QUANTUM GRAVITY REPRESENTED AS DYNAMICAL TRIANGULATIONS*

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We discuss the attempts to represent quantum gravity in more than two dimensions as a sum over random triangulations. The results are mainly of numerical nature, obtained by Monte Carlo simulations in three and four dimensions. In addition to pure Einstein gravity we also consider generalizations where R^2-terms or matter fields are added to the Einstein action.

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1. Introduction

Understanding the theory of quantum gravity remains one of the greatest challenges in theoretical physics. One can try to circumvent the problem by embedding gravity in a larger theory like string theory. This is in many

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respects an appealing approach, but it seems to have lost some of its momentum and there is not much hope that it will be possible in the near future to arrive in a natural way to an effective theory of gravity (and matter) in four dimensions. Discussing basic principles one should maybe not be so worried about our technical inability to deduce the consequences of string theory since this is not the first time in theoretical physics we are unable to extract anything but the simplest perturbative consequences of an otherwise healthy and probably correct theory. It is more worrisome that string theory is not (yet) well defined beyond the loop expansion. Again this might not seem so disastrous since the same was (and to some extent still is) true for ordinary field theory. However, in the last three years there *has* been a significant progress in our understanding of the problems connected with the summation of all loops in string theory. The message has not been encouraging. At the moment we have no general principles which allow us to define in an unambiguous way the summation over all genera in string theory. From this point of view it might be somewhat premature to announce string theory as the fundamental theory. Strictly speaking it is not yet a theory but a set of rules which allows us to calculate certain perturbative quantities.

If we decide to drop string theory as the theory which will teach us the nature of quantum gravity it might be (good ?) conservative policy to stay entirely within the ordinary field theoretical framework. At first glance it does not look too promising either. The four dimensional theory is hampered by being non-renormalizable and we do not at present know any example where such a theory can be defined non-perturbatively, is non-trivial and at the same time satisfies what we usually view as the axioms of quantum field theory. At the same time one of the lessons of the last thirty years is that field theory is deeply connected to the theory of critical phenomena via the path integral and has a natural formulation in Euclidean space-time. But precisely for gravity the Einstein-Hilbert action is unbounded from below due to the conformal mode and the Euclidean path integral is ill-defined. One could try to make sense of the unboundedness of the action either by a contour rotation associated with the conformal mode as suggested by Hawking and others [1] or by stochastic regularization (the so-called fifth time action) as advocated by Greensite and Halpern [2, 3]. It is not the purpose here to enter into a discussion of the virtues and drawbacks of these interesting suggestions. Let us only mention that they do not really have the flavour of general descriptions based on ordinary field theory. Of course it is wise to bear in mind that if any field theory should depart from basic axiomatic principles it is quantum gravity, but lacking a general alternative we have decided to return to analysis of quantum gravity in the context of ordinary field theory.

Field theory suggests one rather simple minded way out of the above mentioned problems. This was already discussed long ago by Weinberg who called it asymptotic safety [4]. The idea is simply that when we, by means of the renormalization group equations, work our way back from the infrared fixed point where the Einstein-Hilbert action seems a good effective description we will at some point reach a non-trivial ultraviolet fixed point. In addition the associated critical surface is assumed to be finite dimensional, which means that only a finite number of parameters are left arbitrary in the theory, which from this point of view can be said not to differ much from ordinary renormalizable field theories. The effective Lagrangian description of the theory by means of fields suitable for the infrared fixed point might then be an infinite series

$$\mathcal{L}_G = \sqrt{g}\left[\Lambda - \frac{1}{16\pi G}R + f_2 R^2 + f_2' R_{\mu\nu}R^{\mu\nu} + \cdots\right] \tag{1.1}$$

which might even be non-polynomial, but which might now (and we will assume this is the case) make sense if we make a formal rotation to Riemannian space where the metric has the signature $\{+1,+1,+1,+1\}$ (the generalization of the rotation to Euclidean space in ordinary field theory).

Of course one weakness in this scenario is that the existence of the ultraviolet fixed point has been entirely hypothetical. Further we have not exactly been flooded with examples of this kind in field theory, as already mentioned. Finally, and we agree that this point is an annoyance, such a solution does not have the appeal and aesthetical beauty of the original theory which we want to quantize. If the quantum theory of gravity offers to us a solution like (1.1) the next task must be to find a simpler description in terms of other variables, maybe somewhat like the switch in hadronic physics from hadrons to quarks and gluons.

A few things have happened since Weinberg outlined the above strategy. There exist now regularizations of the path integral which allow us to define theories like (1.1) non-perturbatively and further the progress in computer science has made it possible to calculate approximately these path integrals. It is therefore possible to explore, by numerical means, the phase diagram of the regularized theory and try to locate phase transition points in the coupling constant space. If the transitions are of second order one can attempt to define a continuum limit. The approach has one virtue: it requires only a finite amount of work to verify whether the idea of a non-trivial ultraviolet fixed point is viable or not.

2. The discretized model

The continuum theory of gravity is reparametrization invariant. If we discretize the theory in order to regularize it we will have to break this invariance provided the action depends on the metric. An alternative is to consider theories which depend only on topology. A very interesting approach in this direction in three dimensions has recently been followed by many people following the work of Turaev and Viro [5] and it has now been generalized to four dimensions [6]. Unfortunately the precise connection with the usual continuum version of Einstein gravity is not yet clear, especially in four dimensions.

If we restrict ourselves to the conservative approach of discretizing Einstein's theory of gravity we will break reparametrization invariance since an action like (1.1) depends explicitly on the metric and without having done anything yet we can already now say that the most important question to answer in case one manages to carry out successfully the program outlined in the introduction is whether the theory defined by approaching in a well defined way the ultraviolet fixed point is really reparametrization invariant. This is by no means obvious since the regularization has broken this invariance explicitly.

2.1. Quantization of Regge calculus

Two rather different regularization schemes have been suggested. The oldest one goes back to Regge [7] and we will call it Regge calculus. It was originally invented as a means to approximate a given smooth manifold by a piecewise flat Riemannian manifold, obtained by a suitable triangulation of the smooth manifold. Regge showed that it was still possible to define in a sensible way the concept of curvature for such a piecewise flat manifold. For d-dimensional manifolds the building blocks would be d-dimensional simplices and the curvature assigned to the $d-2$-dimensional sub-simplices. In this way one has both volume and curvature assigned to the piecewise flat manifold and it is possible to approximate the continuum Einstein–Hilbert action which reads

$$S[g] = \lambda \int d^d\xi \sqrt{g} - \frac{1}{16\pi G} \int d^d\xi \sqrt{g} R, \tag{2.1}$$

by the following discretized expressions:

$$\int d^d\xi\sqrt{g} \quad \to \sum_{n_{d-2}} V_{n_{d-2}}(d) \tag{2.2}$$

$$\int d^d\xi\sqrt{g}R \to \sum_{n_{d-2}} V_{n_{d-2}}(d) \left[2\delta_{n_{d-2}} \frac{V_{n_{d-2}}(d-2)}{V_{n_{d-2}}(d)} \right]. \tag{2.3}$$

In (2.2)-(2.3) the summation is over all $d-2$-dimensional sub-simplices n_{d-2} with volume $V_{n_{d-2}}(d-2)$. The quantity $\delta_{n_{d-2}}$ is the so-called deficit angle associated with the $d-2$ dimensional sub-simplex n_{d-2}, while the d-dimensional volume $V_{n_{d-2}}(d)$ associated with the $d-2$ dimensional sub-simplex n_{d-2} is defined as

$$V_{n_{d-2}}(d) = \frac{2}{d(d+1)} \sum_{n_d \supset n_{d-2}} V_{n_d}(d), \tag{2.4}$$

where the summation is over all d-dimensional simplices n_d which contain the subsimplex n_{d-2}. It is possible to show that for a suitable refinement of the triangulation of the smooth manifold the discretized expressions (2.2) and (2.3) will actually converge to the continuum value. In this way Regge calculus provides a geometrical coordinate-independent description of gravity where it is natural to use the length of the links (the geodesic length of the edges in the triangulation of the given manifold) as dynamical variables since they completely specify the flat d-simplices used as building blocks. Originally the method was used mainly in a classical context where there are no conceptional problems connected with the approach. However, already as early as in 1968 Regge and Ponzano in an impressive paper, which contains also the seed to the recent development in topological gravity mentioned above, pointed out that quantum mechanical amplitudes in three-dimensional Regge calculus can be defined by a functional integral and maybe computed non-perturbatively. However, if we seriously want to apply the Regge calculus directly in the functional integral it looses some of its beauty. In the classical context the geometry was specified by the length of the links of the building blocks and the incidence matrix which specified how the building blocks were glued together. This incidence matrix which determines the topology was fixed and not considered a dynamical variable. When we use the Regge formalism in the path integral the situation is opposite. The task is not to approximate a given continuum Riemannian manifold but (at least) to sum over equivalence classes of metrics associated with a given manifold. Unfortunately there is no simple one-to-one correspondence between link lengths and equivalence classes of metrics, as is easily seen by considering triangulations of the two-dimensional plane. Obviously many triangulations correspond to the same Riemannian geometry. This means that a nasty jacobian is involved if we want to use link lengths as our integration variables. In addition one has to choose an integration measure which ensures that the link lengths satisfy the triangle inequalities and their higher dimensional analogues, which express that the k-dimensional volume of k-dimensional sub-simplices in the given triangulation must be positive. A great deal of work has gone into understanding and repairing these shortcomings of conventional Regge calculus. For a recent excellent review and references we refer to [8]. While the classical Regge calculus gives a coordinate independent geometrical description of gravity it of course has nothing like reparametrization invariance[1]. It is therefore necessary to prove that this invariance is recovered at the point in coupling constant space where the continuum limit is taken. Unfortunately the "quantum Regge calculus" has not, in our opinion, been so successful in this respect. For instance computer simulations in [11] seemingly give the wrong coupling to Ising spins in two dimensions where the coupled spin-gravity system can be solved explicitly in the continuum. Hopefully this is due to problems with the simulations rather than a basic flaw in the approach, but we do not know for sure.

2.2. Dynamically triangulated gravity

Due to the above mentioned problems with the translation of classical Regge calculus to quantum theory we will here use another approach which has been extensively used in the last few years in the study of two-dimensional gravity and non-critical strings (which are nothing but two-dimensional gravity coupled to special matter fields) [12–15]. We will call it dynamically triangulated gravity or (interchangeably) simplicial gravity.

In this approach the fundamental building blocks are regular simplices. In $d = 2$ this means equilateral triangles, in $d = 3$ regular tetrahedra. One now constructs the manifolds by gluing together regular d-dimensional simplices along their $d-1$ dimensional sub-simplices, in such a way that they form a piecewise flat manifold. The dynamics is shifted from the length of the links to the connectivity of the piecewise linear manifold and as we shall see there will not be the over-counting present in the quantum Regge prescription.

The assignment of volume and curvature for a given triangulation T created by gluing together the regular simplices is in this case very simple. Let us introduce the following notation: An i-dimensional (sub)-simplex is denoted n_i, *i.e.* vertices are denoted n_0, links n_1 *etc.* The total number of such (sub)-simplices in T is denoted $N_i(T)$. By the order $o(n_i)$ we understand the number of d-dimensional simplices which share the sub-simplex n_i. We will usually consider only the class of regular simplicial manifolds where we have put the following restrictions on $o(n_i)$:

$$o(n_{d-1}) = 2, \qquad o(n_i) \geq d - i + 1, \qquad (i \leq d-2)\,. \tag{2.5}$$

Then (2.2)-(2.3) reduce to

$$\int d^d\xi\sqrt{g} \propto N_d \tag{2.6}$$

[1] One might try to *define* the analogue of local coordinate transformations, see for instance [10].

and

$$\int d^d\xi\sqrt{g}R \propto \sum_{n_{d-2}} (c_d - o(n_{d-2}))\,. \tag{2.7}$$

The constant c_d in (2.7) should be adjusted in such a way that for a hypothetical triangulation of flat space the sum should give zero. For $d = 2$ one can triangulate flat space with regular triangles. The order of each vertex is 6 and consequently $c_2 = 6$. Higher dimensional flat space does not admit a regular tessellation, but one can still ask for the average value of $o(n_{d-2})$ required to fill up d-dimensional flat space. The angle θ_d between two $d-1$ dimensional simplices belonging to the same d-simplex is given by

$$\cos\theta_d = \tfrac{1}{d} \tag{2.8}$$

and in order to fill up d-dimensional space we need to have

$$o(n_{d-2}) = \tfrac{2\pi}{\theta_d} \equiv c_d\,. \tag{2.9}$$

This is the constant which enters in (2.7). We find

$$c_2 = 6, \qquad c_3 = 5.104, \qquad c_4 = 4.767\,. \tag{2.10}$$

Let us further note that

$$\sum_{n_{d-2}} o(n_{d-2}) = \frac{(d+1)d}{2} N_d\,, \tag{2.11}$$

since there are $\begin{pmatrix} d+1 \\ d-1 \end{pmatrix}$ $d-2$ dimensional sub-simplices in a d-dimensional simplex. The discretized version of the continuum action can now, for a given triangulation T, be written as

$$S_d[T] = \kappa_d N_d(T) - \kappa_{d-2} N_{d-2}(T)\,. \tag{2.12}$$

A more general action would be the following:

$$S_d[T] = \sum_{i=0}^{d} \kappa_i N_i(T)\,, \tag{2.13}$$

involving the fugacities for all different i-dimensional sub-simplices. Of course one can choose to consider actions which cannot be expressed entirely in terms of the N_i's. The higher derivative terms which can be added to the continuum action (2.1) will in general be of this kind, and we are going to consider them later, but let us for the moment discard such terms. Not all N_i's are independent. The relations between the N_i's can be worked out by the requirement that the triangulation should be locally homeomorphic to R^d. This means for instance that all n_d's sharing a given vertex n_0 should be homeomorphic to the unit ball in R^d. Similar restrictions hold for the neighbours to an i-dimensional simplex n_i in the triangulation, and the relations this imposes on the N_i's are summarized in the so-called Dehn–Sommerville [16] relations

$$N_i = \sum_{k=i}^{d} (-1)^{k+d} \binom{k+1}{i+1} N_k \,, \tag{2.14}$$

valid for all $i \geq 0$. These relations are not independent, but allow us to eliminate all N_{2i+1}'s if d is even and all N_{2i}'s if d is odd. In the case of even dimensions we have for a given triangulation T in addition Euler's relation

$$\sum_{i=0}^{d} (-1)^{i+d} N_i(T) = \chi_d(T) \,, \tag{2.15}$$

where $\chi_d(T)$ denotes the Euler characteristic of the piecewise linear manifold which corresponds to the triangulation T. Of course this relation is only useful if we know the Euler characteristic of the triangulation T. As we shall see the restriction of topology is very important, and in case we fix the topology of T we can use (2.15) to eliminate for instance N_0. In case the topology is not fixed we can trade $N_0(T)$ for $\chi_d(T)$. For odd dimensions (2.15) follows from the Dehn–Sommerville relations with

$$\chi_{d=2n+1} = 0 \,. \tag{2.16}$$

In odd dimensions the Euler characteristic is identically zero for any simplicial manifold. It follows just from the requirement of local homeomorphism to R^d.

The recipe for going from the continuum functional integral to the discretized one is now:

$$\int \mathcal{D}[g_{\mu\nu}] \to \sum_{T \sim T} \,, \tag{2.17}$$

$$\int \mathcal{D}[g_{\mu\nu}] \, e^{-S[g]} \to \sum_{T \sim T} e^{-S[T]} \,. \tag{2.18}$$

The formal integrations on the lhs of (2.17) and (2.18) are over all equivalence classes of metrics, *i.e.* the volume of the diffeomorphism group is divided out. $\mathcal{T}$ denotes a suitable class of triangulations. One class of constraints is given by (2.5), but it should be stressed that such short distance restrictions are not expected to be important in the scaling limit.

Since different triangulations give rise to different curvature assignment one can view the above summation as a summation over different Riemannian manifolds. There is no problem with over-counting in this formulation. The idea of the continuum functional integral is precisely to perform such a sum over Riemannian manifolds with weight $e^{-S[g]}$. Of course the discretized sum on the rhs of (2.17) and (2.18) can only be viewed as an approximation to the continuum expression which hopefully "converges" in the scaling limit to the correct expression. The questions which are difficult to answer are whether the class of piecewise flat manifolds is "close" to the class of Riemannian manifolds and whether the piecewise linear manifolds are selected sufficiently uniformly with respect to Riemannian manifolds that (2.17) and (2.18) are good approximations. Unfortunately there is no weak coupling expansion where one can check this, but it is very encouraging that the formalism is known to work in the two-dimensional case, even if one couples conformal matter with central charge $c \leq 1$ to the system. In this case it is possible to solve both the continuum and the discretized system. In particular we see that reparametrization invariance is recovered in the scaling limit.

We will report on work in $d = 3$ and $d = 4$. In these cases the partition function look as

$$Z_{d=3}(\kappa_1, \kappa_3) = \sum_{T \in \mathcal{T}} e^{-\kappa_3 N_3 + \kappa_1 N_1}, \tag{2.19}$$

$$Z_{d=4}(\kappa_2, \kappa_4) = \sum_{T \in \mathcal{T}} e^{-\kappa_4 N_4 + \kappa_2 N_2}. \tag{2.20}$$

These are grand canonical partition functions, where the volume of the universe can vary. It is sometimes convenient to change from the grand canonical ensemble to the canonical ensemble where the volume N_d is kept fixed. The corresponding partition function will be

$$Z(\kappa_{d-2}, N_d) = \sum_{T \sim \mathcal{T}(N_d)} e^{\kappa_{d-2} N_{d-2}(T)}, \tag{2.21}$$

$$Z(\kappa_{d-2}, \kappa_d) = \sum_{N_d} Z(\kappa_{d-2}, N_d)\, e^{-\kappa_d N_d}. \tag{2.22}$$

where $d > 2$ is assumed.

If the entropy, *i.e.* the number of configurations for a given N_d, is exponentially bounded it is easy to prove that there is a critical line

$\kappa_d = \kappa_d^c(\kappa_{d-2})$ in the (κ_{d-2}, κ_d)-coupling constant plane. For a given κ_{d-2} the partition function (2.22) will then be well defined for $\kappa_d > \kappa_d^c(\kappa_{d-2})$. Let us call this domain in the coupling constant plane $\mathcal{D}$. Critical behaviour can be found only when we approach the boundary $\partial\mathcal{D}$ which we denote the critical line, but in general we only expect interesting critical behaviour at certain critical points on $\partial\mathcal{D}$ (*i.e.* at certain values of κ_{d-2}). These are the points we are looking for in the numerical simulations.

Let us end this subsection by discussing a point which is worth emphasizing. We have been deliberately vague defining the class of triangulations $\mathcal{T}(N_d)$ $(d > 2)$ over which the summation is to be performed in a formula like (2.21). Already in two dimensions where the classification of topology is so simple (it is defined by the Euler number χ) an unrestricted summation over manifolds of different topology does cause problems. In fact the two-dimensional analogue of (2.21) and (2.22) which by means of Euler's relation can be written:

$$Z(\tilde{\kappa}_0, N_2) = \sum_{T \sim \mathcal{T}(N_2)} e^{\tilde{\kappa}_0 \chi}, \tag{2.23}$$

$$Z(\tilde{\kappa}_0, \tilde{\kappa}_2) = \sum_{N_2} Z(\tilde{\kappa}_0, N_2) e^{-\tilde{\kappa}_2 N_2}, \tag{2.24}$$

does not make any sense. The well known reason is that the number of triangulations, $\mathcal{N}(N_2)$, which one can make by gluing together a given number N_2 of equilateral triangles to a two-dimensional surface is too large. It grows factorially fast: $\mathcal{N}(N_2) \geq N_2!$. This means that the two-dimensional analogue of (2.22) is never convergent. This is not a spurious result of a perverse discretization. An analogous result has been proven in the continuum two-dimensional theory where the volume of moduli space grows at least factorially with the genus [17]. It is the same effect we observe in the discretized case: As long as we fix the topology, *i.e.* the Euler characteristic χ of the surface, we have: $\mathcal{N}_\chi(N_2) \sim N_2^{\gamma_\chi - 3} \exp(\tilde{\kappa}_{2c} N_2)$, *i.e.* only an exponential growth of the number of surfaces. In this case (2.24) is well defined for a certain range of $\tilde{\kappa_2}$'s. However, an unrestricted summation over topologies makes the subleading pre-exponential factor $N_2^{\gamma_\chi}$ dominant when $|\chi| \sim N_2$ since $\gamma_\chi \sim -5\chi/4$.

In higher dimensions the situation is of course only worse and the best we can hope for is a well defined expression for a fixed (or at least restricted) topology.

The above outlined non-perturbative definition of gravity has nothing to add to our understanding (or rather lack or understanding) of the question of whether or not to sum over topologies in quantum gravity. Apart from the problem that the topologies of non-simply connected four-dimensional manifolds cannot be classified in a sensible way, the partition function does not even make sense if we restrict ourselves to the sub-class of topologies which one can construct by simple analogy to two-dimensional surfaces of genus g. In the rest of this article we will only be interested in the search for non-trivial fixed point of the above defined theory (modified with higher curvature terms or matter fields) where the class of triangulations $\mathcal{T}$ corresponds to manifolds with the topology of S^d.

2.3. Higher curvature terms

There is no straightforward generalization of Regge's work to theories of gravity which involve higher derivative terms like R^2 in the action. The reason is that Regge viewed the curvature of the piecewise flat manifold, not as a discrete approximation to an underlying continuum curvature, but as one which could be defined in a mathematical stringent way, entirely in terms of the geometrical concepts involved in parallel transportation. In this way the curvature occurs in δ-functions on the lattice geometry, with support on the $d-2$-dimensional sub-simplices. Anything other than the Einstein action (and the cosmological term) will then involve higher powers of δ-functions and this means that for piecewise flat manifolds, interpreted as by Regge, terms like $\int \sqrt{g}R^2$ *are* infinite. In order to make sense of higher derivative terms one has to change the perspective on Regge calculus somewhat as advocated by Hamber and Williams [18] and view the lattice geometry as representing an approximation to some smooth geometry and the local curvature as some average curvature for a small volume. In fact our formulas for Regge calculus have already hinted this interpretation in the sense that we have assigned to each $d-2$-dimensional sub-simplex n_{d-2} a volume density $V_{n_{d-2}}(d)$ which can be viewed as an appropriate share of the volumes of the d-dimensional simplices to which the sub-simplex n_{d-2} belongs. In the same way we have written the curvature density R as $\delta_{n_{d-2}} V_{n_{d-2}}(d-2)/V_{n_{d-2}}(d)$ viewing it formally as representing some average value in the volume $V_{n_{d-2}}(d)$. With such an interpretation one can of course write

$$\int d^d\xi \sqrt{g} R^2 \sim \sum_{n_{d-2}} V_{n_{d-2}}(d) \left[\frac{2\delta_{n_{d-2}} V_{n_{d-2}}(d-2)}{V_{n_{d-2}}(d)} \right]^2 . \tag{2.25}$$

This definition must be interpreted with some care if we want convergence to the continuum value for a smooth manifold by successive subdivision [19]. We do not have to worry too much about these subtleties here since our task in the functional integral is not to approximate a given smooth manifold but to select some class of manifolds which can be used in an (approximate) evaluation of the integral. From this point of view we will use the R^2 term as representing typical higher derivative terms which one would have to insert in order to stabilize the Euclidean path integral as explained in the introduction. As is well known from for instance lattice gauge theories discretized versions of higher derivative terms are by no means universal. It is clear that this point of view is not as beautiful as the original geometrical way that Regge viewed piecewise flat manifolds, but our perspective is that a term like (2.25) will probe a universality class of theories which have an effective expansion in terms of higher derivative actions like (2.25).

In the case where we consider the piecewise flat manifolds which can be obtained by the process of dynamical triangulation as described above (2.25) simplifies and we get

$$\int d^d\xi \sqrt{g} R^2 \sim \sum_{n_{d-2}} o(n_{d-2}) \left[\frac{c_d - o(n_{d-2})}{o(n_{d-2})}\right]^2 . \tag{2.26}$$

This formula has a slight problem with the continuum interpretation since flat space does not have a regular tessellation except for $d = 2$. This means that the term can never scale to zero:

$$\sum_{n_{d-2}} o(n_{d-2}) \left[\frac{c_d - o(n_{d-2})}{o(n_{d-2})}\right]^2 \geq \text{const. } N_d . \tag{2.27}$$

If we introduce a scaling parameter a, which is to be identified with the link length, and which is going to be scaled to zero, the physical volume $V \equiv \int d^d\xi \sqrt{g}$ being kept fixed, and if we assume canonical scaling of the terms involved of both sides of (2.27) we get

$$\int d^d\xi \sqrt{g} R^2 \bigg|_{DT} > \frac{\text{const.}}{a^4} \int d^d\xi \sqrt{g} \bigg|_{DT} . \tag{2.28}$$

This means that the leading term on the lhs (2.27) is just a cosmological constant term and by expanding the lhs we see that it also contains an Einstein–Hilbert term *etc.*. Under the assumption of naive scaling we have to write instead of (2.26)

$$\sum_{n_{d-2}} o(n_{d-2}) \left[\frac{c_d - o(n_{d-2})}{o(n_{d-2})}\right]^2 \sim \int d^d\xi \sqrt{g} \left[\frac{c_0}{a^4} + \frac{c_1}{a^2} R + c_2 R^2 \cdots\right] . \tag{2.29}$$

Our lattice "R^2"-term is thus to be considered as a generalized higher derivative term which, when added to the lattice Einstein–Hilbert term, in addition will lead to a redefinition of the bare cosmological coupling constant and the bare gravitational coupling constant.

One interesting aspect of the dynamical triangulation approach is that for a finite lattice volume it automatically provides a cut off for the Einstein action. This is not the case for the conventional Regge calculus where the action can go to infinity without the volume diverging. The reason is that the volume (for $d > 2$) of the $d-2$-dimensional sub-simplices can diverge without the corresponding volume of the d-dimensional simplices going to infinity. In the case of dynamical triangulations we have (for $d > 2$)

$$-\text{const.} \cdot N_d < \sum_{n_{d-2}} (c_d - o(n_{d-2})) < \text{const.} \cdot N_d \,. \tag{2.30}$$

If we assume a conventional scaling in the tentative continuum limit we can rewrite (2.30) as

$$\left| \int d^d\xi \sqrt{g} R \right|_{DT} \leq \frac{\text{const.}}{a^2} \int d^d\xi \sqrt{g} \bigg|_{DT}, \tag{2.31}$$

where again we have have introduced the link length a, which is going to be scaled to zero while the physical volume $V \equiv \int d^d\xi \sqrt{g}$ is kept fixed.

2.4. Coupling to matter

The coupling to matter is incorporated in a simple way in the above formalism since all the d-dimensional simplices which constitute the building-blocks of the d-dimensional piecewise linear manifold have the same internal length scale.

We have considered two cases: the coupling of discrete spin systems to gravity (examplified by the Ising model coupled to gravity) and the coupling of continuous "spin" systems to gravity. In the latter case we used simply a real scalar field as our matter variable. There is an ambiguity in the possible assignment of matter variables to the underlying lattice. Should we for instance place the fields at the vertices of the lattice or in the centers of the d-dimensional simplices? On a regular lattice the choice should not be important and one would indeed expect the two theories to belong to the same universality class except if very special symmetry considerations play a role. In the case of dynamical triangulations the situation could be more complicated. For instance there is a priori no relation between the number of vertices and the number of d-simplices when $d > 2$. We have considered it most reasonable to assign the fields to the centers of the d-dimensional simplices. In this way we are sure that the number of independent fields will grow with the volume of our universe. Eventually one should of course check whether a coupling to vertices instead of d-simplices leads to different results.

The partition function in the case of the Ising model coupled to gravity will be

$$Z(\beta, \kappa_{d-2}, \kappa_d) = \sum_{N_d} e^{-\kappa_d N_d} \sum_{T\sim N_d} \sum_{[\sigma]} \exp\left(\kappa_{d-2} N_{d-2}\right) \exp\Big(\beta \sum_{<i,j>} (\delta_{\sigma_i \sigma_j} - 1)\Big) . \quad (2.32)$$

In this formula $T \sim N_d$ symbolizes the summation over all piecewise linear manifolds which can be formed by gluing together N_d (regular) tetrahedra such that the topology is that of S^d. $\sum_{[\sigma]}$ means the summation over all spin configurations while $\sum_{<i,j>}$ stands for the summation over all neighbouring pairs of d-simplices.

It is seen that the coupling of Ising spin to gravity in $d = 3$ and $d = 4$ is done in the same way as for the two-dimensional case. For $d = 2$ we know that the dynamical triangulated model coupled to Ising spins leads to critical exponents which agree with the ones calculated using continuum formalism. In $d = 2$ it is further known that only at the critical point β_c of the Ising model[2] do we find a coupling between gravity and spins which changes the critical properties of both $2d$ gravity and the spin system. This is in agreement with intuition: only at the critical point of the spin system will we have long range interactions which can influence the pure gravity system.

If we couple a scalar field to gravity we have formally no problem with critical behaviour since a scalar field on a lattice is automatically critical in the infinite volume limit as long as the coupling is purely Gaussian. From this point of view that coupling is more convenient than the spin coupling since we do not have an additional coupling constant like β, which will have to be fine-tuned to its critical value. Any coupling constant in front of the Gaussian scaled field can be scaled out and absorbed in a redefinition of the cosmological constant in the functional integral. Note that this would not a priori be possible if we had attached the scalar field to the vertices rather than the d-simplexes. The partition function in the case of a Gaussian scalar field coupled to gravity is now given by:

$$Z(\kappa_{d-2}, \kappa_d) = \sum_{N_d} \exp\left(-\kappa_d N_d\right) \sum_{T\sim N_d} \exp\left(\kappa_{d-2} N_{d-2}\right) \times \int \prod_{i=1}^{N_d} d\phi_i \, \exp\Big(- \sum_{<i,j>} (\phi_i - \phi_j)^2\Big) \quad (2.33)$$

[2] By β_c we mean here the critical point after coupling to gravity. It differs from the critical point for a fixed lattice.

and the notation is the same as in Eq. (2.32).

3. Observables

In the section above we defined the model. Let us here discuss the observables (see also [20] for a more general discussion).

3.1. Observables associated with gravity itself

Due to diffeomorphism invariance and the fact that we in quantum gravity have to integrate over all Riemannian manifolds the observables which are most readily available are averages of invariant local operators like the curvature $R(x)$ and suitable contractions of powers of the curvature tensor, like $R^2_{\mu\nu}$ or $R^2_{\mu\nu\lambda\sigma}$, and the fluctuation of these averages. In addition one can discuss and measure so-called fractal properties of space-time and finally with some effort define the concept of correlators of local invariant operators.

The simplest observable is the average curvature. If we consider the discretized partition function we have

$$\int d^d\xi \sqrt{g(\xi)} R(\xi) \propto \sum_{n_{d-2}} o(n_{d-2}) \frac{c_d - o(n_{d-2})}{o(n_{d-2})} = c_d N_{d-2} - \frac{d(d+1)}{2} N_d \tag{3.1}$$

and since the volume is

$$\int d^d\xi \sqrt{g(\xi)} \propto \frac{2}{d(d+1)} \sum_{n_{d-2}} o(n_{d-2}) = N_d \,, \tag{3.2}$$

we can define the average curvature per volume as

$$\langle R \rangle = \frac{\int d^d\xi \sqrt{g(\xi)} R(\xi)}{\int d^d\xi \sqrt{g(\xi)}} \propto \frac{2c_d}{d(d+1)} \frac{N_{d-2}}{N_d} - 1 \,. \tag{3.3}$$

In (3.3) $\langle R \rangle$ is defined for a single manifold. We get of course the quantum version by calculating the functional average of $\langle R \rangle$ over all Riemannian manifolds, weighted by e^{-S}. This is what we will do numerically. The average curvature is a bulk quantity which will allow us to get a quick survey of the phase diagram of four-dimensional quantum gravity. In a similar way we can define (with the drawbacks described in the last subsection) $\langle R^2 \rangle$ by

$$\langle R^2 \rangle = \frac{\int d^d\xi \sqrt{g(\xi)} R^2(\xi)}{\int d^d\xi \sqrt{g(\xi)}} \propto \frac{\sum_{n_{d-2}} o(n_{d-2}) \left[\frac{c_d - o(n_{d-2})}{o(n_{d-2})}\right]^2}{N_d} \,. \tag{3.4}$$

Again this average is defined over a single manifold and we have eventually to take the weighted average over all manifolds in our ensemble. A quantity which will have our interest will be $\langle R^2 \rangle - \langle R \rangle^2$.

A more refined, but related observable is the integrated curvature-curvature correlation. In a continuum formulation it would be

$$\chi(\kappa_{d-2}) \equiv \left\langle \int d^d\xi_1 \, d^d\xi_2 \, \sqrt{g(\xi_1)}R(\xi_1) \, \sqrt{g(\xi_2)}R(\xi_2) \right\rangle - \left\langle \int d^d\xi \sqrt{g(\xi)}R(\xi) \right\rangle^2 . \quad (3.5)$$

In a lattice regularized theory one would expect that away from the critical points short range fluctuations will prevail, while approaching the critical point long range fluctuation might be important and would result in an increase in $\chi(\kappa_{d-2})$. The observable $\chi(\kappa_{d-2})$ is the second derivative of the free energy $F = -\ln Z$ with respect to the gravitational coupling constant G^{-1}. In the case where the volume N_d is kept fixed we see that

$$\chi(\kappa_{d-2}, N_d) \sim \langle N_{d-2}^2 \rangle_{N_d} - \langle N_{d-2} \rangle_{N_d}^2 = -\frac{d^2 \ln Z(\kappa_{d-2}, N_d)}{d\kappa_{d-2}^2} . \quad (3.6)$$

From the above discussion we have to look for points along the critical line $\partial\mathcal{D}$ where $\chi(\kappa_{d-2}, N_d)/N_d$ diverges in the infinite volume limit $N_d \to \infty$.

An independent susceptibility is the one associated with volume fluctuations:

$$\chi_V(\kappa_{d-2}, \kappa_d) \sim \langle N_d^2 \rangle - \langle N_d \rangle^2 = -\frac{d^2 \ln Z(\kappa_{d-2}, \kappa_d)}{d\kappa_d^2} . \quad (3.7)$$

Assume that $Z(\kappa_{d-2}, N_d)$ has the form

$$Z(\kappa_{d-2}, N_d) \sim N_d^{\gamma(\kappa_{d-2})-3} e^{\kappa_d^c(\kappa_{d-2})N_d} \left(1 + O(1/N_d)\right) . \quad (3.8)$$

This is the case for $d = 2$. For higher d it is of course a necessity for the existence of the model that $Z(\kappa_{d-2}, N_d)$ is exponentially bounded, but it is by no means clear that subleading corrections should appear in the form of a power-law correction to the exponential factor. If it nevertheless does, we can identify $\gamma(\kappa_{d-2})$ with the critical exponent for the volume fluctuations at the critical point $\kappa_d^c(\kappa_{d-2})$:

$$\chi_V(\kappa_{d-2}, \kappa_d) \sim \frac{1}{(\kappa_d - \kappa_d^c(\kappa_{d-2}))^{\gamma(\kappa_{d-2})}} , \quad (3.9)$$

for κ_d close to κ_d^c.

Another observable is the Hausdorff dimension. One can define the Hausdorff dimension in a number of ways, which are not necessarily equivalent. Here we will simply measure the average volume $V(r)$ contained within a radius r from a given point. In [20] the concept of a *cosmological Hausdorff dimension* d_{ch} was defined. It essentially denotes the power which relates the average radius of the ensemble of universes of a fixed volume to this volume:

$$\langle \text{Radius} \rangle_{N_d} \sim N_d^{1/d_{ch}} . \tag{3.10}$$

From the distribution $V(r)$ we can try to extract d_{ch}. If for large r we have the behaviour

$$V(r) \sim r^{d_h} , \tag{3.11}$$

we can identify d_{ch} and d_h.

By the use of Regge calculus it is straightforward to convert these continuum formulas to our piecewise flat manifolds. Between two points in the piecewise flat manifold there is a geodesic which is a piecewise linear path. Rather than using this definition we will use approximations which are much more convenient from a numerical point of view and which one expects should be sufficient for the purpose of extracting general scaling behaviour and fractal properties: We define the distance between two vertices as the length of the shortest path along links which connects the two vertices, *i.e.* it is essentially the number of links of the shortest path since all links have the same length. We call this distance the "n_1"-distance between vertices and denote its value by l_1. In the same way we can define a shortest path between two d-simplices as the length of the shortest path obtained by moving between the centers of neighbouring d-simplices. We call this distance the "n_d"-distance between d-simplices and denote its value by l_d. The dual graph to a given triangulation, obtained by connecting the centers of neighbouring d-simplices, will be a ϕ^{d+1}-graph and the n_d-distance on the triangulation will be the n_1-distance on the dual ϕ^{d+1}-graph. A priori these distances are not related and it is easy to find triangulations where they differ vastly for specific choices of vertices and associated d-simplices. But for averages over vertices and over triangulations one would expect that they carry the same information about the geometry and we shall see that this is indeed the case.

With the n_1- and n_d-definitions of geodesic distances on the triangulations it becomes trivial to measure numerically relations like (3.11). As we shall see it is less trivial to extract in a reliable way the Hausdorff dimension d_h.

Let us finally discuss the measurements of correlation functions in quantum gravity. If we have local invariant operators $O_1(x)$ and $O_2(x)$ (like for instance $R(x)$) we can define a correlation function of geodesic distance d

by

$$\tilde{G}(r) = \int\int d^d\xi_1 \sqrt{g(\xi_1)} d^d\xi_2 \sqrt{g(\xi_2)} O_1(\xi_1) O_2(\xi_2) \delta(d(\xi_1, \xi_2) - r). \quad (3.12)$$

In this definition $d(\xi_1, \xi_2)$ denotes the geodesic distance between ξ_1 and ξ_2 for a given manifold, but (3.12) makes sense also as a functional average, and therefore in principle in quantum gravity. When we discuss numerical data we will always have in mind the functional average. It might be convenient to divide by a volume element to get the dimension of a point-point correlation function. If $V(r)$ denotes the volume inside a "ball" of radius r we can write

$$dV(r) = V'(r)dr$$

and we define

$$G(r) = \tilde{G}(r)/V'(r). \quad (3.13)$$

In the case where we have a finite Hausdorff dimension the exponential fall off of $G(d)$ and $\tilde{G}(d)$ are identical, but it needs not be the case if we have an infinite Hausdorff dimension.

3.2. Observables associated with matter fields

When matter fields are coupled to gravity we have access to additional observables associated with the these fields. Again reparametrization invariance will limit the class of observables we can define.

We can still discuss thermodynamic functions and their possible critical behaviour. One could define the spin susceptibility for the Ising model coupled to gravity as for the Ising model on the regular lattice. It is however more difficult to relate in a clear manner the divergence of such a quantity to a divergent correlation length since we do not have in a clear manner the concept of a two point correlation function. We can imitate the definition of correlation functions for a fixed lattice in the following way (using the Ising spin as an example): The length between two points, measured as the geodesic distance, is invariant under diffeomorphisms. In the discretized version this means that the geodesic "n_d" distance, as defined above, between two d-simplexes where spin is located should be considered as an invariant. Since we are instructed to sum over all manifolds in the functional integral it makes no sense to talk about the spin-spin correlation function between point (*i.e.* d-simplex) i and point (i.e. d-simplex) j. It does however make sense to talk about the average of such spin-spin correlations over all pairs of points in the manifold separated by a geodesic distance r and averaged over all manifolds, the geodesic distance on each manifold being calculated according to the metric on that manifold. In this way we are lead to define the spin-spin correlation function as

$$g(r) \equiv \left\langle \frac{1}{V_{\text{total}}} \sum_j \frac{1}{dV_j(r)} \sum_{i \in dV_j(r)} \sigma_i \sigma_j \right\rangle, \tag{3.14}$$

where $V_j(r)$ is the volume of a ball of ("n_d") radius r around the simplex j and $dV_j(r)$ denotes the differential volume (in the discretized version) $dV_j(r) = V_j(r) - V_j(r-1)$. Finally V_{total} denotes the total volume of the particular universe in the average. An alternative correlation function would be

$$G(r) \equiv \left\langle \frac{1}{V_{\text{total}}} \sum_j \sum_{i \in dV_j(r)} \sigma_i \sigma_j \right\rangle \tag{3.15}$$

which is related to the magnetic susceptibility $\chi(\beta)$ by

$$\sum_r G(r) = \chi(\beta) \sim |\beta - \beta_c(\kappa_{d-2})|^{-\gamma} \quad \text{for} \quad \beta \to \beta_c(\kappa_{d-2}). \tag{3.16}$$

In principle one can extract the mass gap $m(\beta)$ from the exponential fall off of $g(r)$ and in this way determine the critical exponent ν defined by $m(\beta) = |\beta - \beta_c|^{\nu}$. This seems however difficult to do in a reliable way, in accordance with the experience in two-dimensional quantum gravity coupled to Ising spins, and it is maybe understandable if one keeps in mind that not only is a precise determination of β_c needed in order to extract ν. In addition our data are folded into the distribution $dV(r)$ which determines the Hausdorff dimension, a quantity which by itself is very difficult to measure in a reliable way.

In the same way we can construct $\chi(\beta)$ from $G(r)$, but it does not lead to a precise determination of the critical exponent γ (again in agreement with the experience from two dimensions).

One might instead of measuring the correlation functions directly try to extract results from finite size scaling relations. In two-dimensional gravity this approach seems to work somewhat better than the direct attempts to measure ν and γ mentioned above. The disadvantage of the method is that it only gives us certain combinations of the exponents.

If a given thermodynamic function F has a critical behaviour

$$F(\beta) \sim (\beta - \beta_c)^{-x}, \tag{3.17}$$

one expects in ordinary flat space a finite size dependence of the form

$$F(\beta, L) = L^{x/\nu} f(|\beta - \beta_c| L^{1/\nu}), \tag{3.18}$$

where L denotes the linear size of the system and the exponent ν is determined by the divergence of the correlation length $\xi(\beta)$:

$$\xi(\beta) \sim |\beta - \beta_c|^{-\nu} . \tag{3.19}$$

By a measurement of $F(\beta_c, L) \sim f(0)L^{\frac{x}{\nu}}$ as a function of L we can determine the combination x/ν, while measurements away from β_c would give us x directly for sufficiently large L.

If we want to use these formulae for systems coupled to quantum gravity we must identify the divergent correlation length ξ in terms of geodesic distances, as was already discussed above. We further have to identify the linear extension L. If the system has a finite Hausdorff dimension d_H it is tempting to define

$$L \sim N_d^{1/d_h} . \tag{3.20}$$

These ideas have been used with some success in 2d-quantum gravity [31–33], but could be spoilt if the Hausdorff dimension is infinite.

So far we have discussed spin systems, but similar considerations can be carried over to scalar fields coupled to gravity and in principle one can go beyond the Gaussian approximation discussed so far and include a self-interaction of the scalar field. We have not yet done that.

4. The numerical method

4.1. *The gravity sector*

Unfortunately the analytic methods of two dimensional simplicial gravity have not yet been extended to higher dimensions. The numerical method of "grand canonical" Monte Carlo simulation, which is well tested in two dimensions [21–23], has recently been applied to three dimensions [24–28] and in four dimensions [39–42]. A necessary ingredient for Monte Carlo simulations in simplicial quantum gravity is a set of so-called "moves", *i.e.* local changes of the triangulations, which are ergodic in the class of triangulations we consider. A general set of moves in any dimension has been known for a long time [43]. They are however not optimal for numerical simulations. A more convenient set of moves for higher dimensional gravity was suggested in [25]. The ergodicity of these moves in three dimensions was proved in [26], and the generalization of this proof to $d = 4$ was given in [29]. In d dimensions there are $d+1$ moves. Their general description is as follows: remove an i-dimensional simplex of order $d+1-i$ and the higher dimensional simplices of which it is part, and replace it by a $d-i$-dimensional simplex ("orthogonal" to the removed i-dimensional simplex) plus the appropriate higher dimensional simplices such that we still have a triangulation. Such moves will be allowed provided they do not violate (2.5) and provided they do not create simplices already present, *i.e.* simplices which already have the same vertices as the newly created simplices.

Let us consider (as an example) four dimensions, where there are five moves. The first move consists of removing a four-dimensional simplex $n_4(old)$ and inserting a vertex $n_0(new)$ in the void interior and adding links (and the induced higher dimensional simplices) which connect $n_0(new)$ to the five vertices of $n_4(old)$. In this way $n_4(old)$ is replaced by five new n_4's and the total change of N_i's is

$$\Delta N_0 = 1, \quad \Delta N_1 = 5, \quad \Delta N_2 = 10, \quad \Delta N_3 = 10, \quad \Delta N_4 = 4\,. \tag{4.1}$$

The second move consists of removing a three-dimensional simplex n_3 and the two n_4's sharing it, and then inserting a link ("orthogonal" to n_3) and associated n_2's, n_3's and n_4's. The total change of N_i's is in this case

$$\Delta N_0 = 0, \quad \Delta N_1 = 1, \quad \Delta N_2 = 4, \quad \Delta N_3 = 5, \quad \Delta N_4 = 2\,. \tag{4.2}$$

The third move is a "self-dual" move where $\Delta N_i = 0$. It consists of removing a triangle of order three and associated higher dimensional simplices and inserting the "orthogonal" triangle and its associated higher dimensional simplices such that we still have a triangulation. The fourth move is the inverse of the second move, while the fifth move is the inverse of the first.

The change in the action induced by these moves can now readily be calculated and we are in a position to use the standard Metropolis updating procedure. The weights required for detailed balance are easily determined. Let us only remark here that the nature of the problem naturally suggests to use indirect addressing by pointers since there is no rigid lattice structure. In addition we found it most efficient to keep pointers to vertices of order five, links of order four and triangles of order three since these are the ones used in the updating. Since programs of the above nature are not well suited for vectorization it is optimal to run them on fast workstations or massive parallel machines with independent processors.

Since we are forced to use a grand canonical updating where the volume of the universe N_4 is changing, it is convenient to use the technique first introduced in [30] and used successfully in the simulations in three-dimensional gravity [24, 27]. It allows us to get as close to a canonical simulation as possible and it provides at the same time an estimate of the critical point $\kappa_4^c(\kappa_2)$ for a given value of the coupling constant κ_2. The idea is the following: Assume we want to perform a measurement at some fixed value $N_4(F)$. The task is to constrain the fluctuations of N_4 to the neighbourhood of $N_4(F)$ without violating ergodicity. First we make an approximate estimate of the critical point κ_4^c, which we denote $\kappa_4^c(N_4(F))$.

It can in principle depend on $N_4(F)$. Next we choose the the actual κ_4 used in the simulation as a function of the value of N_4 in the following way:

$$\kappa_4(N_4) = \begin{cases} \kappa_4^c(N_4(F)) - \Delta\kappa_4 & \text{for} \quad N_4 < N_4(F) \\ \kappa_4^c(N_4(F)) + \Delta\kappa_4 & \text{for} \quad N_4 > N_4(F) . \end{cases} \tag{4.3}$$

For sufficiently large values of $N_4(F)$ and small values of $\Delta\kappa_4$ we will get an exponential distribution of N_4's peaked at $N_4(F)$:

$$P(N_4) \sim \exp\left((N_4 - N_4(F))(\kappa_4^c(N_4) - \kappa_4)\right) . \tag{4.4}$$

By monitoring $\Delta\kappa_4$ we can effectively control the width of the distribution of N_4 around $N_4(F)$ without violating the principle of ergodicity. A measurement of the exponential distribution also allows us to determine κ_4^c. If the exponential fall off is different above and below $N_4(F)$ it means that κ_4^c is different from our guess $\kappa_4^c(N_4(F))$ and we can use the optimal value in the next run. Further the measurements of κ_4^c for different values of $N_4(F)$ allow us to extract the subleading correction to the distribution. Assume that the partition function has the form:

$$Z(\kappa_2, \kappa_4) = \sum_{N_4} Z(\kappa_2, N_4) e^{-\kappa_4 N_4} , \tag{4.6}$$

where

$$Z(\kappa_2, N_4) \sim N_4^{\gamma(\kappa_2)-2} \exp\left(\kappa_4^c(\kappa_2) N_4\right) \left(1 + O(\tfrac{1}{N_4})\right) . \tag{4.6}$$

By our method the critical point κ_4^c determined by measurements in the neighbourhood of various $N_4(F)$'s would lead to:

$$\kappa_4^c(\text{measurement}) = \kappa_4^c(\kappa_2) + \frac{\gamma(\kappa_2) - 2}{N_4(F)} \tag{4.7}$$

and a determination of the entropy exponent $\gamma(\kappa_2)$, which in fact governs the volume fluctuations of the system, as already mentioned.

4.2. The matter field sector

While the algorithm for updating of the gravity sector is new and unconventional from the point of view of lattice field theory, the matter fields coupled the geometrical system can be updated using standard techniques. The scalar field will just be updated by the ordinary Metropolis algorithm, while the spin updating is performed by the single cluster variant of the Swendsen–Wang algorithm developed by Wolff [36]. The cluster updating algorithms have been successfully applied to the Ising model coupled to 2d gravity [31–34] and to the ordinary three dimensional Ising model [35].

5. Numerical results

Let us now give an overview of the numerical results obtained so far.

5.1. 3d gravity

In this case there are two coupling constants κ_1 and κ_3. It was established that the number of triangulations of S^3 seems to grow exponentially, but on the other hand it seemed that the subleading corrections were not of the form (3.8), at least for a certain range of κ_1 to be discussed below. The phase-diagram as a function of κ_1 was non-trivial. At a finite value κ_1^c of κ_1 ($\kappa_1^c \approx 4.0$) there was a phase transition between two types of geometry. For $\kappa_1 < \kappa_1^c$ the average universe appeared to be highly connected with a small radius which was almost independent of the volume. The Hausdorff dimension (if finite at all) is quite large in this region, which was denoted the "hot" phase. It was in this region that the subleading correction had a form different from (3.7). For $\kappa_1 > \kappa_1^c$ the geometry appeared to be essentially one-dimensional. Very elongated structures dominate in this coupling constant region, which was called the "cold" phase. Very pronounced hysteresis was observed at the phase transition and it was classified as a first order transition which could not be used for the purpose of defining a continuum limit, at least not in an obvious way. Further the transition took place a a non-zero value of the "bare" curvature, a fact with presents an further obstacle for defining in a naive way the continuum limit at this transition point.

5.2. 4d gravity, including also an R^2 term

Let us now discuss in slightly more detail the extensive simulations we have done over the last half year in four dimensions [37]. The model studied is given by the following action

$$S = \kappa_4 N_4 - \kappa_2 N_2 + \frac{h}{c_4^2} \sum_{n_2} o(n_2) \left\{ \frac{c_4 - o(n_2)}{o(n_2)} \right\}^2 \qquad (5.1)$$

The measurements of average curvature *etc.* were performed for different values of N_4: 4000, 9000, 16000 and 32000. The number of attempted updatings were of the order $10,000 \times N_4$ (sometimes considerably larger at critical points where thermalization was slow). The phase diagram was scanned varying the (inverse) bare gravitational coupling constant κ_2 and the "R^2" coupling constant which we denote h. For each value of κ_2 and h and each value of N_4 this implies a fine-tuning of the value of the bare "cosmological" coupling constant κ_4 to its critical value $\kappa_4^c(\kappa_2, h, N_4)$. As explained in the last section it is convenient to perform the simulations in the neighbourhood of some fixed value of the volume so we choose a specific $N_4(F)$ and limit the fluctuations in volume to some neighbourhood of

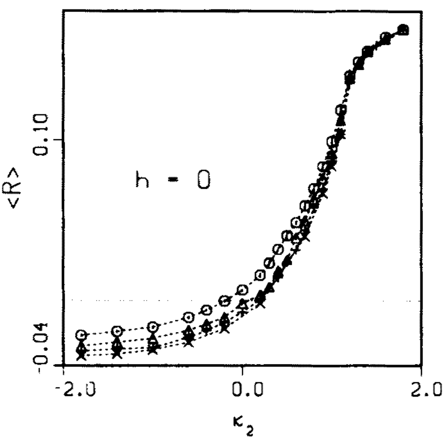

Fig. 1. The average curvature $\langle R\rangle(\kappa_2)$ for $h = 0$ and $N_4 = 4000(\bigcirc)$, $N_4 = 9000(\triangle)$, $N_4 = 16000(+)$ and $N_4 = 32000(\times)$.

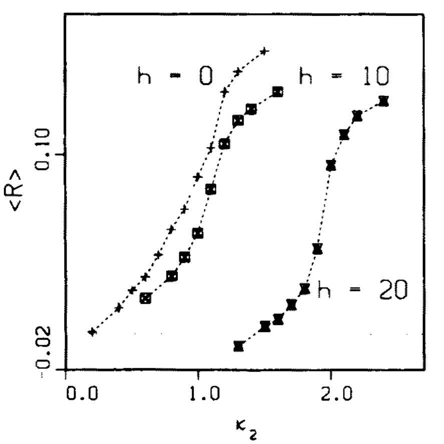

Fig. 2. The average curvature $\langle R\rangle(\kappa_2)$ for $h = 0, 10$ and 20 and $N_4 = 16000$.

$N_4(F)$. If we decide to perform the measurements after a given number, n, of Monte Carlo sweeps, in practise we perform the actual measurement the first time, after the n'th sweep, the system passes a state where the value of N_4 is equal to $N_4(F)$. This is what we mean when we say that the measurements were performed for a given value of N_4.

The result for the average curvature is shown in Fig. 1 for $h = 0$ over a large range of κ_2, while Fig. 2 shows the average curvature for different values of h, ranging from $h = 0$ to $h = 20$ and $N_4 = 16000$. It is not possible with our choice of R^2 term to increase h further since the acceptance rates in the Metropolis updating become too small.

We observe the following: In case there is no coupling constant except the cosmological coupling constant the average curvature $\langle R \rangle \approx 0$, which is a nice result since it shows that the selection of manifolds in the context of dynamical triangulations has no serious bias towards positive or negative curvature. In case we take κ_2 positive the average curvature will be positive (this corresponds to the conventional sign of the gravitational coupling constant). If we take κ_2 negative ("anti-gravity") the average curvature will be negative.

For $h = 0$ and $\kappa_2 \approx 1.1$ we see a change towards large positive curvature. The same change is seen for $h > 0$, only we have to go to larger values of κ_2 when $h > 10$. For a fixed positive value of κ_2 the curvature decreases with increasing h as expected. However, as discussed above, the limit $h \to \infty$ does not really correspond to zero local curvature due to the special form of the our discretized "R^2"-term. The absolute minimum value of the discretized term corresponds to a constant negative curvature: $R = -0.046$.

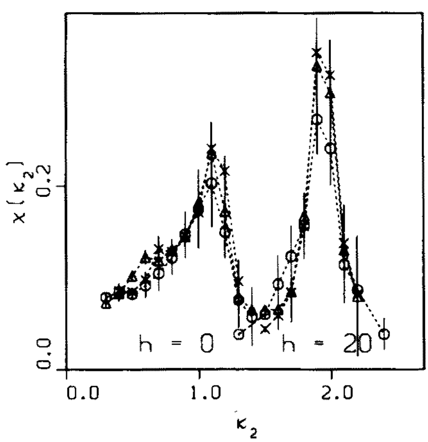

Fig. 3. Susceptibility $\chi(\kappa_2)$ for $h = 0$ and 20 and $N_4 = 4000(\bigcirc), N_4 = 9000(\triangle)$ and $N_4 = 16000(\times)$.

In Fig. 3 we have shown the susceptibility defined by (3.6). It can be measured directly, or as the derivative of the average curvature. We have used the second method, but have also measured the susceptibility directly, with comparable results. One sees a clear peak which grows somewhat with volume. This could be taken as a sign that the system becomes critical in this region, although the system size is not big enough to exclude the possibility of a phase transition of higher order. An independent signal of criticality is found by looking at the observable $\langle R^2\rangle - \langle R\rangle^2$.

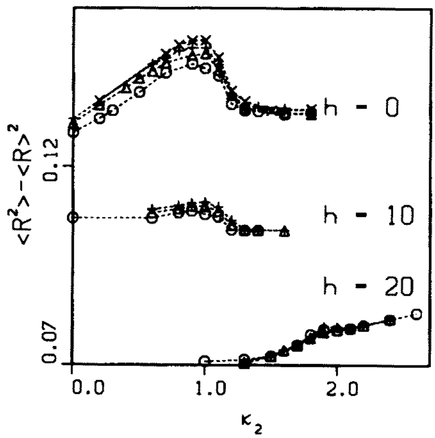

Fig. 4a. $\langle R^2\rangle - \langle R\rangle^2$ as a function of κ_2 for $h = 0, 10$ and 20 and $N_4 = 4000(\bigcirc)$, 9000 ($\triangle$), 16000(+) and 32000 ($\times$).

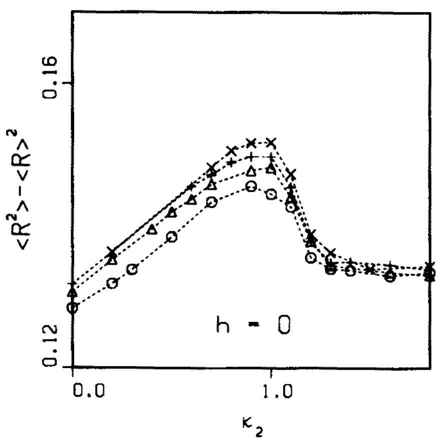

Fig. 4b. $\langle R^2\rangle - \langle R\rangle^2$ as a function of κ_2 for $h = 0$ and $N_4 = 4000(\bigcirc)$, 9000 ($\triangle$), 16000 (+) and 32000 ($\times$).

In Fig. 4a we show its behaviour as a function of κ_2 for h between 0 and 20.

For small h values we see a clear volume dependent peak in the region where the κ_2 susceptibility has a peak too. We note the clear asymmetry between the two sides of the peak, especially for $h = 0$ (Fig. 4b). This explains why the position of the peak seems to be shifted towards smaller values of κ_2 when compared to the susceptibility curve. Here again our system is not big enough to exclude the possibility that the increase with volume is only a finite size effect and that eventually we shall observe only a discontinuity in the derivative at a critical point, which again could signal a phase transition of higher (perhaps 3-rd) order. The signal deteriorates somewhat for large values of h, contrary to the susceptibility signal.

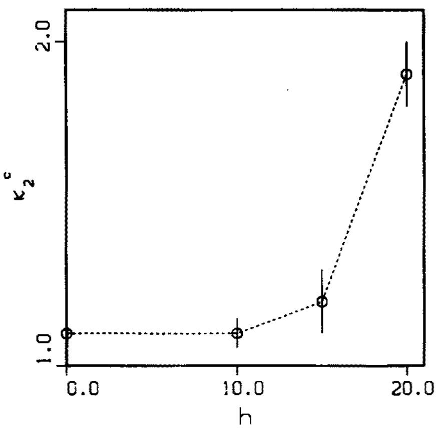

Fig. 5. Critical line $\kappa_2^c(h)$.

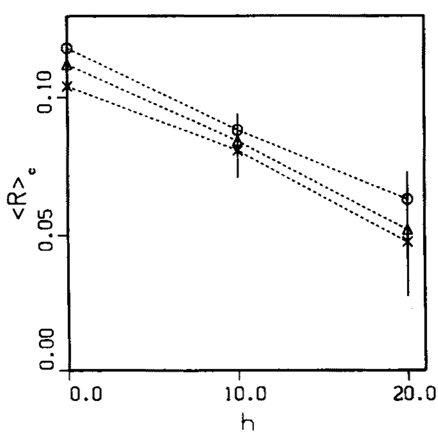

Fig. 6. Average curvature at the phase transition for $N_4 = 4000(\bigcirc), 9000(\triangle)$ and $16000(\times)$.

We can draw a critical line in the (κ_2, h)-coupling constant plane, Fig. 5, and Fig. 6 shows the average curvature at the transition point as a function of h for $N_4 = 4000$, 9000 and 16000. We see that the average curvature gets smaller when h increases and also when N_4 increases, *but for all the values we have been able to probe we have*

$$\langle R \rangle_c \equiv \langle R(\kappa_2^c, h) \rangle > 0 \tag{5.2}$$

and we can not conclude that $\langle R \rangle_c > 0$ is a finite volume effect. The data indicate rather that $\langle R \rangle_c$ remains positive even for $h \to \infty$. For $h > 12$ there seems to be a qualitative change in the distributions, which might indicate a different transition, but we have not found that it cured the problems of the $h = 0$ situation.

As noted above the simulations in three dimensions revealed a similar situation: A transition and a $\langle R \rangle_c > 0$. In three dimensions there was a very strong hysteresis in the same transition, favouring a first order transition. Here we have not seen the same strong hysteresis. For $N_4 = 4000$ there was no problem moving from one phase to the other. For larger N_4 we have observed very slow thermalization and huge fluctuations in geometry close to $\kappa_2 = \kappa_{2c}$, but it did not present itself as clear hysteresis.

Let us now explore the change in geometry along the critical line. Above we have defined the "geodesic" link distance d_1 between vertices and the four-simplex distance d_4 between four-simplices. The average values describe typical radii of our universes. They are shown in Fig. 7 ($\langle d_1 \rangle$) and Fig. 8 ($\langle d_4 \rangle$). Although the d_4 distances are approximately six times larger that the d_1 distances they clearly behave qualitatively in the same way and reveal a drastic change in the geometry as we pass the critical region of κ_2. The nature of the change seems to be independent of h.

The typical universes generated by the computer simulations have small radii, almost independent of the volume if we are below the critical κ_2 region. After we have passed the critical region the radii become quite large and show a very clear volume dependence. In fact it seems as if the radius grows almost linearly with volume. *Qualitatively this implies that the Hausdorff dimension is large below the critical region and small (in fact close to one) above the critical region.* We have not attempted to determine the larger Hausdorff dimension. The growth in radius with volume is so small that one has to go to much larger volumes in order to do it in a reliable way.[3]

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[3] It has been argued that one should not use the "geometrical method" advocated here as a measure of the Hausdorff dimension, but extract it from the random walk representation of the massive propagator, since this seems to give more "reasonable" values. We disagree with this point of view. The two methods are mathematically equivalent [44] and disagreement reflects in our

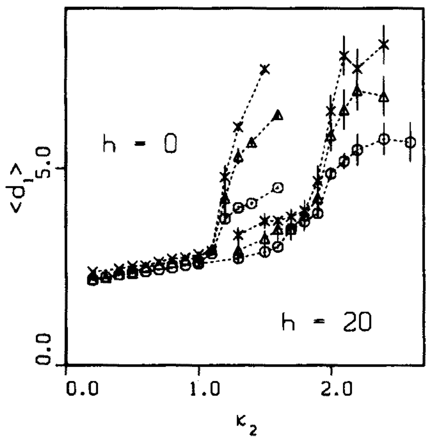

Fig. 7. Average d_1 distance for $h = 0$ and 20 and for $N_4 = 4000(\bigcirc), 9000(\triangle)$ and 16000($\times$).

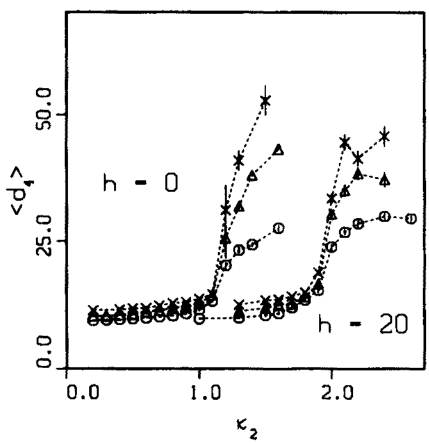

Fig. 8. Average d_4 distance for $h = 0$ and 20 and for $N_4 = 4000(\bigcirc), 9000(\triangle)$ and 16000($\times$).

We get a nice representation of the change in geometry between the two phases by showing the actual distribution of geodesic length in the universes. This is done in Fig. 9 for the d_4 distances for $h = 0$. If space-time has a fractal structure with some Hausdorff dimension d_h the distribution should be like

$$P(d) \sim d^{d_h - 1}\,. \tag{5.3}$$

In Fig. 9 we have shown four curves which correspond to $\kappa_2 = 1.0$, 1.1,

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opinion the fact that it is not possible to extract the Hausdorff dimensions with the desired precision.

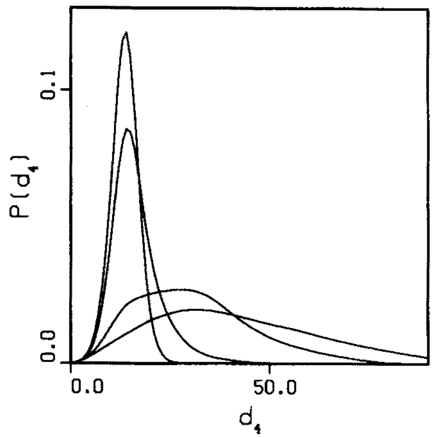

Fig. 9. Distribution of the d_4 distances for $h = 0$ and $\kappa_2 = 1.0, 1.1, 1.2$ and 1.3.

1.2 and 1.3 in the critical region. Since the transition is smooth and extrapolates from large to small d_h it is of course possible to find a κ_2 in the transition region where we get a curve quite similar to (5.3) with $d_h \approx 4$ which is of course amusing, but we do not consider the value as especially well determined by the numerical simulations.

One disturbing aspect of our results if we want to give the above mentioned phase transition a continuum interpretation is the fact that $\langle R \rangle_c > 0$. If we take this result literally it is difficult to attribute any sensible naive continuum scaling to the system. If we introduce a scaling parameter a, which conveniently can be identified with the link length in the triangulation the simplest scaling would be one where $a \to 0$ while the volume $a^4 N_4$ was kept fixed. One would then expect the following relation between the "bare" lattice curvature and the continuum curvature:

$$\langle R \rangle(\text{lattice}) = R(\text{continuum})a^2 \tag{5.4}$$

which shows that if R(continuum) should remain finite in the scaling limit $a \to 0$ $\langle R \rangle$ *must* scale to zero. It does not. We have found no way to repair this and although it is possible to find a scaling

$$|\langle R(\kappa_2, h)\rangle - \langle R(\kappa_2^c, h)\rangle| \sim |\kappa_2 - \kappa_2^c|^{\delta - 1} , \tag{5.5}$$

its significance is not clear to us due to $R_c > 0$.

The appearance of $R_c > 0$ for $h = 0$ was one of the motivations to look at theories with higher curvature terms. The other motivation was to investigate the question of universality in the spirit outlined by Weinberg, as explained in the introduction. However, our results are negative in the sense that even if R_c indeed decreases with increasing h, it does not go to zero and for not too large values of h we clearly are in the same universality class as for $h = 0$. Distributions, the nature of the transition *etc.* seem to be the same except for a displacement in κ_4 and κ_2. As mentioned before, for $h > 12$ we see a qualitative change in the distributions, which might indicate a different transition. This point requires however further investigation.

5.3. Coupling to matter fields

5.3.1. The Ising model coupled to 3d gravity

As mentioned above pure three-dimensional simplicial quantum gravity has two phases depending on the value of κ_1. In the following we report on the results obtained when gravity is coupled to an Ising spin [38]. The first statement we can make is that this is unchanged by the coupling to Ising spin.

In the "hot" phase ($\kappa_1 \leq 4.0$) where the Hausdorff dimension is large the numerical value of the magnetization

$$|\sigma| \equiv \frac{1}{N_3}\left|\sum_{i=1}^{N_3} \sigma_i\right| \tag{5.6}$$

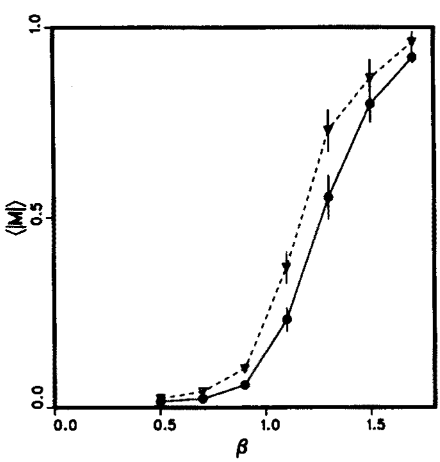

Fig. 10. The magnetization $|\sigma|$ (defined by (5.6)) as a function of β in the "hot" phase ($\kappa_1 = 3.7$) for $N_3 = 4000$ (Δ) and $N_3 = 10000$ ($\circ$).

is shown in Fig. 10 as a function of β. We see a clear signal indicating a phase transition from a disordered phase (small β) where $|\sigma| \approx 0$ to an ordered phase (large β) where $|\sigma| \approx 1$. The transition becomes sharper with increased volume N_3 and seems to be a second order transition. This

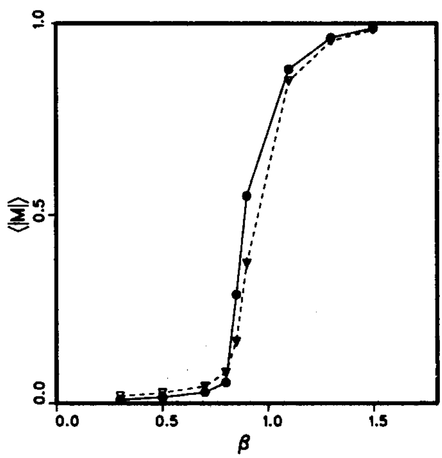

Fig. 11. The magnetization $|\sigma|$ (defined by (5.6)) as a function of β in the "cold" phase ($\kappa_1 = 4.4$)for $N_3 = 4000$ ($\triangle$) and $N_3 = 10000$ ($\circ$).

situation is contrasted by the magnetization curve in the "cold" phase shown in Fig. 11. Here is only a gradual cross over to $|\sigma| \approx 1$ for large β, and the cross over is weakened for increased volume N_3. The situation is precisely as one would expect in the case of a one-dimensional system where there is no spontaneous magnetization. We conclude that the Hausdorff dimension $d_h \approx 1$ measured in the pure gravity case seems to reflect correctly the dimension relevant for coupling to matter.

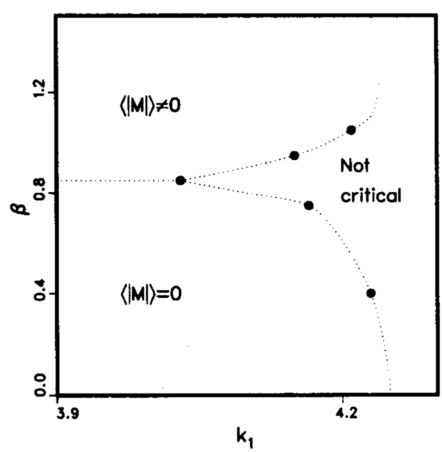

Fig. 12. The phase diagram for 3d quantum gravity coupled to matter. Filled circles are results obtained for $N_3 = 10000$.

In the $\kappa_1 - \beta$ plane we have the phase-diagram shown in Fig. 12. If β is away from the critical value $\beta_c(\kappa_1)$ (which has only a weak dependence on κ_1) the coupling between the fluctuations in geometry and spin seems weak and of course it vanishes in the limits $\beta \to \infty$ and $\beta \to 0$. In these limits we therefore have a strong first order transition between the "hot" and the "cold" phase of three-dimensional quantum gravity, precisely as is the case in the absence of spins [28]. In the "hot" phase, where the Ising system has a second order transition, we have seen an increased coupling between geometry and spins when we approach the critical $\beta_c(\kappa_1)$. This is shown in Fig. 13 where we plot the average curvature $\langle R \rangle$ as a function of β. A clear peak is seen at β_c. This enhanced coupling between geometry and spins at the critical point is qualitatively in agreement with the 2d results, where we have a change in the string susceptibility exponent γ_{string} (not to be confused with the magnetic susceptibility exponent γ_{mag}) from the pure gravity value $-1/2$ to $-1/3$, precisely when $\beta = \beta_c$. Unfortunately it is not clear, as mentioned above, that the entropy exponent analogous to γ_{string} exists in the hot phase of three-dimensional quantum gravity ([24, 26–28]) so we have no obvious exponent with which we can compare the effect of the spin coupling, but the enhanced coupling between spin and geometry leaves open the possibility that the transition between the "hot" and "cold" phase changes from a first order to a second order transition. We have looked for hysteresis when changing κ_1 and adjusting β to the critical value $\beta_c(\kappa_1)$. While the hysteresis is indeed weaker when measured this way, we still see a clear hysteresis (Fig. 14) and we conclude that there is never a second order transition in geometry.

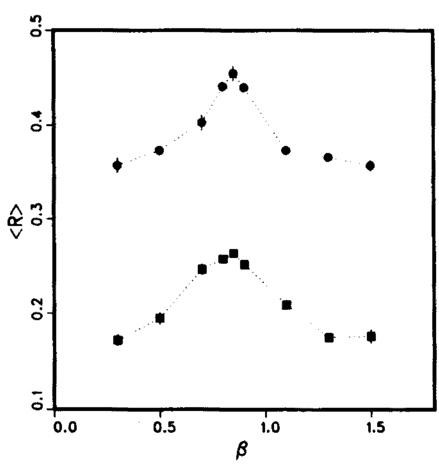

Fig. 13. The average curvature $\langle R \rangle$ as a function of β for $N_3 = 4000$ (circles) and $N_3 = 10000$ (squares). The value of κ_1 is 3.7. The position of the peak coincides with the value of β_c determined from the magnetization curve.

Let us make the following remark concerning the determination of the

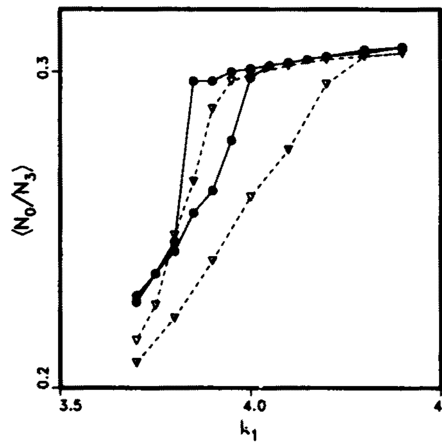

Fig. 14. The hysteresis curve for pure gravity (triangles) and in the case where the Ising spin system is critical *i.e.* where it couples in a maximal way to gravity (circles). $N_3 = 10000$.

phase diagram shown in Fig. 12: Due to the strong hysteresis it is somewhat ambiguous. We have used the following procedure: Well inside the "hot" phase the system follows a unique path when changing κ_1 and keeping β fixed as illustrated in Fig. 14. The precise location depends on the value of β. We have extrapolated these paths until they intersect the parts of the hysteresis curves which correspond to the "cold" phase.

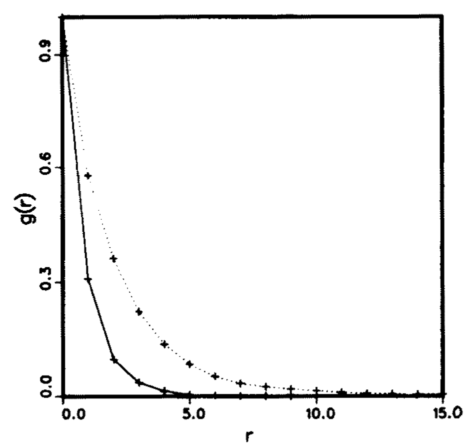

Fig. 15. The spin-spin correlation function $g(r)$ (defined by (3.14)) as a function of the geodesic distance r for $\beta = 0.5$ (full drawn curve) and $\beta = 0.8$ (dotted curve), $k_1 = 3.7$ and $N_3 = 4000$. The best estimate of the critical value of β is: $\beta_c = 0.85$

In Fig. 15 we have shown the spin-spin correlation (3.14) $g(r)$ as a function of geodesic distance "n_3", for two different values of β. In principle one can extract the mass gap $m(\beta)$ from the exponential fall off of $g(r)$, but as already remarked above we had difficulties extracting reliable results.

5.3.2. The Ising model coupled to 4d gravity

We have repeated the measurements in the case of the Ising model coupled the four-dimensional gravity. As in three dimensions the coupling between gravity and spins is not strong enough to change the phase structure of quantum gravity. We still have two phases: a "hot", highly connected one and a "cold" elongated phase. In the first phase the Ising model as a function of the spin coupling has a transition between a magnetized phase (large β) and a phase which the average magnetization is zero (small β). The transition is second order or higher. From the large Hausdorff dimension of gravity in this phase one would conjecture that it should be second order (mean field result), but we have not yet been able to distinguish between this and a higher order transition. In the elongated phase where the Hausdorff dimension of gravity is small (probably close to one as in three dimensions) there seems to be no transition. Again this is in agreement with intuition. We have not yet been able to make detailed statements about the region close to the transition between the two gravity phases. Although this is still work in progress we feel nevertheless that it is safe to make the statement that the coupling between gravity and spin seems weak away from the transition point between the highly connected and the elongated phase of gravity. Its strength is similar to what was found in three dimensions and although an enhancement of the coupling between the two degrees of freedom is observed in the "hot" phase of gravity when the Ising model undergoes a phase transition we doubt that it influences the critical properties of any of the two systems. The influence of the spin system on for instance the average curvature is similar to the one shown in three dimensions.

We should issue a warning, though. In two dimensions we know from analytic calculations that there *is* an interaction between spin and gravity which at the critical point of the spin system changes both gravity and the spin system. In addition it is non-trivial to extract this change from numerical data simply because the change is small. The quality of our data are not yet compatible with the two-dimensional ones needed to verify the different critical behaviour of the Ising in the presence of 2d-gravity. There is still a window open in both three and four dimensions for a different critical behaviour of at least the spin systems.

5.3.3. Scalar fields coupled to gravity

The coupling between a (Gaussian) scalar field and four dimensional gravity has so far not shown any surprises. We have mainly looked at the back-reaction present in the gravity sector. Since the four dimensional Ising model has a second order phase transition with mean field exponents one would expect that the nature of the back-reaction of the Gaussian scalar field should agree with the one of the Ising model at its critical point. This seems to be the case.

It would be interesting to investigate the coupling of a non-trivial field theory (like for instance an SU(2)-gauge theory) to gravity. The different infrared behaviour of asymptotically free theories might lead the a more non-trivial coupling to gravity.

6. Discussion

The notation of a "hot" and a "cold" phase in quantum gravity in $d = 3$ was introduced in [27]. In the hot phase the large entropy of "quantum" universes was dominant. These quantum universes were characterized by a large Hausdorff dimension and a high connectivity and the hot phase was continuously connected to "anti-gravity", where the (bare) gravitational coupling constant is negative. In the cold phase extended structures dominated. In fact the Hausdorff dimension seemed close to one, suggesting some kind of linear structure. This phase was interpreted as representing the dominance of the conformal mode. In the regularized theory the action is not unbounded from below, but instead some lattice configurations which are pure artifacts without any connection to the continuum will dominate. These were the extended structures observed in three dimensions. The interesting question was asked, whether it was possible at the transition point to have truly extended structures, relevant for continuum physics. In [28] it was shown that the transition in three dimensions was of first order, and a continuum limit was ruled out from this point of view.

From a superficial point of view the situation does not look so different in four dimensions. We have two phases which we again can call hot and cold. The hot phase is continuously connected to the "anti-gravity" region where the bare gravitational coupling constant is negative. As in three dimensions the cold phase is characterized by an almost linear, extended structure, while the hot phase has a larger Hausdorff dimension and much larger connectivity. In the hot phase the average order of vertices is much larger and the average curvature changes from being large positive in the cold phase to small positive or even negative in the hot phase. However, the nature of the transition seems different in four dimensions. We have not seen any true hysteresis, but there are very long thermalization times in the cold phase where the linear structures developed. This is in contrast to three dimensions where a very pronounced hysteresis was observed [28]. Our data are not incompatible with a second order transition, and this opens for the possibility that a continuum limit can be associated with the transition. The scenario is from this point of view quite nice: In three dimensions a first order transition rules out a continuum limit, but we do not really want the continuum limit in a usual sense in three dimensions since we would be confronted with the embarrassing question of a three dimensional graviton. The physical Hilbert space of *pure* three-dimensional quantum gravity is most likely finite dimensional [4] and does not allow for true dynamical fields. The situation in four dimensions is probably very different and it is interesting that the discretized model seems to hint at such a difference.

It is also encouraging that our data share some similarities with the results obtained by Regge calculus ([9] and references therein). As explained in the introduction the philosophy of the two methods is quite different and it would be a strong argument in favour of universality if one manages to obtain the same results by the two methods. In Regge calculus one also observes the two phases, and the phase with large positive curvature is characterized by very singular spiky configurations. They seem similar to our linear structures, which however cannot arise by single points moving away from the rest, as is the case in the Regge formalism. The "hot" phase is in the Regge formalism characterized by a small negative curvature which however (contrary to our results) scales to zero at the critical point. The latest results [9] indicate that one actually has a *first* order transition for $h = 0$ and only for a finite $h > 0$ it changes to a second order transition. This transition for finite h might have some similarity with the change we have seen for large h, but we still have $R_c > 0$. We postpone the discussion of this point to later publication.

At this point we should emphasize again that *we see one major obstacle to taking a continuum limit at the critical point and that is the fact that the average curvature does not scale to zero.* As is seen from the fig.1 we have $\langle R \rangle \approx 0$ for N_4 large in the case where there is no gravitational coupling constant. When we add a gravitational coupling constant κ_2 the expectation value $\langle R \rangle = R_0(\kappa_2)$ is essentially a linear function of κ_2 for κ_2 not too large. Since the only interesting critical behaviour takes place for $\kappa_2 > 0$ we need at least a reinterpretation of the scaling limit in order to be able to claim we can make contact with continuum physics. One possibility is that the expectation value of R_c has its root in the missing tessellation of flat space, but we feel that would be a surprise in a quantum theory of gravity. It is only slightly more likely that we see a finite size effect and if we go to much large lattices indeed the curvature would drop to zero. Although the curvature at the critical point decreases with the volume there is a long

[4] The situation might be different if we include matter fields.

way to zero and we cannot say that our data support such an explanation (although it is not ruled out completely).

Two radical views are more appealing. The first one is that there is no real theory of gravity without coupling to matter. This was our motivation for investigating the coupled system of gravity and matter. So far this line of thought has no support in the data, as reported above. If anything, the average curvature increases by coupling to matter. We are then left with the alternative that the scaling limit should be taken in a truly non-conventional way. At the moment we have no idea precisely how this should be done, but it is important to keep in mind that the situation is not so different from two dimensional gravity. Although a kind of conventional scaling *is* possible in that case, the corresponding continuum theory is topological in nature, contrary to the discretized formulation, reflecting in some, not yet fully understood way, that it is impossible to introduce a genuine length scale two dimensional gravity, probably due to reparametrization invariance of the underlying continuum theory. Nevertheless the discretized version of the theory contained all information about this topological theory if only the scaling limit was viewed correctly. Hopefully the same is the case in higher dimensions.

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