PAIR PRODUCTION IN A STRONG ELECTRIC FIELD WITH BACK-REACTION*

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A summary is presented of the current status of efforts to solve the problem in which pairs are produced in a strong electric field, are accelerated by it, and then react back on it through the counter-field produced by their current. A review of recent developments in this back-reaction problem is given. A simple version of the theory of pair tunneling from a fixed electric field, is first presented and then a sketch is provided as to how this has been applied to the quark-gluon plasma. Then we turn to a field formulation of the problem for charged bosons, which leads to the need to carry out a renormalization program, outlined again in simple terms. Numerical results for this program are presented for one and for three spatial dimensions, and the corresponding physical behavior of the system is discussed We exhibit a phenomenological transport equation embodying physics that is essentially identical to that of the field formulation Last, we present the extension to the fermion case and to the formulation in terms of boost-invariant variables (as required for the quark-gluon plasma).

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1. Introduction

The main motivation for this work arose from studies [1] of the behavior of the quark-gluon plasma, and especially its oscillations, using transport equations. In order to establish a firmer basis for the use of the transport method, we have attempted to study exact field formulations for a quantized boson or fermion field in interaction with a homogeneous, classical, infinite

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electric field. Pairs tunnel out of the electric field, are accelerated by it, and react back on it — the so-called back-reaction problem. Results from this program have begun to appear in the literature [2–4], based heavily on the doctoral thesis at Tel Aviv University of Yuval Kluger [3], and so the emphasis in this paper is on a didactic summary of the current status of the back-reaction problem; technical details are generally suppressed in the hope of setting forth the main physical issues more clearly.

In a very simplistic scenario for the first stage of the production of a quark-gluon plasma at ultra-relativistic energies, one might imagine that two highly contracted nuclei collide, generate color charges on each other, and then pass through each other. In their wake they leave a chromoelectric field produced by the color charges on them. In our treatment this chromoelectric field is taken in a radically simplified view: We treat it as an Abelian — and therefore an ordinary electric — field; we further take it to be a classical field [4], and regard it as filling all space homogeneously. These are, of course, highly unrealistic assumptions insofar as the quark-gluon plasma application is concerned, but, as we shall see, even with these grotesque simplifications the back-reaction problem remains remarkably recalcitrant. Thus we exploit all of them in order to make progress with back-reaction, and hope to restore a more realistic framework after that has been done. Out of the (chromo)electric field there now tunnel pairs of partons, quarks and gluons of opposite "charge", that are eventually to comprise the plasma.

The tunneling mechanism in question has been extremely well known for over 60 years now [5], and an exact solution [6], for the pair creation rate in the presence of a fixed, external electric field has been available for some 40 years. In simple terms [7], what is happening in the tunneling process can be thought of in the following terms: The pair we are considering is initially latent in the Dirac sea. We imagine a fictitious potential that binds this latent pair at the combined rest-mass energy $2mc^2$. The electric field provides a further potential $-eEx$, and the overall potential then allows the pair to tunnel through to the outside. The point at which they emerge is $x = 2m/eE$, implying, for small fields, a long tunneling distance. A rough estimate of the rate of pair production is then given by

$$\begin{aligned} \text{rate} \sim \exp\left[-\int |p|\,dx\right] &\sim \exp\left[-2\int_0^{2m/eE} \frac{1}{2}(2m\cdot 2m)^{1/2}\,dx\right] \\ &\sim \exp\left[-\frac{4m^2}{eE}\right] \to \exp\left[-\frac{\pi m^2}{eE}\right], \end{aligned} \tag{1}$$

where the last expression is that to emerge from a more precise treatment. As is to be expected, the tunneling process is quantal by its nature, and no perturbative expansion about $E = 0$ is possible for it.

Now the application [1, 8, 9] of this pair tunneling rate to the scenario for quark-gluon plasma production has been made through the use of a Boltzmann equation in which the pair rate serves as a source term on the right-hand side,

$$\frac{\partial f}{\partial t}+\frac{\vec{p}}{(\vec{p}^2+m^2)^{1/2}}\cdot\frac{\partial f}{\partial \vec{x}}+e\vec{E}(t)\cdot\frac{\partial f}{\partial \vec{p}}= \text{ pair rate} = \cdots \exp\left[-\frac{\pi m^2}{eE(t)}\right], \quad (2)$$

where, of course, $f(\vec{x}, \vec{p}, t)$ is the density of particles at position $\vec{x}$, with momentum $\vec{p}$ and at time t. We have again indicated only very loosely the structure of the equation; more precise forms for our present context will eventually be given below. Further, the applications to the quark-gluon plasma are generally made in terms of boost-invariant variables and coordinates. We shall return to a consideration of this point towards the end of this paper. (We note that a very closely analogous back-reaction problem arose years ago in inflationary cosmology [4, 10]; there the time-dependent metric plays the role of the time-dependent electric field here.)

In allowing the electric field in Eq. (2) to be time-dependent, we have anticipated the inevitable appearance of back-reaction: Once the charged pairs make their appearance, they will be accelerated by the electric field, producing a current, which in turn produces an electric field. This field will oppose the direction of the original field, and eventually field and plasma oscillations will be set up. This effect enters our theoretical description through a simple application of Maxwell's laws for this case,

$$\dot{\vec{E}}(t) = -\vec{j}(t) = -e\int d\vec{p}\frac{\vec{p}}{(\vec{p}^2+m^2)^{1/2}}\, f(\vec{x}, \vec{p}, t), \quad (3)$$

where, for the case of a homogeneous system filling all of space, only a constant magnetic field can arise, which we ignore. The two equations (2) and (3) now form a coupled set which must be solved to incorporate back-reaction and exhibit oscillatory behavior.

It is clear, however, that there is an inconsistency built into the construction of this set of coupled equations: The expression for the rate of pair production used on the right-hand side of Eq. (2) is derived [5–7] for the case of a field *fixed* (by an external agent) in time, while back-reaction necessarily involves a changing field. Furthermore, one might easily suppose that pairs produced directly from the time variation of the field will be generated at a faster rate than those that emerge from tunneling [11]. In fact, Eq. (2) has not really been "derived" from any basic field equations through the use of a Wigner representation, say, but instead has been put together on the grounds of a healthy physical intuition for the problem. This immediately raises the two questions, *(i)* How would back-reaction emerge in a description of this same physical system based on field theory, and *(ii)* can one derive a transport equation resembling Eq. (2) from the field-theory formulation? In this work, we provide an answer for the first of these questions, and are able to find a transport equation very much along the lines of Eq. (2) above, whose solutions bear a striking resemblance — at the quantitative level — to those of the field theory. The question of a direct derivation of the transport equations from the field equations is touched upon briefly at the end.

2. Formulation of back-reaction in field theory

In line with our didactic purpose here, we shall continue to formulate the back-reaction problem in its simplest form; a far more complete discussion is given in Ref. [4]. We take a system of charged bosons of mass m satisfying a Klein-Gordon equation

$$-(\partial - ieA)^{\mu}(\partial - ieA)_{\mu}\phi + m^2\phi = 0\,, \tag{4}$$

where for the homogeneous electric field filling all space we take a vector potential (in a particular gauge)

$$A_{\mu} = (0; \vec{A}(t))\,, \tag{5}$$

which satisfies the Maxwell equation

$$\ddot{\vec{A}}(t) = \vec{j} = -ie[\phi^{\dagger}\vec{\nabla}\phi - (\vec{\nabla}\phi^{\dagger})\phi] - 2e^2\vec{A}\phi^{\dagger}\phi\,. \tag{6}$$

The form of the boson field after second quantization is

$$\phi(\vec{x}, t) = \int \frac{d\vec{k}}{(2\pi)^3}\left[f_{\vec{k}}(t)\, a_{\vec{k}} \exp[i\vec{k}\cdot\vec{x}] + f^*_{-\vec{k}}(t)\, b^{\dagger}_{\vec{k}} \exp[-i\vec{k}\cdot\vec{x}]\right], \tag{7}$$

where $a_{\vec{k}}$ and $b^{\dagger}_{-\vec{k}}$ are the particle annihilation and antiparticle creation operators, respectively. The forms $f_{\vec{k}}$ are the mode amplitudes for bosons with momentum $\vec{k}$, which, upon substituting Eq. (7) into Eq. (4), are seen to satisfy

$$\ddot{f}_{\vec{k}}(t) + \omega^2_{\vec{k}}(t) f_{\vec{k}}(t) = 0\,, \tag{8}$$

where

$$\omega^2_{\vec{k}}(t) = (\vec{k} - e\vec{A}(t))^2 + m^2\,, \tag{9}$$

just as one would anticipate. The boson commutation relations imply a constraint on the mode amplitudes,

$$[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = [b_{\vec{k}}, b^{\dagger}_{\vec{k}'}] = (2\pi)^3\,\delta(\vec{k} - \vec{k}') \Rightarrow f_{\vec{k}}\dot{f}^*_{\vec{k}} - f^*_{\vec{k}}\dot{f}_{\vec{k}} = i\,. \tag{10}$$

Last, using Eq. (7) the Maxwell equation becomes

$$\ddot{\vec{A}}(t) = \langle \vec{j}(t) \rangle = e \int \frac{d\vec{k}}{(2\pi)^3} (\vec{k} - e\vec{A}(t))\; 2|f_{\vec{k}}(t)|^2 , \tag{11}$$

where the brackets on $\vec{j}(t)$ are necessary in order to yield a classical electric field, according to the restriction we have chosen. They will imply, for our present purposes, an expectation value in the initial vacuum, although more general formulations are, of course, possible [4]. There immediately arises the question as to whether this integral for the current converges.

3. Renormalization

As we shall in a moment see, the convergence of the integral in Eq. (11) is by no means guaranteed, and a renormalization procedure is required for the back-reaction problem [4]. It is a rather intricate case since the dynamics of the system are intrinsically interwoven with the renormalization. Put another way, there is no simple answer to the question of the convergence of the integral for the current, since that convergence is governed by the behavior of $f_{\vec{k}}(t)$ at large $\vec{k}$, which in turn is to be known only after the solution of the coupled equations describing the system dynamics. And, of course, no such solution is possible so long as there is no finite result for the current. The general considerations pertinent to the problem of renormalizing — an issue having a long history — are presented very lucidly in Ref. [4], where previous literature on the subject is also noted. The only viable approach appears to be to study the high-k behavior of $f_{\vec{k}}(t)$ using a WKBJ-like ansatz, which, as one might expect, rather readily lends itself to the investigation of the high-energy limit, and an assumption of adiabaticity. We therefore make the general ansatz

$$f_{\vec{k}}(t) = \frac{1}{[2\Omega_{\vec{k}}(t)]^{1/2}} \exp\left[-i \int^t \Omega_{\vec{k}}(t')\, dt'\right] , \tag{12}$$

where it is easily seen that, as for the usual WKBJ treatment in quantum mechanics, $\Omega_{\vec{k}}(t)$ satisfies

$$\Omega_{\vec{k}}^2(t) + \frac{\ddot{\Omega}_{\vec{k}}}{2\Omega_{\vec{k}}} - \frac{3\dot{\Omega}_{\vec{k}}^2}{4\Omega_{\vec{k}}^2} = \omega_{\vec{k}}^2(t) . \tag{13}$$

The Maxwell equation (11) then becomes

$$\ddot{\vec{A}}(t) = \langle \vec{j}(t) \rangle = e \int \frac{d\vec{k}}{(2\pi)^3} (\vec{k} - e\vec{A}(t)) \frac{1}{\Omega_{\vec{k}}(t)} . \tag{14}$$

In order to proceed with an investigation of the behavior of $f_{\vec{k}}(t)$ at high $\vec{k}$, we have no choice but to suppose that $\Omega_{\vec{k}}(t)$ varies adiabatically in this limit. This is in some sense consistent with one's expectations for a renormalization program since a violent variation in the time dependence of the physical quantities at large momenta would certainly appear to rule out any hope of renormalizing them meaningfully. In the present case, we shall, at the end, find a kind of *a posteriori* justification for this procedure in that the physics that emerges from it is consistent with a quite different phenomenological formulation of the problem. Assuming adiabaticity, we replace the time derivatives of $\Omega_{\vec{k}}(t)$ with those of $\omega_{\vec{k}}(t)$, and suppose $\dot{\omega}_{\vec{k}}(t)/\omega^2_{\vec{k}}(t) \ll 1$, $\ddot{\omega}_{\vec{k}}(t)/\omega^3_{\vec{k}}(t) \ll 1$; we then expand in these small ratios. For the quantity entering in the Maxwell equation, we have

$$\frac{1}{\Omega_{\vec{k}}(t)} = \frac{1}{\omega_{\vec{k}}(t)}\left[1 - \frac{3\dot{\omega}^2_{\vec{k}}}{8\omega^4_{\vec{k}}} + \frac{\ddot{\omega}_{\vec{k}}}{4\omega^3_{\vec{k}}} + \cdots\right]. \tag{15}$$

To the order in $1/k$ that we need, terms in the first and second time derivatives suffice, and, again to the necessary order, these derivatives are

$$\dot{\omega}_{\vec{k}} = \frac{-e\dot{\vec{A}}\cdot(\vec{k}-e\vec{A})}{\omega_{\vec{k}}} \quad \text{and} \quad \ddot{\omega}_{\vec{k}} = \frac{-e\ddot{\vec{A}}\cdot(\vec{k}-e\vec{A})}{\omega_{\vec{k}}} + \mathcal{O}\frac{1}{k}. \tag{16}$$

It is then clear from the antisymmetry of the integral in (14) in the variable $(k - eA)$ that the first term on the right-hand side of Eq. (15)—the "1" in the brace—makes no contribution. The second term gives a finite integral. But the third term diverges logarithmically. We eliminate this divergence by adding and subtracting the logarithmically divergent integral

$$e\int \frac{d\vec{k}}{(2\pi)^3}\frac{k_3^2\, e\ddot{\vec{A}}(t)}{4\omega^5_{\vec{k}}(0)} = e\int \frac{d\vec{k}}{(2\pi)^3}\frac{k^2 e\ddot{\vec{A}}(t)}{12\omega^5_{\vec{k}}(0)}.$$

This added integral can then be regrouped with that of Eq. (14) to produce a finite result, and the subtraction of the identical integral is absorbed into the definition of the renormalized charge, $e_R^2 = Ze^2$, where the infinite renormalization constant is

$$Z = \left[1 + e^2\int \frac{d\vec{k}}{(2\pi)^3}\frac{k^2}{12\omega^5_{\vec{k}}(0)}\right]^{-1}.$$

The electromagnetic field is correspondingly renormalized by $\vec{A}_R = \vec{A}/Z^{1/2}$, so that the combination eA which appears throughout is unchanged, and we therefore need not bother to label all the quantities e and A with a subscript R. The renormalized Maxwell equation now reads

$$\ddot{\vec{A}}(t) = e \int \frac{d\vec{k}}{(2\pi)^3} \left[\frac{\vec{k} - e\vec{A}(t)}{\Omega_{\vec{k}}(t)} + \frac{k^2 e\ddot{\vec{A}}(t)}{12\omega_{\vec{k}}^5(0)} \right], \tag{17}$$

where it is to be understood that all the electromagnetic quantities appearing refer to their renormalized values, and the integral on the right-hand side is finite.

Still a problem remains, however: There is no guarantee that, for a given choice of initial values, the resulting solutions for $f_{\vec{k}}(t)$, or, equivalently, for $\Omega_{\vec{k}}(t)$, will remain adiabatic at all future times, a property upon which the entire renormalization scheme rests through the use of Eq. (15). Cooper and Mottola[4] therefore proposed to require that at all times these quantities fulfill

$$\frac{1}{\Omega_{\vec{k}}(t)} = \frac{1}{\omega_{\vec{k}}(t)} - \frac{e(\vec{k} - e\vec{A}) \cdot \ddot{\vec{A}}}{4\omega_{\vec{k}}^5(t)} + r(\vec{k}, t), \tag{18}$$

where $r(\vec{k}, t)$ must fall off faster than $1/k^4$, thus guaranteeing adiabaticity at all times. Then

$$\ddot{\vec{A}}(t) = e \int \frac{d\vec{k}}{(2\pi)^3} (\vec{k} - e\vec{A}(t))\, r(\vec{k}, t), \tag{19}$$

where the factor preceding the integral on the right-hand side results from explicitly carrying out the integrals that multiply $\ddot{\vec{A}}$. This integral is, of course, finite, and the coupled equations (13) and (19) can now, in principle, be solved, at each stage in time imposing Eq. (18). In fact, it would appear, rather peculiarly, that Eq. (19) is no longer required except at $t = 0$, since Eq. (18) will allow the determination of $\ddot{\vec{A}}$ at each time by isolating — as we must do anyway to carry through this procedure — the part of $1/\Omega_{\vec{k}}(t)$ that falls off with $1/k^4$ at large momentum. It is as if the main use of the Maxwell equation were to establish the need for renormalization, after which that stiff requirement alone suffices to generate solutions. In reality it emerges that this strange situation is not the one encountered physically.

4. Results for the case of one spatial dimension

Both because of the intricacy of the renormalization procedure and the numerical difficulties of the full three-dimensional problem, it has proved important first to study [2] the back-reaction problem in one spatial dimension $(1 + 1)$ in order to gain some insight in preparation for attacking the problem with three spatial dimensions (3 + 1). In this case, insofar as the adiabatic assumption for $\Omega_k(t)$ is valid, no renormalization is needed. Let us start by looking at the integrand in Eq. (14) as a function of momentum k after some time has passed in this $1+1$ case. The result is shown in Fig. 1, where we see on the left-hand side a highly oscillatory result for it. At the technical numerical level this oscillatory behavior means that a very fine grid in k is required from the start of the problem at $t = 0$. For this reason reliable numerical results are tedious to come by. Furthermore, it rules out a renormalization program based on isolating terms that decay as a reciprocal power in k, which becomes impossible in the face of the oscillations. We return to this point briefly below.

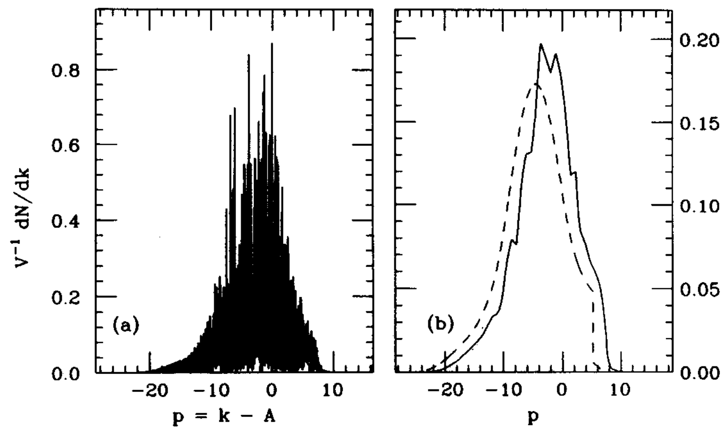

Fig. 1. The number of pairs produced per unit volume (or length, since we have only one spatial dimension here) and momentum (in units of the particle mass m) interval dk at $p = k - eA$ for $e^2/m^2 = 1$ and $\tilde{E}(0) = eE(0)/m^2 = 1$ at time $\tau = mt = 550$, all in scaled, dimensionless units as indicated. In the right-hand figure, the solid line shows the result of smoothing the exact numerical solution on the left by combining 100 bins into one, and the dashed line is the solution of Eq. (20) for $f(p,t)$ under these same conditions.

Figure 2 shows $\tilde{E}(t)$, and its derivative $\tilde{j}(t)$, as functions of time for an initial value $\tilde{E}(0) = 4$ and $e^2/m^2 = 0.1$. The quantities are scaled for $1+1$ such that they become dimensionless, i.e., $\tilde{E} \equiv eE/m^2$, $\tilde{j} \equiv ej/m^3$, and $\tau \equiv mt$, and adiabatic initial values have been taken for $\Omega_k(0) = \omega_k(0)$ and $\dot{\Omega}_k(0) = \dot{\omega}_k(0)$. All these quantities show plasma oscillations with slightly increasing frequency as time goes by, corresponding to the additional production of pairs in the electric field, mainly at its peak values. The current $j(t)$ shows a quite flat plateau where its first (and, in some cases, second) oscillatory peak is expected. This occurs because the initial number of pairs produced is not very great, and the subsequent acceleration of the

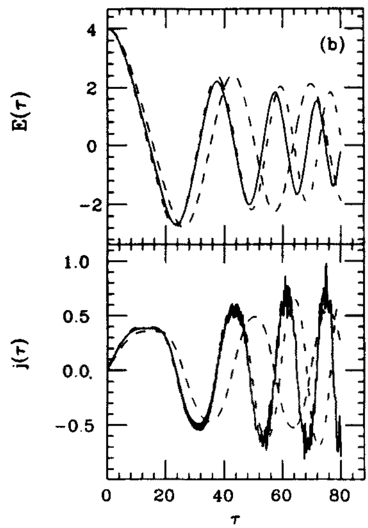

Fig. 2. Results for $\tilde{E}(\tau)$ and $\tilde{j}(\tau)$ for $\tilde{E}(0) = 4$ and $e^2/m^2 = 0.1$ in the same units defined in the caption to the previous figure. The solid line shows the solutions to the coupled field equations; the dashed line is for the Boltzmann equation (20); and the dot-dash line — which for short times is almost indistinguishable from the solid line — is the Boltzmann equation modified by a boson enhancement factor.

particles brings them to the speed of light, leading to a saturation of the current. This is in fact a rather useful property since it gives us a way to measure the number of particles present at the early times: The current is given by $j = 2nev$, where n is the density of particles (or antiparticles) of charge e, and $\pm v$ is their velocity. At the plateau $|v| = 1$, and n may be read off from the plateau height. Since the precise definition of particle number is unclear in a field formulation so long as the electric field is nonzero, this gives us a physical way to extract the operative number. At later times we may use the connection between the relativistic plasma frequency and the particle number for this purpose.

5. A classical Boltzmann model yielding equivalent results

All of these physical points may be sharpened considerably by considering a Boltzmann model [2] close to that of Eqs. (2) and (3), but now with benefit of the field solutions of Eqs. (13) and (14) to test it. The model in question has the transport equation

$$\frac{\partial f}{\partial t} + eE(t)\frac{\partial f}{\partial p} = \frac{|eE(t)|}{2\pi} \log\left[1 + \exp\left[-\frac{\pi m^2}{|eE(t)|}\right]\right]\delta(p)\,, \qquad (20)$$

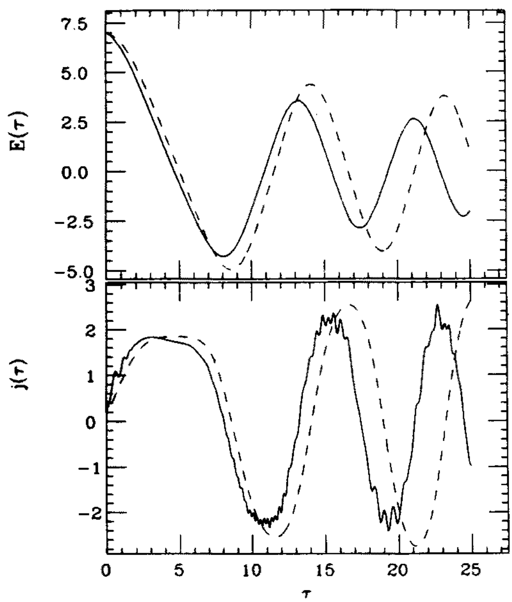

Fig. 3. Results in three spatial dimensions (3 + 1) for $\tilde{E}(\tau)$ and $\tilde{j}(\tau)$ for $\tilde{E}(0) = 7$ and $e^2/m^2 = 4$ in the same units defined in the caption to Fig. 1. The solid line shows the solutions to the coupled field equations and the dashed line is for the Boltzmann equation (20).

where the right-hand side is the pair-production rate of the tunneling mechanism in one spatial dimension, with a distribution in momentum space suggested by microscopic arguments on pair tunneling (see especially the first paper noted in Ref. [7]). This can be inserted into the Maxwell equation (3), and the resulting single equation in $A(t)$ and its derivatives is easily solved. A comparison of this solution, shown by the dashed line, with that obtained from the field equations is given in Fig. 2, and is seen to reproduce the initial oscillatory and plateau behavior quite remarkably. Later, the oscillations drift out of phase, presumably because the classical Boltzmann equation has no mechanism for direct production of pairs through time variations in $E(t)$, as is allowed in the field case. The agreement is made even more striking — shown in Fig. 2 by the dot-dash line — if an enhancement factor for induced boson emission $(2f + 1)$ is inserted to multiply the right-hand side of Eq. (20). Of course, the highly oscillatory numerical results for $1/V\, dN/dk$ bear no resemblance to the smooth distribution $f(p, t)$, but if we smooth the former, say by grouping every 100 momentum points into one bin, we obtain the curve shown in Fig. 1 on the right-hand side, which again shows a remarkable resemblance to the result for $f(p, t)$ given on the right-hand side of Fig. 1 by the dashed line. Thus for purposes of physical

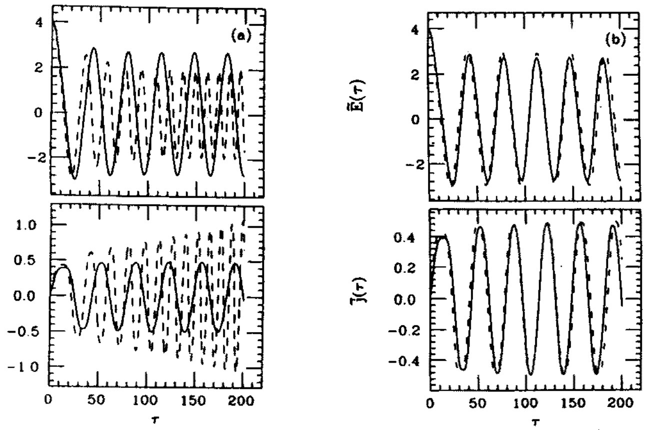

Fig. 4. Results for fermions in $1+1$ dimensions with an initial field $\tilde{E}(0) = 4$. The dashed curve on the left-hand side is for the unblocked Boltzmann equaiton, while the dot-dash curve on the right includes blocking.

interpretation the Boltzmann problem can in a major degree replace the field formulation, very much along the lines of the applications [1, 8, 9] to the quark-gluon plasma.

A representative calculation in three spatial dimensions is shown in Fig. 3 for $\tilde{E}(0) = 7$ and $e^2/m^2 = 4$. It has features very similar to those of the $1+1$ calculations, including very good agreement between the field-theory calculation and the result of the (Bose-enhanced) Boltzmann calculation. This case requires renormalization of course. A consistency loop, based on Eq. (19) and guaranteeing that Eq. (18) is indeed satisfied for all t, is inserted into the coupled equations. The already tedious numerical work of the $1+1$ problem becomes, rather more difficult in this case.

6. Fermions

The main question of importance in turning to the case of fermions is whether there exists any hope for the success of a classical transport-equation description in that situation. We do not discuss here in any detail the technical difficulties that arise in carrying out a similar program for fermions [2]. The main issue that needs to be addressed there has to do with the necessity, in our approach, of having sensible initial conditions that incorporate adiabaticity, since otherwise, as we have seen above, we have no hope of carrying out a sensible renormalization program. To do this, one must use a representation for the Dirac spinors that embodies the symmetry between particles and antiparticles in a convenient manner. Once this is done, results very similar to those for the boson case ensue, including once again the good agreement with the Boltzmann description. A sample case in $1+1$ dimension is shown in Fig. 4, for an initial field strength of 4 units. Perhaps more striking is the result for the particle distribution in this case, shown in Fig. 5. In the upper part of that figure, one sees that the field solution quite properly confines itself to particle occupation between 0 and 1 particle per momentum bin. This quite naturally averages over bins to a distribution that fluctuates very close to the value 1/2, as shown in the lower figures. Of course, the Boltzmann distribution has no knowledge of the Pauli principle and is quite prepared to put 6 fermions in a bin, as seen in the figure at the lower left. But once the Boltzmann formula of Eq. (20) is modified with a Pauli-blocking facotr of $(1-2f)$, as shown on the lower right, a surprising degree of agreement is once again restored.

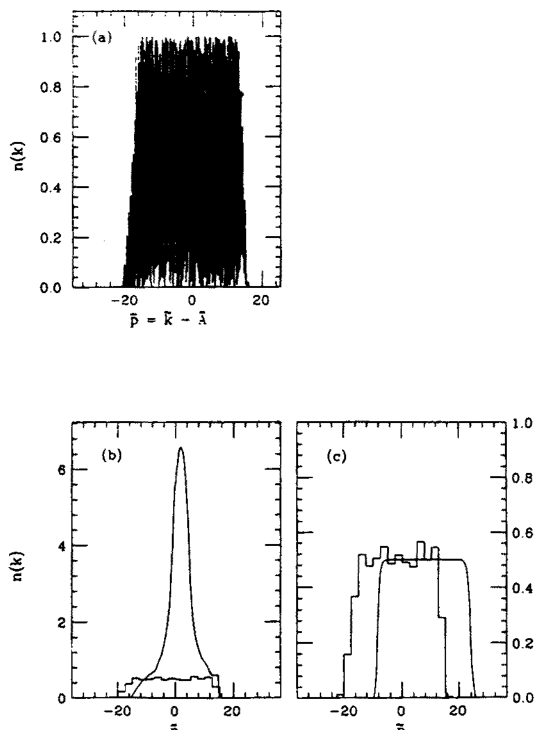

Fig. 5. Particle distributions for the case of the previous figure at time $\tau = 200$. The upper figure is the result of the field-theory calculation. The lower left curve shows this distribution with rebinning to smooth the rapid fluctuations as a function of momentum; it is compared with the unblocked Boltzmann distribution. The lower right shows the smoothed field-theory result in comparison with the Pauli-blocked Boltzmann calculation.

7. Boost-invariant variables

The study of the use of boost-invariant variables for the back-reaction is important because the application for the quark-gluon plasma requires it [1], because it is an opportunity to see what may happen when spatial dependence — however specialized — is introduced, and because the expansion of the system implied by such variables will lead to a much more rapid and much more physical decay of the electric field involved. Again there is a technical difficulty, and again we do not address it here [3, 12]. This time the difficulty has to do with the inevitable singularity arising in such variables at proper time $\tau = 0$. An easy way to deal with this problem is by making the same substitution of variables $\tau = \exp(u)$ that is used [13] in the usual three-dimensional WKBJ approximation, for instance. This sends the singularity at $\tau = 0$ off to $u \to -\infty$, where it is harmless. A sample result for bosons is shown in Fig. 6, and again the transport-equation description is seen to work very well as a stand-in for the full field-theory solution.

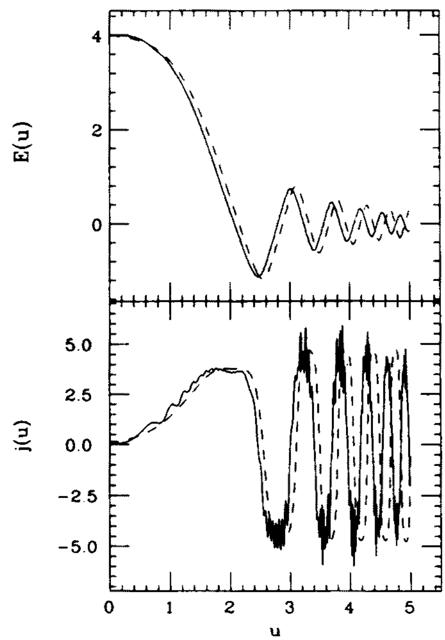

Fig. 6. Results for boost-invariant variables starting with a field strength $\tilde{E}(u = 0) = 4$.

8. Summary and outlook

We believe that the present calculation provides, for the first time, reliable numerical results for the formulation of the back-reaction problem as one of coupled fields in $1+1$ dimensions. It allows the elucidation of the main physical effects of plateaux in the current at early times, and plasma oscillations throughout. It also implies that a modification of the renormalization scheme based on Eq. (19) is required. Last it provides a mapping to an equivalent classical and phenomenological Boltzmann formulation for the problem, which may be of great value for considering the physics of systems where back-reaction is important, but quantum details may not be essential. These results pertain for bosons and for fermions, in cartesian and in boost-invariant variables.

Much has yet to be done before one can feel satisfied with our mastery of even this simplified problem:

1. The question remains as to how to derive a transport equation directly from the field formulation by use of a Wigner representation. In fact this issue becomes rather more mysterious in the light of our results here, since it is well known [14, 15] that transport equations derived in that fashion — and exhibiting pair production — are generally *homogeneous* in the transport function, which makes the appearance of a source term on the right-hand side of Eq. (20) quite surprising. It does seem to be possible, however, to carry out such a program [16], based on a single-time formalism for the Wigner function and a proper identification of the pair-production mechanism.

2. It is, of course, essential for physical applications and for our general understanding of the back-reaction problem to make the extension to a problem with spatial dependence [17], and, in particular, to the case of a finite volume. Work on this is currently underway.

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