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ON THE TWO-LOOP CALCULATION OF THE ANOMALOUS MAGNETIC MOMENT OF THE ELECTRON*

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A new method of calculating two-loop Feynman integrals, proposed by Broadhurst *et al.*, is discussed. As an example of its application the calculation of the anomalous magnetic moment of the electron is carried out. The renormalization procedure is illustrated in detail.

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1. Introduction

Theory and experiment are in excellent agreement as to the value of the electron magnetic moment [1]. This has been achieved on the theory side by computing radiative corrections to the interaction of an electron with a constant magnetic field up to the eighth order in the electromagnetic coupling constant e. These corrections can be expressed as a power series in $\alpha/\pi = e^2/4\pi^2$:

$$\frac{g-2}{2} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + \dots. \quad (1)$$

The first correction, $C_1 = 1/2$, has been computed by Schwinger [2] and the second, C_2, by Sommerfield [3] and Petermann [4]. Ref. [5] contains a detailed description of a calculation of C_2 from which one can get an idea of the enormous effort one has to make in calculating C_2 using traditional methods. Matters get even worse when one proceeds to C_3 [6]. A large

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part of the diagrams in this order has been calculated analytically but there are still 8 which have to be integrated numerically. This is also the only known method to obtain C_4 [1]. Error in the numerical calculation is one of the sources of uncertainty in the theoretical prediction for $g-2$. It is therefore important to look for new algorithms which will enable us to calculate multi-loop corrections analytically.

One such method has been proposed recently by Gray, Broadhurst, Grafe and Schilcher (GBGS) [7]. It is based on the idea of the integration by parts pioneered by Chetyrkin and Tkachov [8] in the context of massless Feynman integrals. The GBGS method can be applied to massive propagator-type integrals. In the previous paper [9] we have discussed how this can be extended to the calculation of the anomalous magnetic moment where we have to deal with vertex diagrams at zero momentum transfer.

The purpose of this paper is twofold. First, we want to give a practical summary of the GBGS method in order to facilitate its applications. Second, we believe that the example of the two-loop calculation of the anomalous magnetic moment illustrates well the renormalization of QED and, once the computational burden is taken care of by a computer, it is of significant pedagogical value.

The essence of the GBGS method is the observation that in the framework of dimensional regularization all integrals needed to compute two-loop on-shell propagator diagrams are of two types: $M(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$ and $N(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$ (this notation will become clear in the next section). Both types can be reduced to 3 known integral structures with the help of a set of recurrence relations. In Sections 2, 3, and 4 we discuss these two classes of integrals, show how the recurrence relations are derived, and present formulas for the three basic integral structures. In section V we show how this apparatus can be applied to the calculation of two-photon corrections to the magnetic moment.

2. Integrals $M(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$

The integral $M(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$ is defined by

$$M(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \pi^{-D}(p^2)^{-D+\sum\alpha_i} \times \iint \frac{d^Dk_1 d^Dk_2}{k_1^{2\alpha_1}(k_1-k_2)^{2\alpha_2}k_2^{2\alpha_3}\left(k_1^2+2p\cdot k_1\right)^{\alpha_4}\left(k_2^2+2p\cdot k_2\right)^{\alpha_5}}, \qquad (2)$$

where $D=4-2\epsilon$. There are 6 recurrence relations for the integrals M which can be derived from the identity:

$$\iint d^Dk_1 d^Dk_2 \frac{\partial}{\partial k^\mu}\left[q^\mu f(k_1,k_2,p,\{\alpha_i\})\right] = 0\,, \qquad (3)$$

where f is the integrand on the RHS of (2), $k \in \{k_1, k_2\}$ and $q \in \{k_1, k_2, p\}$. These relations will be labeled according to Table I. Two more relations can be obtained by expressing integrals:

$$\iint d^D k_1 d^D k_2 k_j^\mu f\left(k_1, k_2, p, \{\alpha_i\}\right), \qquad j = 1, 2, \tag{4}$$

in terms of $M\left(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\right)$ and then differentiating such expressions with respect to p^μ. We will label these relations M_7 and M_8 for j in Eq. (4) equal to 1 and 2 respectively.

TABLE I

Notation used for labeling the recurrence relations

k	q: k_1	q: k_2	q: p
k_1	M_1	M_2	M_3
k_2	M_4	M_5	M_6

Before we derive the recurrence relations, we write down formulas for the derivatives needed in the identity (3). We adopt the operator notation [7]: $1^\pm f\left(\alpha_1, \alpha_2, \cdots\right) \equiv f\left(\alpha_1 + 1, \alpha_2, \cdots\right)$. Using this notation the action of the derivative operator can be expressed as follows:

$$\begin{aligned}
\frac{\partial}{\partial k_1^\mu} \frac{1}{k_1^{2\alpha_1}} &= -2\alpha_1 k_{1\mu} \frac{1}{k_1^{2(\alpha_1+1)}} \equiv -2\alpha_1 k_{1\mu} 1^+ \frac{1}{k_1^{2\alpha_1}}, \\
\frac{\partial}{\partial k_1^\mu} \frac{1}{\left(k_1 - k_2\right)^{2\alpha_2}} &= -2\alpha_2 \left(k_{1\mu} - k_{2\mu}\right) 2^+ \frac{1}{\left(k_1 - k_2\right)^{2\alpha_2}}, \\
\frac{\partial}{\partial k_1^\mu} \frac{1}{\left(k_1^2 + 2p \cdot k_1\right)^{\alpha_4}} &= -2\alpha_4 \left(k_{1\mu} + p_\mu\right) 4^+ \frac{1}{\left(k_1^2 + 2p \cdot k_1\right)^{\alpha_4}}, \\
\frac{\partial}{\partial p^\mu} \frac{1}{\left(k_1^2 + 2p \cdot k_1\right)^{\alpha_4}} &= -2\alpha_4 k_{1\mu} 4^+ \frac{1}{\left(k_1^2 + 2p \cdot k_1\right)^{\alpha_4}}, \\
\frac{\partial}{\partial p^\mu} \frac{1}{\left(k_2^2 + 2p \cdot k_2\right)^{\alpha_5}} &= -2\alpha_5 k_{2\mu} 5^+ \frac{1}{\left(k_2^2 + 2p \cdot k_2\right)^{\alpha_5}}.
\end{aligned} \tag{5}$$

The above formulas contain four-vectors which we will contract with $k_{1\mu}$, $k_{2\mu}$ and p_μ. The resulting scalars in the numerator of the integrand can be cancelled with similar scalars in the denominator, so their appearance is equivalent to the action of following operators:

$$
\begin{aligned}
k_1^2 &\rightsquigarrow \mathbf{1}^- \\
k_2^2 &\rightsquigarrow \mathbf{3}^- \\
2k_1 \cdot p &\rightsquigarrow \mathbf{4}^- - \mathbf{1}^- \\
2k_2 \cdot p &\rightsquigarrow \mathbf{5}^- - \mathbf{3}^- \\
2k_1 \cdot k_2 &\rightsquigarrow \mathbf{1}^- + \mathbf{3}^- - \mathbf{2}^- .
\end{aligned} \tag{6}
$$

In the following we derive relations M_1 and M_7. The relations M_2 and M_3 can be found in the same way, and the remaining ones — by a change of indices $1 \leftrightarrow 3$, $4 \leftrightarrow 5$, which is a consequence of a symmetry of $M(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$.

2.1. Relation M_1

For this relation equation (3) reads:

$$
\begin{aligned}
0 = &\iint d^D k_1 d^D k_2 \frac{\partial}{\partial k_1^\mu} \left[k_1^\mu f(k_1, k_2, p, \{\alpha_i\}) \right] \\
= &D\pi^D (p^2)^{D - \sum \alpha_i} M(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \\
&+ \iint d^D k_1 d^D k_2 k_1^\mu \frac{\partial}{\partial k_1^\mu} f(k_1, k_2, p, \{\alpha_i\}) .
\end{aligned} \tag{7}
$$

The action of $k_1^\mu \partial / \partial k_1^\mu$ is equivalent to

$$
\begin{aligned}
&k_1^\mu \left[-2\alpha_1 k_{1\mu} \mathbf{1}^+ - 2\alpha_2 (k_{1\mu} - k_{2\mu}) \mathbf{2}^+ - 2\alpha_4 (k_{1\mu} + p_\mu) \mathbf{4}^+ \right] \\
&\rightsquigarrow -2\alpha_1 \mathbf{1}^+ \mathbf{1}^- - \alpha_2 \mathbf{2}^+ \left(2\mathbf{1}^- - \mathbf{1}^- - \mathbf{3}^- + \mathbf{2}^- \right) - \alpha_4 \mathbf{4}^+ \left(2\mathbf{1}^- + \mathbf{4}^- - \mathbf{1}^- \right) .
\end{aligned} \tag{8}
$$

Thus we obtain:

$$
\begin{aligned}
&\left[D - 2\alpha_1 - \alpha_2 - \alpha_4 - \alpha_2 \mathbf{2}^+ \left(\mathbf{1}^- - \mathbf{3}^- \right) \right. \\
&\qquad \left. - \alpha_4 \mathbf{4}^+ \mathbf{1}^- \right] M(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = 0 .
\end{aligned} \tag{9}
$$

2.2. Relation M_7

This time we begin with

$$
\begin{aligned}
&\iint d^D k_1 d^D k_2 k_1^\mu f(k_1, k_2, p, \{\alpha_i\}) \\
&\qquad = p^\mu \pi^D (p^2)^{D - \sum \alpha_i} \mathcal{O} M(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) ,
\end{aligned} \tag{10}
$$

where $\mathcal{O}$ is an operator which we determine by contracting both sides of (10) with $2p_\mu$:

$$\mathcal{O} = \tfrac{1}{2}\left(\mathbf{4}^- - \mathbf{1}^-\right) . \tag{11}$$

Now we differentiate (10) with respect to p^μ and get:

$$\begin{aligned} &k_1^\mu\left(-2k_{1\mu}\alpha_4\mathbf{4}^+ - 2k_{2\mu}\alpha_5\mathbf{5}^+\right) M\left(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\right) \\ &\quad = \tfrac{1}{2}\left(D + 2D - 2\sum\alpha_i\right)\left(\mathbf{4}^- - \mathbf{1}^-\right) M\left(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\right) \end{aligned} \tag{12}$$

or

$$\begin{aligned} \Big[&-2\alpha_4\mathbf{4}^+\mathbf{1}^- - \alpha_5\mathbf{5}^+\left(\mathbf{1}^- + \mathbf{3}^- - \mathbf{2}^-\right) \\ &\quad - \frac{3D - 2\sum\alpha_i}{2}\left(\mathbf{4}^- - \mathbf{1}^-\right)\Big] M\left(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\right) = 0 . \end{aligned} \tag{13}$$

3. Integrals $N(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5)$ and a summary of recurrence relations

Another class of integrals which we have to consider is defined by

$$\begin{aligned} N\left(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\right) &= \pi^{-D}(p^2)^{-D+\sum\alpha_i} \iint \frac{d^Dk_1 d^Dk_2}{k_1^{2\alpha_1}k_2^{2\alpha_2}} \\ &\times \frac{1}{\left(k_1^2 + 2p\cdot k_1\right)^{\alpha_3}\left(k_2^2 + 2p\cdot k_2\right)^{\alpha_4}\left((k_1+k_2)^2 + 2p\cdot(k_1+k_2)\right)^{\alpha_5}} . \end{aligned}$$

In a manner similar to the one described for integrals $M\left(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\right)$ we can derive eight recurrence relations $N_{1,\ldots,8}$.

It turns out that it is convenient to use certain linear combinations of the recurrence relations. Below we summarize these combinations which in our opinion are the most useful:

$$\begin{aligned} N_1 :\ &\left[D - 2\alpha_1 - \alpha_3 - \alpha_5 - \alpha_3\mathbf{3}^+\mathbf{1}^- \right. \\ &\left. +\alpha_5\mathbf{5}^+\left(\mathbf{4}^- - \mathbf{1}^-\right)\right] N = 0 , \\ N_1 + N_2 + N_3 :\ &\left[D - \alpha_1 - \alpha_3 - 2\alpha_5 + \alpha_1\mathbf{1}^+\left(\mathbf{4}^- - \mathbf{5}^-\right)\right. \\ &\left. +\alpha_3\mathbf{3}^+\left(\mathbf{2}^- - \mathbf{5}^- - 2\right) - 2\alpha_5\mathbf{5}^+\right] N = 0 , \\ N_1 + N_3 + N_5 :\ &\left[2D - \alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - 2\alpha_5 - \alpha_1\mathbf{1}^+\mathbf{3}^- \right. \\ &\left. - \alpha_4\mathbf{4}^+\mathbf{2}^- - 2\alpha_3\mathbf{3}^+ - 2\alpha_5\mathbf{5}^+\right] N = 0 , \end{aligned}$$

$$
\begin{aligned}
N_4 + N_5 + N_6 : & \left[D - \alpha_2 - \alpha_4 - 2\alpha_5 + \alpha_2 2^+ \left(3^- - 5^-\right)\right. \\
& \left. + \alpha_4 4^+ \left(1^- - 5^- - 2\right) - 2\alpha_5 5^+ \right] N = 0\,, \\
N_5 : & \left[D - 2\alpha_2 - \alpha_4 - \alpha_5 - \alpha_4 4^+ 2^-\right. \\
& \left. + \alpha_5 5^+ \left(3^- - 2^-\right)\right] N = 0\,, \\
N_1 + N_5 + N_6 : & \left[2D - 2\alpha_1 - \alpha_2 - \alpha_3 - 2\alpha_4 - 2\alpha_5\right. \\
& \left. - \alpha_2 2^+ 4^- - \alpha_3 3^+ 1^- - 2\alpha_4 4^+ - 2\alpha_5 5^+ \right] N = 0\,, \\
N_7 - 2N_1 : & \left[- 2D + 4\alpha_1 + 2\alpha_3 + \alpha_4 + \alpha_5 - \alpha_4 4^+ \left(5^- - 3^-\right)\right. \\
& \left. + \alpha_5 5^+ \left(3^- - 4^-\right) \frac{3D - 2\sum \alpha_i}{2} \left(3^- - 1^-\right) \right] N = 0\,, \\
N_8 - 2N_5 : & \left[- 2D + 4\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 - \alpha_3 3^+ \left(5^- - 4^-\right)\right. \\
& \left. + \alpha_5 5^+ \left(4^- - 3^-\right) - \frac{3D - 2\sum \alpha_i}{2} \left(4^- - 2^-\right) \right] N = 0\,,
\end{aligned}
$$

$$
\begin{aligned}
M_1 : & \left[D - 2\alpha_1 - \alpha_2 - \alpha_4 + \alpha_2 2^+ \left(3^- - 1^-\right)\right. \\
& \left. - \alpha_4 4^+ 1^-\right] M = 0\,, \\
M_2 - M_1 : & \left[-D + \alpha_1 + 2\alpha_2 + \alpha_4 + \alpha_1 1^+ (2^- - 3^-)\right. \\
& \left. + \alpha_4 4^+ (-5^- + 2^-)\right] M = 0\,, \\
M_1 + M_3 : & \left[D - \alpha_1 - \alpha_2 - 2\alpha_4 - \alpha_1 1^+ 4^-\right. \\
& \left. + \alpha_2 2^+ (5^- - 4^-) - 2\alpha_4 4^+\right] M = 0\,, \\
M_5 + M_6 : & \left[D - \alpha_2 - \alpha_3 - 2\alpha_5 + \alpha_2 2^+ (4^- - 5^-)\right. \\
& \left. - \alpha_3 5^- 3^+ - 2\alpha_5 5^+\right] M = 0\,, \\
M_4 - M_5 : & \left[-D + 2\alpha_2 + \alpha_3 + \alpha_5 + \alpha_3 3^+ (-1^- + 2^-)\right. \\
& \left. + \alpha_5 5^+ (2^- - 4^-)\right] M = 0\,, \\
M_1 + M_5 : & \left[2D - 2\alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4 - \alpha_5 - \alpha_4 4^+ 1^-\right. \\
& \left. - \alpha_5 3^- 5^+\right] M = 0\,, \\
M_7 : & \left[- 2\alpha_4 4^+ 1^- - \alpha_5 5^+ \left(1^- + 3^- - 2^-\right)\right. \\
& \left. - \frac{3D - 2\sum \alpha_i}{2} \left(4^- - 1^-\right) \right] M = 0\,, \\
M_8 : & \left[- 2\alpha_5 5^+ 3^- - \alpha_4 4^+ \left(3^- + 1^- - 2^-\right)\right. \\
& \left. - \frac{3D - 2\sum \alpha_i}{2} \left(5^- - 3^-\right) \right] M = 0\,.
\end{aligned}
$$

4. Analytic formulas for some of the integrals

It turns out that the recurrence relations described so far are sufficient to reduce all the integrals of the type (2) and (14) to products of one-loop integrals and three types of two-loop integrals for which we have closed formulas:

$$
\begin{aligned}
M(0,\alpha,0,\beta,\gamma) &= (-1)^{1+\alpha+\beta+\gamma} \\
&\quad\times \frac{\Gamma(\alpha+\beta+\gamma-D)\,\Gamma\left(\frac{D}{2}-\alpha\right)\Gamma\left(\alpha+\beta-\frac{D}{2}\right)\Gamma\left(\alpha+\gamma-\frac{D}{2}\right)}{\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma\left(\frac{D}{2}\right)\Gamma(2\alpha+\beta+\gamma-D)}, \\
M(\alpha,\beta,\gamma,\delta,0) &= (-1)^{1+\alpha+\beta+\gamma+\delta}\frac{\Gamma\left(\frac{D}{2}-\beta\right)\Gamma\left(\frac{D}{2}-\gamma\right)\Gamma\left(\beta+\gamma-\frac{D}{2}\right)}{\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\delta)} \\
&\quad\times \frac{\Gamma(2D-2\alpha-2\beta-2\gamma-\delta)\,\Gamma(\alpha+\beta+\gamma+\delta-D)}{\Gamma(D-\beta-\gamma)\,\Gamma\left(\frac{3D}{2}-\alpha-\beta-\gamma-\delta\right)}, \\
N(1,1,1,1,1) &\equiv I(\epsilon),
\end{aligned}
\tag{14}
$$

where in the last formula we will only need the value of $I(0)$ at $D=4$ dimensions, for which this integral has been calculated by Broadhurst [10].

For completeness we also give a formula for a single-loop integral:

$$
\begin{aligned}
S(\alpha,\beta) &\equiv \int \frac{d^D k}{k^{2\alpha}\left(k^2+2k\cdot p\right)^{\beta}} \\
&= i\pi^{\frac{D}{2}}(-1)^{\alpha+\beta}(p^2)^{\frac{D}{2}-\alpha-\beta}\frac{\Gamma\left(\alpha+\beta-\frac{D}{2}\right)\Gamma(D-2\alpha-\beta)}{\Gamma(\beta)\,\Gamma(D-\beta-\alpha)}
\end{aligned}
\tag{15}
$$

5. Calculation of the anomalous magnetic moment

To fourth order in the coupling constant e the contribution to the anomalous magnetic moment of the electron comes from diagrams shown in Fig. 1. Diagrams (1c-f) should be accompanied by their mirror reflections, not shown in the figure. We will not consider here the effect of the vacuum polarization effect on the photon propagator which we have discussed in a previous paper [9].

The GBGS method has been designed to compute propagator type diagrams. In order to use it for the anomalous magnetic moment we note

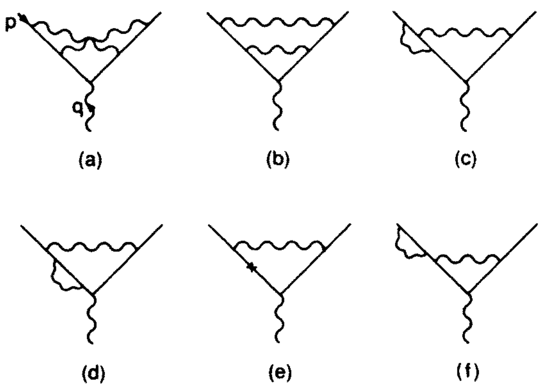

Fig. 1. Feynman diagrams for the two-loop correction to the magnetic moment of the electron

that we only need values of diagrams in Fig. 1 for an infinitesimally small external momentum q, for which we can expand the propagator:

$$\frac{1}{\not p + \not q + \not k_1 - m} \approx \frac{\not p + \not q + \not k_1 + m}{k_1^2 + 2k_1 \cdot p} - 2k_1 \cdot q \frac{\not p + \not k_1 + m}{(k_1^2 + 2k_1 \cdot p)^2} . \qquad (16)$$

so that all the denominators depend only on one external momentum p. Since the electron is on-shell we can express all the integrals in terms of $N-$ and $M-$functions discussed in the previous section. We are now going to discuss the treatment of divergences in the loop integrations convenient for this calculation.

Diagrams of the types (1b-d) contain infinite one-loop corrections to the propagator and vertex functions. In order to get a sensible finite result we carry out the renormalization procedure [12]. In the case of the vertex this consists in subtracting from it its value with electron legs taken on-shell and momentum transfer equal to zero. For example, the unrenormalized amplitude corresponding to the diagram (1c) is, in the Feynman gauge (for simplicity we drop the four-spinors representing the electron):

$$M_c = e^3 \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} \gamma^\alpha \frac{1}{\not p + \not k + \not q - m} \gamma^\mu \frac{1}{\not p + \not k - m} \Lambda_\alpha (p, p + k; k) \ , \qquad (17)$$

where the unrenormalized vertex function is defined by:

$$\Lambda_\alpha (p, p + k_1; k_1) = e^2 \int \frac{d^D k_2}{(2\pi)^D} \frac{1}{k_2^2} \gamma^\beta \frac{1}{\not p + \not k_1 + \not k_2 - m} \gamma_\alpha \frac{1}{\not p + \not k_2 - m} \gamma_\beta , \qquad (18)$$

and the renormalized one is obtained by making the following subtraction:

$$\Lambda_\alpha^R (p, p + k; k) = \Lambda_\alpha (p, p + k; k) - \Lambda_\alpha (p_0, p_0; 0) \ , \qquad (19)$$

with $p_0^2 = m^2$.

Similarly, the renormalized propagator is defined by:

$$\Sigma^{\mathrm{R}}(p) = \Sigma(p) - \Sigma(p_0) - (\not p - m)\left.\frac{\partial\,\Sigma(p)}{\partial\not p}\right|_{p=p_0}. \tag{20}$$

Thanks to the Ward–Takahashi identity:

$$\Lambda_\alpha\,(p_0,p_0;0) = -\left.\frac{\partial\,\Sigma(p)}{\partial p^\alpha}\right|_{p=p_0}, \tag{21}$$

part of the infinities cancels after adding diagrams (1c,d) and their mirror counterparts to the diagram (1b). What remains are the mass counterterms $\Sigma(p_0)$, represented by (1e), and half of the divergences of the vertex functions (1c), for which the subtraction is depicted by (1f). We now explain briefly how these two diagrams are conveniently evaluated.

The mass counterterm is

$$\begin{aligned}\Sigma(p_0) &= e^2 m^{1-2\epsilon}\frac{3-2\epsilon}{1-2\epsilon}\frac{\Gamma(\epsilon)}{(4\pi)^{\frac{D}{2}}}\\ &\equiv e^2 m\frac{3-2\epsilon}{1-2\epsilon}\int\frac{d^Dk}{(2\pi)^D i}\,\frac{1}{\left(k^2+2k\cdot p\right)^2}\end{aligned} \tag{22}$$

which corresponds to a tadpole diagram inserted in place of the cross in the figure (1e), and this is just a product of one-loop integrals (15). The contribution symbolized by (1f) is a product of the wave-function renormalization constant:

$$\left.\frac{\partial\,\Sigma(p)}{\partial\not p}\right|_{p=p_0} = -e^2\frac{\Gamma(1+\epsilon)}{(4\pi)^{\frac{D}{2}}}\left(\frac{3}{\epsilon}+4\right) \tag{23}$$

and the one-loop correction to the magnetic moment, which in $(4-2\epsilon)$-dimensions is:

$$\frac{\alpha}{\pi}C_1\,(\epsilon) = \frac{\alpha}{\pi}\frac{\pi^\epsilon}{2^{1-2\epsilon}}\Gamma(1+\epsilon)\frac{1+2\epsilon}{1-2\epsilon}. \tag{24}$$

After this discussion it is clear how diagrams (1a-e) are expressed in the form of integrals (2) and (14). To perform this calculation we have used symbolic manipulation programs. FORM [13] was used to carry out the Dirac algebra, and the integrals were calculated using Mathematica [14].

In Table II we have summarized the results of the calculation for each diagram in Fig. 1 (together with their mirror counterparts, where applicable), in the limit $\epsilon \to 0$. We have used the value of the integral $N(1,1,1,1,1) \equiv I(0) = \pi^2 \ln 2 - \frac{3}{2}\zeta(3)$, computed in [10]. For the sake of completeness we

TABLE II
Contributions of diagrams in Fig. 1 to the anomalous magnetic moment of the electron.

Diagram	Coefficient of $\left(\frac{\alpha}{\pi}\right)^2$
1a	$\frac{1}{6} - \frac{5}{6}I(0) + \frac{13}{36}\pi^2$
1b	$\frac{3}{8\epsilon} - \frac{3}{4}\gamma + \frac{107}{48} + \frac{1}{18}\pi^2$
1c	$\frac{1}{4\epsilon} - \frac{1}{2}\gamma - \frac{19}{24} + \frac{1}{3}I(0) + \frac{1}{18}\pi^2$
1d	$\frac{1}{2\epsilon} - \gamma + \frac{5}{24} - \frac{1}{18}\pi^2$
1e	$-\frac{3}{4\epsilon} + \frac{3}{2}\gamma - \frac{7}{4}$
1f	$-\frac{3}{8\epsilon} + \frac{3}{4}\gamma - 2$
Vac. pol.	$\frac{119}{36} - \frac{1}{3}\pi^2$
Total	$\frac{197}{144} + \frac{1}{12}\pi^2 + \frac{3}{4}\zeta(3) - \frac{1}{2}\pi^2 \ln 2$

have added in Table II the contribution from the vacuum polarization diagram calculated in [9]. The total correction is the same as the one computed by Sommerfield [3] and Petermann [4].

It can now be seen that the GBGS method greatly simplifies the calculation of the two-loop correction to the magnetic moment of the electron. In principle it can also be extended to calculations of three-loop diagrams and first steps in this direction have been taken [11]. Calculation of two-loop corrections to the decay of a heavy quark can become another important application. This is worth studying both in the effective field theory approach [15] and in the exact QCD in the limit of the very large quark mass.

Note added: After this work had been completed we learned about Ref. [16], in which two loop contribution to the anomalous magnetic moment of the electron is evaluated using the FORM program SHELL2.

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