THE INFRARED LIMIT OF QCD EFFECTIVE STRING*

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Some model-independent properties of the effective string of gauge field systems in the confining phase, for very large quark separations, are described in terms of two-dimensional conformal field theories. The constraints induced by the gauge theory of the boundaries of the effective string induce a Coulomb-like term in the interquark potential which is universal, but different from the one proposed by Lüscher. Some universal relations among the string tension, the thickness of the colour flux tube, the location of the deconfining temperature and the mass of the lowest glueball state are discussed.

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The possibility of describing the long-distance dynamics of gauge theories in the confining phase by an effective string theory is a fascinating, twenty years old conjecture [1], which is resisting against numerous attempts to prove (or disprove) it [2].

It is based on the very intuitive assumption that the colour flux connecting a pair of distant quarks is concentrated, in the confining phase, inside a thin flux tube, which then generates the linear rising of the confining potential. Actually this flux tube has been recently observed in lattice simulations [3, 4].

According to the common lore, this thin flux tube should behave, when the quarks are pulled very far apart, as a free vibrating string. This is also supported by the strong coupling expansion of the lattice gauge theories, which can be formulated as a sum of weighted random surfaces with quark lines as boundary.

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Unfortunately, the action describing this effective string in the continuum limit is substantially unknown. The simplest assumption is that this action is described by the Nambu–Goto string [5] which is the conformal gauge can be described in terms of $D-2$ free bosonic fields associated to the transverse displacements of the string.

There are, however, two kinds of difficulties in applying the bosonic string outside the critical space-time dimension of 26.

The first is that, depending on the quantization method, one finds either the breaking of the Lorentz invariance or appearance of the conformal Liouville mode, destroying the simple free field description of the string. If one insists in formulating the effective string theory only in terms of transverse modes, one can replace the Liouville field with a non-polynomial interaction term [6] which looks rather cumbersome.

The other difficulty for the Nambu–Goto string is its instability against the formation of pinches for space-time dimension $D > 2$. As a consequence, the string world sheet takes the shape of a branched polymer, which is very different from the behaviour of the colour flux tube. In order to control this crumpling transition it has been considered another kind of string action in which a new scale invariant, non-conformal interaction term proportional to the square of the extrinsic curvature of the world sheet is added to the Nambu–Goto term [7, 8]. This string, sometimes called rigid string, favors more realistic smooth configurations of the flux tube at short distances.

Note that these modifications, if on one hand transform the Nambu–Goto action into a consistent theory, on the other hand make it very difficult to evaluate, even approximately, physical observables, which should be the final goal of the effective string picture of the gauge theories.

Luckily, these interaction terms modify the string theory only at short distance: indeed it has been shown that the Lorentz invariance is asymptotically restored [9] at large distance and that the rigid string does not modify [10] the infrared behaviour of the interquark potential predicted by the Nambu–Goto action. Thus one is lead to conclude that the effective string is asymptotically described [5] by a two-dimensional conformal field theory formed by $D-2$ massless free bosonic fields[1].

We shall see in this lecture that this description is too drastic an approximation because there are constraints dictated by the gauge system [11] which cannot be obeyed by the free bosonic string. There is indeed a simple modification of this picture, consisting in a suitable compactification of the bosonic fields, which fulfills the constraints and fit well the numerical simulations of the gauge systems in three and four space-time dimensions and with various gauge groups. We shall see also that this new asymptotic form

[1] This is the infrared limit of the Nambu–Goto action called in the following the free bosonic string.

of the effective string accounts for the observed finite thickness of the colour flux tube, gives a good lower bound to the glueball masses and, finally, suggests a universal value for the transition temperature to the quark–gluon plasma.

We start with the rather general assumption that the infrared limit of the effective string is described by a two-dimensional conformal field theory (CFT). Then, the vacuum expectation values of gauge invariant quantities involving large loops are expressible as the partition function of this CFT on a Riemann surface with these loops as boundaries. In particular, in the study of the interquark potential two kinds of loops are considered: the rectangular Wilson loop and the Polyakov loop. The rectangular Wilson loop is expressed in terms of the contour integral of the gauge field $A_\mu(x)$ along a rectangle ρ of sides L and R as follows

$$\langle W(R,L)\rangle = \left\langle \mathrm{tr}\ \mathcal{P}\exp\left(\oint_\rho igA\cdot dl\right)\right\rangle , \tag{1}$$

where $\mathcal{P}$ is the path-ordering and g is the gauge coupling constant.

The Polyakov loop $P(x)$ can be defined in a gauge system at finite temperature or, equivalently, confined in a box with periodic boundary conditions. Then $P(x)$ is given again as the trace of the path-ordered exponential of the contour integral of A_μ along a line parallel to the periodic direction and crossing the point x.

According to our assumption on the asymptotic form of the effective string theory, we can write

$$\langle W(R,L)\rangle = \mathrm{e}^{-F(R,L)} , \tag{2}$$

where $F(R,L)$ is the free energy of a suitable conformal theory defined on a rectangle. Similarly, in a D-dimensional gauge system at finite temperature $T = 1/L$ or, equivalently, on a box of size $L\times\infty^{D-1}$, the correlation function of two Polyakov lines $P(x)$ parallel to the periodic time axis at a distance R is given by

$$\langle P(x)P^\dagger(x+R)\rangle = Z_{\mathrm{cyl}} = \mathrm{tr}\ \mathrm{e}^{-LH} = \mathrm{e}^{-F_{\mathrm{cyl}}(R,L)} , \tag{3}$$

where H and $F_{\mathrm{cyl}}(R,L)$ are now the Hamiltonian and the free energy of the same conformal theory on the cylinder of height R, bounded by the two Polyakov lines P and $P^\dagger$ of length L, which represent the world lines of a quark and antiquark, respectively.

In conformal field theory a central role is played by the conformal anomaly c, which measures the response of the dynamical system to curving of the surface and also controls the finite size scaling through a sort of Casimir effect [12]. It can be simply derived by the transformation law of the holomorphic component of the energy momentum tensor $T(z)$ under the conformal transformation $z \to w(z)$

$$T(z) = w'^2 T(w) + \frac{c}{12}\{w, z\}, \tag{4}$$

where the Schwarzian derivative $\{w, z\}$ is given by

$$\{w, z\} = \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2. \tag{5}$$

A physical state $\mid \psi\rangle$ of CFT is characterized by the Virasoro constraints

$$L_n \mid \psi\rangle = 0, \quad n > 0, \qquad L_0 \mid \psi\rangle = h \mid \psi\rangle,$$
$$L_n = \frac{1}{2i\pi}\oint \frac{dz}{z} z^{n+2} T(z), \tag{6}$$

where h is the conformal weight of $\mid \psi\rangle$. Consider now an infinite strip of width R parametrized by $0 \leq \operatorname{Im} z \equiv y \leq R$. It can be considered as the limit $L \to \infty$ of the cylindric world-sheet of Eq. (3), where

$$H = \frac{1}{2\pi}\int_0^R T_{00} dy = \frac{1}{2\pi}\int_0^R (T(z) + \bar{T}(\bar{z}))dy. \tag{7}$$

The transformation $z \to w = \exp\left(\frac{\pi z}{R}\right)$ maps conformally the infinite strip into the upper half plane $\operatorname{Im} w \geq 0$ with $\{w, z\} = -\frac{\pi^2}{2R^2}$ and $\bar{T}(\bar{w})$ may be taken as the analytic continuation of $T(w)$ in the lower-half plane [15]. Then H becomes

$$H = \frac{1}{2i\pi}\oint \frac{dw}{w}\,\frac{\pi}{R}\left(w^2 T(w) - \frac{c}{24}\right) = \frac{\pi}{R}\left(L_0 - \frac{c}{24}\right). \tag{8}$$

The spectrum of the physical states which can propagate along the strip depends on the boundary conditions (BC) on either side of the strip. Labelling this pair of conditions with α and β, and with $h_{\alpha\beta}$ the conformal weight of the lowest physical state $\mid \psi_{\alpha\beta}\rangle$, we have obviously

$$H \mid \psi_{\alpha\beta}\rangle = \frac{\pi}{R}\left(h_{\alpha\beta} - \frac{c}{24}\right) \mid \psi_{\alpha\beta}\rangle. \tag{9}$$

In the limit $L \to \infty$ only this state contributes to the free energy. Then we have

$$V(R) = \lim_{L\to\infty} \frac{F_{\text{cyl}}(R, L)}{L} = \sigma R + k_{\alpha\beta} - \left(\frac{c}{24} - h_{\alpha\beta}\right)\frac{\pi}{R} + O\left(\frac{1}{R^2}\right), \tag{10}$$

where $V(R)$ is the interquark potential, the first term σR is the bulk contribution which defines the string tension σ and $k_{\alpha\beta}$ is a non-universal constant.

Similarly, if we take the other limit $R \to \infty$ keeping L fixed in Eq. (3), we get the free energy of an infinitely long cylinder having the asymptotic expansion [12]

$$\frac{F_{\rm cyl}(R,L)}{R} = \sigma L - \frac{\tilde{c}}{6}\frac{\pi}{L} + O\Big(\frac{1}{L^2}\Big)\,, \tag{11}$$

where the combination $\tilde{c} = c - 24h$ is known as the effective conformal anomaly [13] and h is the lowest conformal weight of the states (closed string states) which can propagate along the cylinder. If the theory is unitary, the lowest state is the vacuum with $h = 0$ [2].

In the case of the strip (open string) the analysis of the conformal spectrum is more delicate, because it depends on the choice of α and β in an essential way. In particular, if we take the same boundary conditions on either side of the strip, it is possible to show, as we shall see shortly, that the lowest propagating state is the vacuum, *i.e.* $h_{\alpha\alpha} = 0$. Later we shall argue that in the asymptotic effective string one should have $h_{\alpha\beta} > 0$, , which implies $\alpha \neq \beta$. On the contrary, within the Nambu–Goto action one gets $h_{\alpha\beta} = 0$ [5].

The universal $1/R$ term of Eqs (10) and (11) , generated by the non-homogeneous part of the transformation law of $T(z)$, may also be viewed as a two-dimensional analog of the Casimir effect, *i.e.* a universal contribution to the zero-point energy due to the finite size of the dynamical system. Indeed Eq. (9) tells us that the zero-point energy of the CFT on the strip is given by

$$E_o = -\frac{\pi(c - 24h_{\alpha\beta})}{24R}\,. \tag{12}$$

In a free field theory there is another simple, instructive way to evaluate E_0. For instance, in a free bosonic string of length R with fixed boundary conditions, the physical degrees of freedom are the $D-2$ transverse normal modes of vibration described by $D-2$ families of free harmonic oscillators of frequency $\omega_n = \frac{\pi}{R}n$. Then E_0 is the sum of the zero-point energy $\frac{1}{2}\hbar\omega_n$ of these oscillators:

$$E_o = (D-2)\frac{\hbar}{2}\sum_{n=1}^{\infty}{}' \omega_n = (D-2)\frac{\hbar\pi}{2R}\sum_{n=1}^{\infty}{}' n\,, \tag{13}$$

[2] The term $-\pi\tilde{c}/6L$ can be also viewed as a universal correction of the string tension due to the finite temperature $T = 1/L$, *i.e.* [14] $\sigma(T) = \sigma - \pi\tilde{c}T^2/6 + O(T^3)$.

where the apex indicates that the divergent sum has been regularized in some way. In most cases regularization introduces a cut-off in the theory; here it is possible to evaluate unambiguously Eq. (13) without using any cut-off procedure. We only assume that there is a regularization $f(a) = \sum'(n+a)$ which shares with the infinite sum $\sum(n+a)$ some of its formal properties, namely that the (regularized) sum from 1 to ∞ is equal to the finite sum from 1 to a plus the (regularized) sum from $a+1$ to ∞ and that the (regularized) sum of the integers is equal to the sum of the even integers plus the sum of the odd integers.

The first condition yields

$$\sum_{n=1}^{\infty}{}' n = \sum_{n=1}^{a} + \sum_{n=a+1}^{\infty}{}' = \frac{a(a+1)}{2} + \sum_{n=1}^{\infty}{}' (n+a) , \tag{14}$$

then

$$\sum_{n=1}^{\infty}{}' (n+a) = \sum_{n=1}^{\infty}{}' n - \frac{a(a+1)}{2} . \tag{15}$$

The second and last condition gives

$$\sum_{n=1}^{\infty}{}' n = 2\sum_{n=1}^{\infty}{}' n + 2\sum_{n=1}^{\infty}{}' (n - \frac{1}{2}) = 4\sum_{n=1}^{\infty}{}' n + \frac{1}{4} ,$$

where it has been assumed that Eq.(15) is true also for non integer a. As a result we finally get [3]

$$\sum_{n=1}^{\infty}{}' (n+a) = -\frac{1}{12} - \frac{a(a+1)}{2} , \tag{17}$$

which coincides with the value obtained with the ζ function method [16, 17] and other regularizations [18, 5].

Inserting Eq.(17) in Eq.(13) we have

$$E_o = -(D-2)\hbar \frac{\pi}{24R} . \tag{18}$$

Owing the fixed boundary conditions on either side of the strip, we put $h_{\alpha\alpha} = 0$ in Eq.(12), then

$$c = D - 2 , \tag{19}$$

[3] Similarly one may derive, in the same way, $\sum' n^{2k} = 0$ and $\sum' n^{2k-1} = (-1)^k \frac{B_k}{2k}$, where the B_k are the Bernoulli numbers. These formulas are useful to evaluate the Casimir effect (or, equivalently, the Stefan–Boltzmann law) in any space dimension.

which states the well known fact that each free boson contributes with $c = 1$ to the conformal anomaly.

It is worthwhile to note that the parameter a in Eq.(17) is determined by the choice of α and β on the two sides of the strip. For instance, one might take for α fixed (or Dirichlet) BC and for β free (or Neumann) BC; this yields $a = -\frac{1}{2}$ and then Eqs (12) and (17) together tell us that the lowest physical state propagating along the strip has conformal weight $h_{\alpha\beta} = \frac{1}{16}$. This is just an example of a very general principle of string theory and CFT, stating that [15] there is an isomorphism

$$\alpha \leftrightarrow |\psi_\alpha\rangle \tag{20}$$

between the conformally invariant boundary conditions and the physical states [4] of the theory: for instance, the free ends of an open string can consistently interact with a background field only if this latter describes a physical string state [25]. Another illuminating example is the critical Ising model [15]; here there are only three boundary conditions which are invariant with respect to the renormalization group; namely, the spins on the boundary can be chosen all up, or all down, or at random (free BC), and there are only three physical states on the spectrum of the $c = \frac{1}{2}$ CFT which describes the $2D$ Ising model at criticality.

The isomorphism mentioned above preserves the fusion algebra of the CFT, in the sense that the physical states which can propagate along the strip with boundary conditions $\alpha \leftrightarrow |\ \psi_\alpha\rangle$ and $\beta \leftrightarrow |\ \psi_\beta\rangle$ are of the form [15] $N^i_{\alpha\bar\beta}\ |\ \psi\rangle_i$, where the integers N^i_{jk} define the fusion algebra $\phi_j\phi_k = N^i_{jk}\phi_i$ and $\bar\beta$ denotes the representation conjugate to β. Using this piece of information we can now write the general form of the partition function on the cylinder defined in Eq.(3)

$$Z_{cyl}(\tau) = \mathrm{e}^{-\sigma RL} N^i_{\alpha\beta}\chi_i(\tau)\,, \tag{21}$$

where $\tau = \frac{i2R}{L}$ and χ_i is the Virasoro character of the representation i propagating along the periodic direction.

Now we come back to the problem of finding the asymptotic string picture of the gauge system. From the foregoing discussion, we see that the effective conformal theory of a gauge system is completely specified not only

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[4] In CFT one works in general in terms of primary fields $\phi(z)$ (*i.e.* BRST-covariant vertex operators in the language of string theory) which are associated to the irreducible representations of the Virasoro algebra. The physical state $|\ \psi\rangle$ associated to $\phi(z)$ is created out of the vacuum $|\ 0\rangle$ simply by $|\ \psi\rangle = \phi(0)\ |\ 0\rangle$.

by the conformal anomaly c and by the spectrum of the physical states, but also by a specific choice of the boundary conditions α and β on either side of the strip. Gauge theory poses two important constraints on the latter two.

Suppose deforming the strip in such a way that the two sides overlap: if the two boundary conditions α, β were compatible, we should get the conformal theory on a cylinder. On the other hand, from the point of view of the gauge theory, when a quark line overlaps an antiquark one, the free energy vanishes identically; this is possible only if α and β are incompatible, i.e. the set of states obeying both α and β on the same side is empty. We can represent this situation symbolically by

$$\alpha \cap \beta = \emptyset\,, \tag{22}$$

which implies, in particular

$$h_{\alpha\beta} > 0\,, \tag{23}$$

because the ground state cannot propagate if $\alpha \neq \beta$ [15].

The other constraint comes simply from the fact that if we perform the symmetry transformation $A_\mu(x) \to -A_\mu(x)$ in the gauge system, the quark and the antiquark sources are exchanged: $q \leftrightarrow \bar{q}$, so there must exist a Z_2 automorphism of the CFT which transposes α and β

$$Z_2 : \alpha \leftrightarrow \beta \qquad F(L,R) \leftrightarrow F(L,R)\,. \tag{24}$$

In order to have an effective string which behaves like a colour flux tube, it is necessary to supplement the theory with some conserved quantum number, which makes it possible to distinguish the two ends of the string, as Eq. (22) and (24) require. One possibility, suggested by the superstring, is to introduce new degrees of freedom, besides the transverse displacements, which however modify the conformal anomaly (19) and are difficult to justify on general grounds.

There is a simpler, more convincing way to modify the bosonic string without changing c, based on the observation that the CFT at fixed $c \geq 1$ are not isolated but depend on a set of parameters (called the moduli of the CFT). For instance, most of the $c = 1$ theories are described [19] by a free boson compactified on a circle of radius r. The free bosonic string corresponds to $r \to \infty$. The spectrum of physical states is a function of r and it is possible to fix r in such a way that the BC are fulfilled.

Note also that the real flux tube connecting a pair of quarks has a small, but not vanishing, thickness. It could be described classically by a solution of the type of the Nielsen–Olesen vortex [1], which has a finite thickness. Then it is reasonable to expect that the quantum fluctuations around this classical solution $x_i^q = X_i - X_i^{\rm cyl}$ are formed not only by the local deformations of the string, but also by topological excitations, characterized by the number of times the string wraps around the flux tube. We can implement these winding modes of the string by assuming the quantum fluctuations compactified on a circle of length $2\pi r = L_c$ [20]

$$x_i^q \equiv x_i^q + nL_c\,, \qquad i = 1, \dots D-2\,. \tag{25}$$

It follows that in the functional integral all the string configurations are equivalent to ones with $|x_i^q| \leq L_c$, then they fill a tube of radius

$$R_f \simeq \sqrt{D-2}\frac{L_c}{2} \tag{26}$$

around the classical solution. We identify this tube with the colour flux tube of the gauge theory in the confining phase.

An interesting feature of this one-parameter family of conformal theories is that, in many cases [5], there is a free fermion in the spectrum. This is a common property of dynamical systems which can sustain local and topological modes. A free fermion in the spectrum accounts for another property of the color flux tubes: they cannot self-overlap freely but must obey to some constraints which depend on the nature of the gauge group.

The two limiting cases are SU(∞) and Z_2, respectively. In the former case there is no limitation on the overlapping and the bosonic string could be a good asymptotic description [22]. On the contrary, in the Z_2 case, the flux tube describes in the space-time self-avoiding surfaces. One may now ask whether the QCD is better described by a free bosonic string or by a self-avoiding one. Actually it has been shown [23] the exact equivalence between a model of random self-avoiding surfaces embedded in a three-dimensional lattice and a $O(N)$ lattice gauge theory for any N. Moreover this model of self-avoiding surfaces has a phase transition belonging to the same universality class of the Z_2 gauge model. A similar construction has also been found for particular $3D$ $U(N)$ lattice gauge theories [24].

This suggests assuming that the effective string, at least at large distances, is described, for any gauge group, by one of those free fermion theories mentioned above. The boundary conditions of such a fermion is a function of R_f. We show now that Eq.(22) and (24) fix these boundary conditions as well as R_f.

Consider a free Dirac fermion $\psi(\varsigma,\tau) = \psi^1 - i\psi^2$ on an infinite strip of width R with action

$$S = -\frac{i}{2}\int \mathrm{d}\tau \int_{-\frac{R}{2}}^{\frac{R}{2}} \mathrm{d}\varsigma\, \bar{\psi}\, \partial\!\!\!/\, \psi\,. \tag{27}$$

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[5] This happens when $\nu = \frac{\sqrt{\sigma}R_f}{\sqrt{\pi(D-2)}}$ is rational.

Equations of motion say that

$$\psi^a_\pm(\varsigma,\tau) = \psi^a_\pm(\tau \pm \varsigma)\,, \tag{28}$$

where ψ^a_+ and ψ^a_- are the up and down components of the Majorana fermion ψ^a. Because of the finite width, in the variation of S arises also a boundary term

$$\psi^a_+ \delta\psi^a_+ - \psi^a_- \delta\psi^a_-\,, \qquad \varsigma = \pm\tfrac{R}{2}\,, \tag{29}$$

which implies $\psi(\varsigma,\tau) \equiv 0$ unless one makes the rather arbitrary assumption $\delta\psi_+ = \pm\delta\psi_-$.

A better way to treat this problem is to add a boundary term S_B [26] to the bulk action (27) in order to compensate the contribution (29). The most general conformal invariant, hermitian term, has the form

$$S_B\big(\varsigma = \tfrac{R}{2}\big) = \tfrac{i}{2}\cos(2\pi\nu)\int \mathrm{d}\tau\, \psi^a_-\psi^a_+ + \tfrac{i}{2}\sin(2\pi\nu)\int \mathrm{d}\tau\, \epsilon_{ab}\psi^a_-\psi^b_+\,, \tag{30}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$, $\epsilon_{aa} = 0$ and ν is a not yet specified boundary phase. A similar expression holds for the other side at $\varsigma = -\frac{R}{2}$, with another phase ν', but it is always possible to redefine the field ψ such that

$$S_B(\varsigma = -\frac{R}{2}) = \frac{i}{2}\int \mathrm{d}\tau\, \psi^a_+\psi^a_-\,. \tag{31}$$

When δS_B is combined with Eq.(28) and (29) we get

$$\psi_\pm(\varsigma + 2R,\tau) = \mathrm{e}^{\pm 2\pi i\nu}\psi_\pm(\varsigma,\tau)\,. \tag{32}$$

We come now to the constraints (22) and (24). Comparing Eq.(30) and Eq.(31) shows that the term proportional to $\cos(2\pi\nu)$ at $\varsigma = \frac{R}{2}$ is compatible with that at $\varsigma = -\frac{R}{2}$, therefore the amplitude for a direct open-closed string transition is forbidden, as required by Eq.(22), only if $\cos(2\pi\nu) = 0$, *i.e.* $\nu = \frac{1}{4}$ or $\nu = \frac{3}{4}$.

To see that also Eq.(24) is fulfilled, consider the reparametrization $\varsigma \to -\varsigma$,which is a symmetry of the bulk action and transposes the two sides of the strip. We may implement this symmetry with the following field transformation

$$\mathcal{F}\,:\;\; \psi^a_+ \to \Omega^a{}_b(\vartheta)\psi^b_-\,, \quad \psi^a_- \to \Omega^a{}_b(-\vartheta)\psi^b_+\,, \tag{33}$$

where $\Omega \in SO(2)$ is a rotation of an angle ϑ. Choosing $\vartheta = \pi\nu$ exchanges also the two boundary terms (30) and (31), as Eq.(24) requires.

In conclusion, a universal string picture describing the large-distance behaviour of gauge theories in the confining phase emerges rather naturally.

It can be formulated simply as a free fermion theory (one Dirac fermion for each transverse dimension). Then the normal modes are now fermi harmonic oscillators with

$$\begin{aligned} \omega_n &= \tfrac{\pi}{R}(n - \tfrac{1}{4}), \\ \omega'_n &= \tfrac{\pi}{R}(n - \tfrac{3}{4}), \end{aligned} \tag{34}$$

so we may apply again Eqs (13) and (17) to evaluate the zero-point energy of this theory:

$$E_0^{\text{fermi}} = -(D-2)\left[\frac{\hbar}{2}\sum_{n=1}^{\infty}{}' \omega_n + \frac{\hbar}{2}\sum_{n=1}^{\infty}{}' \omega'_n\right] \tag{35}$$

$$= -(D-2)\hbar\frac{\pi}{96R}, \tag{36}$$

which is just one fourth of that of the bosonic string. Comparison with Eq.(12) yields

$$h_{\alpha\beta} = \frac{D-2}{32}. \tag{37}$$

Thus, in order to find the value of the compactification radius we have to go back to the corresponding bosonic formulation and look for $c = 1$ theories in which the minimal positive conformal weight of the spectrum is $\frac{1}{32}$. In such a description the primary fields can be written in the form of vertex operators $: e^{ipx^q} :$ with a momentum p given by [20]

$$p = mr + \frac{n}{2r}, \quad m, n \in Z, \tag{38}$$

where the adimensional compactification radius r is related to the scale L_c of Eq.(25) as

$$r = \frac{\sqrt{\sigma}L_c}{2\sqrt{\pi}}. \tag{39}$$

The corresponding conformal weight is $h = \frac{p^2}{2}$. The theory of compactified boson has an obvious symmetry $r \to \frac{2}{r}$, known as duality transformation, which prevents to fix unambiguously the compactification radius: for each allowed conformal spectrum there are, in general, two distinct radii, at least, which reproduce it. In the effective string picture this duality symmetry is broken at short distance because of the coupling of the Liouville mode to the conformal matter [21], so either one of the compactification radii represents an unstable solution. It is then reasonable to assume that the physical compactification radius is the smallest of the possible solutions . It is easy to see that the theory has a discrete spectrum only if r^2 is rational *i.e.* $r^2 = \frac{p}{q}$. In such a case the gap h_{min} of the theory can be written in the general form

$$h_{\min} = \frac{5+3(-1)^q}{16\,pq}\,; \quad r^2 = \frac{p}{q}\,. \tag{40}$$

Comparison with Eq. (37) gives $r = \frac{1}{4}$. We can now evaluate the thickness of the colour flux tube by translating this number into physical units. Indeed using Eqs (39) and (26) yields

$$\sqrt{\sigma}R_f = \frac{\sqrt{\pi(D-2)}}{4}\,. \tag{41}$$

Taking for $\sqrt{\sigma}$ the conventional value of 420 MeV and $D = 4$, we get $R_f \simeq 0.3$ fermi, which agrees with the value observed in various numerical lattice simulations [3, 4, 27, 28].

The free energy $F(R,L)$ of these CFT's can be written not only in the asymptotic regions $R/L \ll 1$ or $L/R \ll 1$ described by Eqs (10) and (11), but for any value of R and L. In particular, for the rectangular Wilson loop (2) we get the following expression

$$F_{r=\frac{1}{4}}(R,L) = \sigma RL + p(R+L) + k + q(R,L)\,, \tag{42}$$

where the first three terms are the usual area, perimeter and constant term ascribed to the classical solution, while the information on the CFT is contained in the term $q(R,L)$ produced by the quantum fluctuations of the string

$$q(R,L) = -(D-2)\log\frac{\vartheta\begin{bmatrix}1/4\\1/4\end{bmatrix}(0|\tau)}{\eta(\tau)}\,, \tag{43}$$

where $\tau = iR/L$, $\vartheta\begin{bmatrix}\alpha\\\beta\end{bmatrix}(0|\tau)$ is the Jacobi theta function with characteristic $\begin{bmatrix}\alpha\\\beta\end{bmatrix}$, and $\eta(\tau)$ is the Dedekind eta function. Eq. (42) should reproduce the vacuum expectation value of the Wilson loop only for values of R and L very large with respect to the other physical scales entering into the game, so that the conformal hypothesis is asymptotically true:

$$-\log\langle W(R,L)\rangle \simeq F_{r=\frac{1}{4}}(R,L)\,, \quad \Lambda R\,,\ \Lambda L \gg 1\,, \tag{44}$$

where Λ is the mass scale of the theory.

Notice that the string picture which emerges from this description is exactly that which was proposed some time ago [27] on purely phenomenological grounds, where the boundary phase $\nu = \frac{1}{4}$ were determined only by a best fit of Eq.(42) to the numerical data on the expectation values of Wilson loops. This picture as been proven [27, 28] to be quite good for all those $3D$ and $4D$ gauge systems where accurate numerical data are available. In particular, the values of the string tension σ determined by fitting Eq.(42) to the data, show a better approach to the asymptotic scaling with respect to more conventional fitting procedures, mostly based on the arbitrary assumption $q(R, L) \equiv 0$.

The idea that the asymptotic effective string is described by a compactified boson allows us also to get approximately the lowest mass of the glueball spectrum. This spectrum can be evaluated by studying the exponential decay of the correlation function of small quark loops at large distance. If γ is a small circular loop with center on a point x and $W_x(\gamma)$ is the associated Wilson loop operator, we have, asymptotically

$$\langle W_x(\gamma) W_{x+L}(\gamma)\rangle \sim e^{-m_G L}, \tag{45}$$

where m_G is the mass of the lowest glueball. In a string picture this expectation value can be written as the partition function of a CFT on a surface with the topology of a cylinder of length L. However in the present case , at variance of what happens for the correlation function of two Polyakov lines, the minimal radius $R_{\min}$ of this cylinder is not determined by the geometry of the system, rather it should be generated dynamically. If we assume that $R_{\min}$ is large enough to apply CFT formulas, we get from Eq. (11)

$$\langle W_x(\gamma) W_{x+L}(\gamma)\rangle \sim \exp\left(-\sigma 2\pi R_{\min} L + \frac{\tilde{c}\pi}{6}\frac{L}{2\pi R_{\min}}\right), \tag{46}$$

where the first term at the exponent is the usual area term and the second one is due to the universal quantum correction. Comparing Eq. (45) with Eqs (46) and (19) yields

$$m_G \simeq 2\pi\sigma R_{\min} - \frac{D-2}{12 R_{\min}}. \tag{47}$$

In the bosonic string picture there is no natural lower bound for $R_{\min}$: the minimal area is obtained for $R_{\min} \to 0$, where the quantum contribution diverges. On the contrary, the self- avoiding string picture we are describing gives obviously $R_{\min} \simeq R_f$, otherwise there is an overlapping of the colour flux tube. From Eqs (41) and (47) we get

$$\frac{m_G}{\sqrt{\sigma}} \simeq \sqrt{D-2}\,\frac{3\pi^2 - 2}{6\sqrt{\pi}}. \tag{48}$$

For $D = 4$ we have $\frac{m}{\sqrt{\sigma}} \simeq 3.67$, which reproduces rather accurately the numerical results of the lattice simulations: indeed for the pure SU(2) gauge theory one finds [29, 30] $\frac{m}{\sqrt{\sigma}} \simeq 3.7(2)$, and similarly for pure SU(3) one has [31] $\frac{m}{\sqrt{\sigma}} \simeq 3.5(2)$.

The asymptotic string picture we are describing in this lecture allows us to gain some information also on the phase transition to the quark gluon plasma [32, 38] . Consider indeed a pair of quarks propagating in a gauge medium at a temperature $T = 1/L$ below the deconfining point T_c. According to Eq. (10) this system is described by the static potential

$$V(R) = \sigma(T)\,R - c'\frac{\pi}{24R} + O(1/R^2)\,, \tag{49}$$

where for the moment we do not commit ourselves with the value of the effective conformal anomaly $c' = c - 24h_{\alpha\beta}$. At the deconfining point $T = T_c$ the string tension $\sigma(T_c)$ vanishes and the flux connecting the two quarks cannot longer be described by an effective string. As a consequence, the long distance Coulomb-like term $c'\pi/24R$ looses its very justification: indeed we have seen that this universal behaviour is produced by the quantum fluctuations of the string, but now the string has faded away . Consistency requires vanishing of c' at $T = T_c$, i.e. there must be in the CFT theory a physical state with a conformal weight

$$h_{\alpha\beta}(T_c) = \frac{c}{24}\,. \tag{50}$$

Note that c' , according to the way it has been calculated in Eqs (13) and (16), measures the number of local degrees of freedom of the CFT. Its vanishing tells us that at the deconfining point the effective string theory has at most a discrete set of degrees of freedom, *i.e.* it behaves like a topological conformal field theory (TCFT). Actually most TCFT's may be formulated as (twisted) $N = 2$ superconformal theories (SCFT) [34]. It turns out [36] that in any $N = 2$ SCFT there is a physical state of conformal weight $h = c/24$ [6] Conversely one is led to conjecture that any CFT with a weight $h = c/24$ is promoted to a $N = 2$ SCFT. This is almost trivially true for $c = 1$ and $c = 2$, *i.e.* $D = 3$ and $D = 4$, which are the cases we are interested in.

As an example, consider the whole set of $c = 1$ CFT's that can be written in terms of a compactified boson $x_\perp(z)$, identified later with the field describing the transverse displacements of the $D = 3$ effective string. The conformal spectrum is given by Eq.(38). If there is a primary field $:e^{p_o x_\perp(z)}:$ with $h = p_o^2/2 = 1/24$, then also $p = 6p_o = \pm\sqrt{3}$ belongs to the set of the allowed momenta, of course. The corresponding primary fields

[6] It is generated by the spectral flow [35] of the Neveu–Schwarz ground state. For further details, see Ref. [36].

$G(z) =: \mathrm{e}^{\sqrt{3}x_\perp(z)}$ and $\bar{G}(z) =: \mathrm{e}^{-\sqrt{3}x_\perp(z)}$ are precisely the two supercurrents which generate the $c = 1$ representation [37 of the $N = 2$ superalgebra [26] [7].

On the other hand, using Eq.(40) with $h_{\min} = 1/24$, we select four possible compactification radii:

$$r_1 = \frac{1}{2\sqrt{3}}\,,\ r_2 = \frac{1}{\sqrt{3}}\,,\ r_3 = \frac{\sqrt{3}}{2}\,,\ r_4 = \sqrt{3}\,. \tag{51}$$

They exactly correspond to the only points where the conformal symmetry is promoted to a $N = 2$ extended supersymmetry. The spectrum of the primary fields is the same for these four radii, and it is now described in terms of two quantum numbers: the conformal weight h and the charge q of the U(1) current. The Neveu-Schwarz (NS) and the Ramond (R) sectors hold each three primary fields, namely

$$\begin{aligned} NS:&\quad \left\{\left(h = 0\,, q = 0\right),\quad \left(h = \tfrac{1}{6}\,, q = \pm\tfrac{1}{3}\right)\right\}, \\ R:&\quad \left\{\left(h = \tfrac{3}{8}\,, q = \tfrac{1}{2}\right),\quad \left(h = \tfrac{1}{24}\,, q = \pm\tfrac{1}{6}\right)\right\}. \end{aligned} \tag{52}$$

A possible choice of the boundary conditions α and β for the two sides of the infinite strip obeying to the constraints (22) and (24) is

$$\alpha \leftrightarrow \left(h = \tfrac{1}{6}\,, q = \tfrac{1}{3}\right);\ \beta \leftrightarrow \left(h = \tfrac{1}{24}\,, q = -\tfrac{1}{6}\right). \tag{53}$$

The Z_2 automorphism is simply generated by the symmetry between the two sectors NS and R : $NS \leftrightarrow R \Rightarrow \alpha \leftrightarrow \beta$.

Consider now a Wilson loop orthogonal to the imaginary time at a temperature $T = 1/L$ as drawn in Fig. 1.

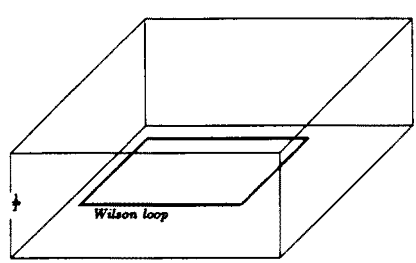

Fig. 1. A Wilson loop orthogonal to the imaginary time.

[7] The other conserved currents of this superalgebra are the energy momentum tensor $T(z)$ and the $U(1)$ current $J(z) = \partial x_\perp(z)$.

Now the field $x_\perp$ describing the displacements of the effective string in the direction of the imaginary time axis is obviously compactified on a circle of length $L = 1/T$, *i.e.*

$$x_\perp \equiv x_\perp + L\,. \tag{54}$$

Increasing the temperature of the system reduces the phase space of the colour flux tube until, for a critical value $T = \Theta$, the inverse temperature L coincides with the width $2R_f(T)$ of the flux tube:

$$2R_f(\Theta) = \frac{1}{\Theta}\,, \tag{55}$$

where we have taken into account that the thickness of the flux tube may be a function of the temperature. At this point the flux tube fills the circle of the imaginary time and is then definitely trapped in it, so it cannot longer fluctuate. Then the zero-point energy must vanish and we recover Eq. (50). So we are led to identify Θ with the deconfinement temperature T_c and the compactification radii (51) as the possible values of the size of the flux tube, according to Eq. (39).

We are now in a position to evaluate explicitly T_c. Indeed, Eq.(51) singles out , through Eqs (54) and (39), four special temperatures. It is reasonable to assume that the deconfining temperature corresponds to the minimal radius r_1, while the others correspond to metastable solutions because, as we have already pointed out, the degeneracy associated to the Eq. (40) is removed at short distance by the coupling to the Liouville mode. This gives

$$\frac{T_c}{\sqrt{\sigma}} = \frac{\sqrt{3}}{\sqrt{(D-2)\pi}}\,. \tag{56}$$

The D dependence has been inserted to take into account also the other interesting case of $D = 4$, which can be treated in the same way.

Indeed at $c = 2$ there are two sets of special points in the space of the CFT's where the symmetry is promoted to an extended $N = 2$ supersymmetry.

A set of these special points corresponds to the direct product of two $c = 1$ $N = 2$ theories, where the string is described by two identical compactified bosons x_i , $i = 1, 2$ associated to the two transverse directions. This kind of $N = 2$ SCFT is not the correct description of the asymptotic effective string associated with the Wilson loop perpendicular to the imaginary time, because in this configuration the transverse displacements along the two orthogonal axes are not on the same footing, being only one direction compactified.

The other set of $N = 2$ SCFT with $c = 2$ corresponds to the $c = 2$ element of $N = 2$ minimal series, which is described by two free fields: the first is a compactified bosonic field $x_\perp(z)$, with a compactification radius related to Eq. (39) by $r_c = \sqrt{2}r_i$, which should describe the string displacements along the imaginary time; then, using the same arguments as before, we get for T_c the value stated in Eq. (56). The other free field is a Z_4 parafermion $\psi(z)$, which we can associate to the string displacements along the other direction. Note that the latter displacements are described by a fermionic field, according to the effective string theory at $T = 0$, where the target space has no compactified directions.

Remarkably enough, our determination of T_c coincides with the value predicted for the Nambu–Goto string [14]. Our argument suggests that this temperature is universal and does not depend on the gauge group.

In Table I the values of the observables we have determined in Eqs (41), (48) and (56) through our asymptotic effective string scheme, are compared with the corresponding data from numerical simulations on lattice gauge theories with various gauge groups.

TABLE I

Comparing string values with data from numerical simulations.

Dimension	D=3			D=4		
model	string	Z_2	SU(2)	string	SU(2)	SU(3)
$T_c/\sqrt{\sigma}$	0.977	1.17(10)[32]	0.94(3)[39]	0.691	0.69(2) [40]	0.56(3) [40]
$m_G/\sqrt{\sigma}$	2.596			3.671	3.7(2) [29]	3.5(2) [29]
T_c/m_G	0.376			0.188	0.180(16)[40]	0.176(20)[40]
$\sqrt{\sigma}R_f$	0.443			0.627	$0.4 \div 0.6$[3]	

The agreement to the numerical simulations on LGT is in general rather good and it is conceivable that the few discrepancies are due essentially to the poor scaling of the current lattice experiments. It would be interesting to do new numerical simulations in order to complete the table and to test the universality of the string formulas with other gauge groups.

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