2 - \frac{3}{4}\left(\frac{\frac{dV}{dr}}{E-V}\right)^2\right.$$
$$\left. - \frac{1}{2}\frac{\frac{d^2V}{dr}}{E-V}\right]\frac{r\chi^0_M}{\sqrt{E-V}} = 0 \tag{12}$$

and for ${}^3P_0$ states

$$\left[\frac{1}{4}(E-V)^2 + \frac{d^2}{dr^2} - \frac{2}{r^2} - (m+\tfrac{1}{2}S)^2 - \frac{3}{4}\left(\frac{\frac{dV}{dr}}{E-V}\right)^2\right.$$
$$\left. - \frac{1}{2}\frac{\frac{d^2V}{dr^2}-\frac{2}{r}\frac{dV}{dr}}{E-V}\right]\frac{r\chi_E}{\sqrt{E-V}} = 0\,. \tag{13}$$

When applying to Eqs (11), (12) and (13) the perturbative approach, one may use the convenient identity

$$\int_0^\infty r^2 dr \frac{F}{\sqrt{E-V}}(E-V)\frac{1}{r}\frac{d^2}{dr^2}r\frac{F}{\sqrt{E-V}}$$

$$= \int_0^\infty r^2 dr F\left[\frac{1}{r}\frac{d^2}{dr^2}r + \frac{1}{4}\left(\frac{\frac{dV}{dr}}{E-V}\right)^2\right]F \tag{14}$$

valid for $F(r)$ real.

If in these equations the two last terms are omitted, they go over into the radial equation corresponding to the (two-body) Klein–Gordon equation with the potentials $V(r)$ and $S(r)$. Then, in the case of Coulombic potential $V = -\alpha/r$ and $S \equiv 0$ one gets for bound states the familiar energy spectrum

$$E_{\mathrm{KG}} = 2m\left[1+\left(\frac{\alpha/2}{n_r+\gamma_{\mathrm{KG}}}\right)^2\right]^{-\frac{1}{2}} \quad \text{with } \gamma_{\mathrm{KG}} = \tfrac{1}{2} + \left[(j+\tfrac{1}{2})^2 - (\alpha/2)^2\right]^{\frac{1}{2}}, \tag{15}$$

where $n_r = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$. Thus, in general, Eqs (11), (12) and (13) can be viewed as radial equations corresponding to (two-body) Klein–Gordon equations with some energy-dependent effective interactions. For instance, such an interaction in Eq. (11) is

$$I = \frac{1}{2}EV - \frac{1}{4}V^2 + mS + \frac{1}{4}S^2 + \frac{3}{4}\left(\frac{\frac{dV}{dr}}{E-V}\right)^2 + \frac{1}{2}\frac{\Delta V}{E-V}. \tag{16}$$

In the case of Coulombic potential $V = -\alpha/r$ and $S \equiv 0$, Eq. (16) gives

$$I = -\frac{1}{2}\frac{\alpha E}{r} - \frac{1}{4}\frac{\alpha^2}{r^2} + \frac{3}{4}\frac{\alpha^2}{r^2(Er+\alpha)^2} \tag{17}$$

since $\Delta V = 4\pi\alpha\delta^3(\vec{r})$ and $r\delta^3(\vec{r}) = 0$ (unless $(^1/_2)\Delta V/(E-V)$ in Eq. (16) is perturbatively replaced by $(^1/_2)\Delta V/2m = (\pi\alpha/m)\delta^3(\vec{r})$, while $(^3/_4)(dV/dr)^2/(E-V)^2$ is perturbatively neglected). So, in the Coulombic case Eq. (11) describing parastates takes the form:

$$\left[\frac{1}{4}E^2 + \frac{d^2}{dr^2} - \frac{j(j+1)-(\alpha/2)^2}{r^2} - m^2 + \frac{1}{2}\frac{\alpha E}{r} - \frac{3}{4}\frac{\alpha^2}{r^2(Er+\alpha)^2}\right]\frac{r\phi^0}{\sqrt{E+\alpha/r}} = 0. \tag{18}$$

This equation implies for $r \to 0$

$$\frac{r\phi^0}{\sqrt{E+\alpha/r}} \sim r^\gamma \text{ with } \gamma = \tfrac{1}{2} + [1 + j(j+1) - (\alpha/2)^2]^{1/2} \qquad (19)$$

and for $r \to \infty$

$$\frac{r\phi^0}{\sqrt{E+\alpha/r}} \sim e^{-\kappa r} \text{ with } \kappa = [m^2 - (E/2)^2]^{1/2}, \qquad (20)$$

the latter when bound states are considered. By the substitution $r\phi^0/\sqrt{E+\alpha/r} = r^\gamma \exp(-\kappa r) g$ one can transform out from Eq. (18) the behaviour of ϕ^0 at $r \to 0$ and $r \to \infty$, obtaining the equation for g:

$$\left\{ \frac{d^2}{dr^2} + 2\left(\frac{\gamma}{r} - \kappa\right)\frac{d}{dr} + \frac{1}{2}\frac{\alpha E - 4\kappa\gamma}{r} + \frac{3}{4}\frac{1}{r^2}\left[1 - \left(\frac{\alpha}{Er+\alpha}\right)^2\right] \right\} g = 0\,. \quad (21)$$

Here, the last term can be rewritten as $(3/4)E(Er+\alpha)/[r(Er+2\alpha)^2]$. Hence, one gets $g \sim 1$ for $r \to 0$ and

$$g \sim r^A\left(1 + \frac{a}{2\kappa r}\right) \text{ with } A = \frac{\alpha E}{4\kappa} - \gamma\,,\; a = A(A - 1 + 2\gamma) - \frac{3}{4} \qquad (22)$$

for $r \to \infty$. Thus, $r\phi^0/\sqrt{E+\alpha/r} \sim r^{\alpha E/4\kappa} \exp(-\kappa r)$ when $r \to \infty$. Evidently, $E = 2m\left[1 + (\alpha/2)^2/(A+\gamma)^2\right]^{-1/2}$ is the exact form of energy spectrum, but A stands here for an unknown parameter (dependent on the state).

It is not difficult to derive from Eq. (21) its other equivalent forms:

$$\left[\frac{d^2}{dr^2} + 2\left(\frac{\gamma_{KG}}{r} - \kappa\right)\frac{d}{dr} + \frac{1}{2}\frac{\alpha E - 4\kappa\gamma_{KG}}{r} - \frac{3}{4}\frac{\alpha^2}{r^2(Er+\alpha)^2}\right] r^{\gamma-\gamma_{KG}} g = 0 \qquad (23)$$

as well as

$$\left[\frac{d^2}{dr^2} + 2\left(\frac{\gamma - \frac{1}{2}}{r} - \kappa\right)\frac{d}{dr} + \frac{1}{2}\frac{\alpha E - 4\kappa(\gamma - \frac{1}{2})}{r} - \frac{\gamma - \frac{3}{2}}{r^2} + \frac{\alpha}{r(Er+\alpha)}\left(\frac{d}{dr} + \frac{\gamma - \frac{3}{2}}{r} - \kappa\right)\right] f = 0 \qquad (24)$$

and

$$\left[\frac{d^2}{dr^2} + 2\left(\frac{\gamma_{KG}}{r} - \kappa\right)\frac{d}{dr} + \frac{1}{2}\frac{\alpha E - 4\kappa\gamma_{KG}}{r} + \frac{\alpha}{r(Er+\alpha)}\left(\frac{d}{dr} + \frac{\gamma_{KG}-1}{r} - \kappa\right)\right] r^{\gamma-\frac{1}{2}-\gamma_{KG}} f = 0\,, \qquad (25)$$

where $f = \sqrt{Er+\alpha}g = r^{\frac{1}{2}}\sqrt{E+\alpha/r}g$. Here, $r\phi^0 = r^\gamma \exp(-\kappa r) \times \sqrt{E+\alpha/r}g = r^{\gamma-\frac{1}{2}}\exp(-\kappa r)f = r^{\gamma_{KG}}\exp(-\kappa r)r^{\gamma-\frac{1}{2}-\gamma_{KG}}f$. Note that $f \sim 1$ both for $r \to 0$ and $r \to -\alpha/E$, while $f \sim r^{A+\frac{1}{2}}$ for $r \to \infty$ (*cf.* Eq. (22)).

Since in the analytic extension of Eq. (21) there is a (regular) singularity at $r = -\alpha/E$ in addition to the familiar regular singularity at $r = 0$ and irregular singularity at $r = \infty$, this equation can be reduced neither to the hypergeometric equation nor to the confluent hypergeometric equation. In consequence, there is no argument to look for its exact bound-state solutions g in the form of polynomials and so to make the power A in the asymptotic formula (22) equal to $n_r = 0, 1, 2, \ldots$. It seems that numerical methods remain as a way out, if the coupling constant α becomes too large to keep the perturbative expansion in α reliable. Of course, a general qualitative discussion like that in Refs [1, 2] may be useful for various potentials $V(r)$ and $S(r)$.

Nonstatic corrections to the Coulombic potential, responsible for hfs of energy levels modifying considerably their fs, are another serious problem. As is known, in order to obey the hole theory the Breit terms must not be treated beyond the lowest-order perturbative calculations, if handled in the framework of the two-body Dirac equation [5, 3]. On the other hand, the Salpeter equation, though fully consistent with the hole theory, is much more complicated. Such are also the resulting radial equations (*cf.* two last Refs [6]).

For an illustration, in the case of Coulombic potential $V = -\alpha/r$ and $S \equiv 0$, the perturbative energy spectrum for parastates described by Eq. (7) or (11) has the form

$$E = 2m - \frac{\alpha^2 m}{4n^2} + \frac{\alpha^4 m}{16n^4}\left[\frac{3}{4} - \frac{n(1-\delta_{j0})}{j+\frac{1}{2}}\right] + O(\alpha^6)\,. \qquad (26)$$

For $j > 0$ it coincides up to the order $O(\alpha^4)$ with the Coulombic spectrum (15) following from the (two-body) Klein–Gordon equations with the potentials $V = -\alpha/r$ and $S \equiv 0$. The lowest-order nonstatic correction for parastates from the Breit terms

$$V_B = \frac{1}{2}\frac{\alpha}{r}\left[\vec{\alpha}_1 \cdot \vec{\alpha}_2 + (\vec{\alpha}_1 \cdot \hat{r})(\vec{\alpha}_2 \cdot \hat{r})\right], \tag{27}$$

where $\hat{r} = \vec{r}/r$, is (*cf. e.g.*, Ref. [3])

$$E_B = \frac{\alpha^4 m}{16n^4}\left[2 - \frac{n(3+\delta_{j0})}{j+\frac{1}{2}}\right] + O(\alpha^6). \tag{28}$$

This gives

$$E + E_B = 2m - \frac{\alpha^2 m}{4n^2} + \frac{\alpha^4 m}{4n^4}\left(\frac{11}{16} - \frac{n}{j+\frac{1}{2}}\right) + O(\alpha^6), \tag{29}$$

what is the well known perturbative result for parastates [5]. Radiative corrections come in the order $O(\alpha^5)$ [9].

The influence of the fatal singularity at $r = -\alpha/E$ can be neglected not only in the weak coupling limit of $\alpha/Er \to 0$, but also in the less familiar strong coupling limit of $\alpha/Er \to \infty$. In the latter case, where $(Er+\alpha)^{-1} \to \alpha^{-1}$ and

$$\left(\gamma - \frac{3}{2}\right)\left[\frac{1}{r^2} - \frac{\alpha}{r^2(Er+\alpha)}\right] = \left(\gamma - \frac{3}{2}\right)\frac{E}{r(Er+\alpha)} \to \left(\gamma - \frac{3}{2}\right)\frac{E}{\alpha r},$$

Eq. (24) tends to

$$\left[\frac{d^2}{dr^2} + 2\left(\frac{\gamma}{r} - \kappa\right)\frac{d}{dr} + \frac{1}{2}\frac{(\alpha + 3/\alpha - 2\gamma/\alpha)E - 4\kappa\gamma}{r}\right] f = 0, \tag{30}$$

implying the energy spectrum

$$E = 2m\left[1 + \left(\frac{1}{2}\frac{\alpha + 3/\alpha - 2\gamma/\alpha}{n_r + \gamma}\right)^2\right]^{-\frac{1}{2}}, \tag{31}$$

where $n_r = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots$, while γ is given as in Eq. (19). This formula is valid asymptotically in the strong coupling limit, but, on the other hand, $\alpha = 2$ at most, as $\alpha = 2$ is here the Klein–paradox critical value at $j = 0$. In the case of $\alpha = 2$, for the ground state ($n_r = 0$, $j = 0$) Eq. (31) gives $E_0 = 0.632$m.

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