THE NUCLEON–NUCLEON INTERACTION FROM EFFECTIVE CHIRAL LAGRANGIANS*†

J. WAMBACH**

Department of Physics
University of Illinois at Urbana-Champaign
1110 W. Green St., Urbana, IL 61801, USA

(Received October 23, 1992)

Recent work on the derivation of the nucleon-nucleon potential from interacting solitons is discussed. Solitons emerge from chiral effective theories which represent QCD at low energy. A general introduction is given to such theories, their classical solutions and the inclusion of quantum effects. Results from various studies of the two-soliton interaction are discussed which are largly based on numerical evaluation. After semiclassical quantization via a restricted number of collective coordinates the resulting nucleon-nucleon interaction compares well with phenomenological potentials.

PACS numbers: 13.75.Cs

1. Introduction

More than fifty years ago Yukawa proposed that the force between two nucleons is mediated by a meson whose Compton wavelength is sufficiently small to explain the short-range nature of the nucleon-nucleon (NN) interaction [1]. That meson was later identified as the pion. Yukawa's idea has proven an extremely successful starting point for understanding many phenomena in nuclear physics. Inclusion of multi-pion exchange and massive

---

* Presented at the XXXII Cracow School of Theoretical Physics, Zakopane, Poland, June 2–12, 1992.

† Supported by NSF grant PHY89-21025.

** also: Institut für Kernphysik Forschungszentrum Jülich, D-5170 Jülich, Germany.

vector mesons in modern meson-exchange potentials has led to a quantitatively accurate description of two-body scattering observables over a wide range of energies [2].

As we know now, however, baryons and mesons are not elementary particles and the meson-exchange picture cannot explain their internal structure. The fundamental theory of the strong interactions is quantum chromodynamics (QCD), and the basic building blocks are colored quarks and gluons. For the past decade it has therefore become a major challenge in intermediate-energy physics to understand the meson-exchange picture from these fundamental degrees of freedom. This is far from trivial since at low energies quarks and gluons are strongly coupled and thus evade a perturbative treatment. For further progress an observation by 't Hooft [3] has proven very useful. He argued that, in spite of strong coupling, QCD has an expansion parameter $1/N_C$, where N_C is the number of colors. This has led to the hope that the theory simplifies in the large-N_C limit. It turns out that such a limit indeed exists, and that for large N_C QCD may be regarded as a local theory of mesons and glueballs with effective interactions of order $1/N_C$ [4]. To date no large-N_C Lagrangian has been derived directly from QCD; nevertheless, any candidate theory is greatly limited by the restrictions that it respects the symmetries and anomaly structure of the QCD Lagrangian and reproduces meson phenomenology. The simplest version of such a theory is realized by a non-linear chiral Lagrangian of interacting pions. It shares an important property of the non-perturbative QCD vacuum, namely, the spontaneous breakdown of chiral $SU(2)_L \times SU(2)_R$ symmetry. Another feature of the vaccuum is the spontaneous breakdown of scale invariance due to the presence of a gluon condensate. This introduces an effective gluonium field whose coupling to the pions is determined by the trace anomaly.

In large-N_C effective theories baryons will appear as solitonic excitations of the meson field as was already realized by Skyrme around 1960 [5, 6]. These field configurations are stabilized by the geometrical properties of maps from ordinary space into the group space of internal symmetry. The characteristics of these maps lead quite naturally to the concept of a conserved baryon number B. Thus a baryon is described as classical soliton which is stabilized geometrically: a topological soliton. In honor of Skyrme such solitons have been dubbed "skyrmions." The nucleon is a fermion and its identification is made complete since spin and isospin are acquired by the soliton through the quantization of collective coordinates, a well known procedure from the many-body theory of finite quantum systems [7].

Starting from a solitonic picture of the nucleon as the proper realization of QCD in the large-N_C limit, the description of the nucleon-nucleon potential needs to consider soliton interactions. Such studies are interesting

non-linear dynamics problems in their own right with applications in many branches of physics. To obtain the the NN potential is particularly interesting since the internal baryon structure and interactions are put on an equal footing. This is not possible in conventional meson-exchange models. As one can imagine, however, a solitonic description is quite challenging. First of all, the classical field equations are non-linear and one has to apply advanced numerical techniques to deal with this properly. Secondly one needs to account for the quantum nature of the interaction which is even more difficult. Part of the problem arises from the fact that the interaction energy at large and intermediate distances is small compared to the nucleon mass.

I shall try to introduce the complexity of the problem and to display some attempts to deal with it. Section 2 starts by introducing effective chiral Lagrangians as a minimal realization of QCD at low energy. Solitonic solutions corresponding to a single baryon, $B = 1$, and its semiclassical quantization will be discussed in Section 3. Having discussed the basic concepts I then move on to the $B = 2$ sector. Section 4 starts with a discussion of collective coordinates necessary for semiclassical quantization [8] and their specific choice for the problem at hand. This leads to an adiabatic interaction hamiltonian whose exact treatment requires nontrivial numerical evaluation. A particular approximation, known as the "product ansatz", will be discussed in some detail since it guides such evaluations. It has a serious deficiency, however, the missing of the medium-range attraction, which is vital for the binding of nuclei. This problems can be traced back to inadequacies at shorter distances. For a quantitative assessment I discuss in Section 5 exact classical $B = 2$ configurations obtained through numerical analysis. First some time-dependent simulations of soliton collisions are presented which help guide intuition on the complexity of such events as well as the proper choice of collective coordinates. Time-dependent solutions are very difficult to quantize, however. Static methods via collective coordinates are more developed at present. These require exact static solutions with Lagrange constraints from which classical potential energies and inertial masses for selected configurations are extracted. From a limited number of such configurations, a relatively straightforward quantization scheme can be implemented. Such calculations represent the state-of-the-art for the solitonic description of the NN-potential, and they begin to make quantitative contact with phenomenological potentials.

2. Effective low-energy Lagrangians

The exact form of the large-N_C effective Lagrangian is not known. At present, one has to rely on models which are guided by the symmetries

and the anomaly structure of QCD. The most relevant symmetry in this connection is chiral symmetry. In the limit of massless quarks the QCD Lagrangian is invariant under a $SU(N_f)_L \times SU(N_f)_R$ chiral transformation, where N_f is the number of flavors. For the case of two flavors this symmetry is respected in nature to a good approximation. At low energies it is spontaneously broken, giving rise to nearly massless pions.

2.1. The non-linear σ model

A natural starting point is therefore the σ-model [9]

$$\mathcal{L}_\sigma = \tfrac{1}{2}(\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi} + \partial_\mu \sigma \partial^\mu \sigma) + V(\boldsymbol{\pi}^2 + \sigma^2)\,. \tag{1}$$

For spontaneously broken symmetry the potential $V(c^2)$ acquires a minimum not for $c^2 = 0$ but at some finite value $c^2 = \boldsymbol{\pi}^2 + \sigma^2$. The constant c is to be identified with the pion weak decay constant ($f_\pi = 93$ MeV) . When the fields are restricted to lie on a 3-sphere in the space of internal symmetry as

$$\boldsymbol{\pi}^2(x) + \sigma(x)^2 = f_\pi^2\,, \tag{2}$$

we can combine the four-vector $(\boldsymbol{\pi}, \sigma)$ at each space-time point $x = (\boldsymbol{x}, t)$ into an SU(2)-matrix

$$U(x) = \exp\left(\frac{i\boldsymbol{\tau}\cdot\boldsymbol{\phi}(x)}{f_\pi}\right)\,. \tag{3}$$

Then Eq. (1) becomes

$$\mathcal{L}_\sigma = \frac{f_\pi^2}{4}\,\mathrm{Tr}\left(\partial_\mu U \partial^\mu U^\dagger\right)\,, \tag{4}$$

which is the non-linear σ model for massless pions [10]. The vacuum is described by $U = 1$, that is, $\sigma = f_\pi$ and $\boldsymbol{\pi} = 0$. Even though $\mathcal{L}_\sigma$ is invariant under chiral transformations, the vacuum is obviously not. Chiral symmetry is spontaneously broken. To be more realistic a pion mass term should be added. A non-vanishing m_π breaks chiral symmetry explicitly and can be understood in terms of small but non-vanishing current quark masses together with a non-zero value for the chiral condensate of the QCD vacuum. Expansion around the vacuum value, $U = 1$,

$$U = 1 + i\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{f_\pi} - \tfrac{1}{2}\left(\frac{\boldsymbol{\tau}\cdot\boldsymbol{\pi}}{f_\pi}\right)^2 + \ldots \tag{5}$$

gives

$$\mathcal{L}_\pi = \tfrac{1}{2}\partial_\mu \boldsymbol{\pi}\cdot\partial^\mu\boldsymbol{\pi} - \tfrac{1}{2}m_\pi^2\boldsymbol{\pi}^2 + O(\boldsymbol{\pi}^4)\,, \tag{6}$$

which describes a system of weakly self-interacting pions and is well founded in low-energy $\pi\pi$-scattering phenomenology.

2.2. Inclusion of vector mesons

It has long been known from Vector Dominance that vector mesons play an important role in strong interaction physics. They can be incorporated following the basic ideas of Sakurai [11], who attempted to model the strong interaction as a massive gauge theory, before the advent of QCD . Starting from the non-linear σ-model (4) vector mesons are introduced as gauge bosons of a local $\mathrm{U(2)_L \times U(2)_R}$ symmetry by defining the covariant derivative as

$$\partial_\mu U \to \mathcal{D}_\mu U \equiv \partial_\mu U - igA_\mu^{\mathrm{L}} U + igUA_\mu^{\mathrm{R}}, \tag{7}$$

where $A_\mu^{\mathrm{L,R}} \equiv \tau \cdot A_\mu^{\mathrm{L,R}}$ and g is the gauge coupling. From large N_c counting it is of order $1/\sqrt{N_C}$. The vector field V_μ and the axial vector field A_μ have been expressed as the left- and right-handed gauge fields $A_\mu^{\mathrm{L}} = \frac{1}{2}(V_\mu + A_\mu)$ and $A_\mu{}^{\mathrm{R}} = \frac{1}{2}(V_\mu - A_\mu)$. Kinetic energy terms are then added to the gauged $\mathcal{L}_\sigma$ as

$$\mathcal{L}_{\mathrm{kin}} = -\tfrac{1}{4}\,\mathrm{Tr}(F_{\mu\nu}^{\mathrm{L}} F_{\mathrm{L}}^{\mu\nu} + F_{\mu\nu}^{\mathrm{R}} F_{\mathrm{R}}^{\mu\nu}), \tag{8}$$

where

$$F_{\mu\nu}^{\mathrm{L,R}} = \partial_\mu A_\nu^{\mathrm{L,R}} - \partial_\nu A_\mu^{\mathrm{L,R}} - ig[A_\mu^{\mathrm{L,R}}, A_\nu^{\mathrm{L,R}}]. \tag{9}$$

In accord with the requirements of low-energy phenomenology, gauge invariance is broken explicitly by further adding mass terms. Therefore this theory is not a fundamental gauge theory. The mass parameters are to be chosen so that the particular form of the Lagrangian conforms to conventional theories of vector mesons.

There is a subtlety that needs to be taken into account. Witten [4] observed that chiral effective Lagrangians of the type above possess an extra discrete symmetry that is not a symmetry of QCD. Consider $\mathcal{L}_\sigma$ (4) for simplicity. Under parity transformation, QCD requires the pion to be a pseudoscalar $P\boldsymbol{\pi}(\boldsymbol{x}, t) = -\boldsymbol{\pi}(-\boldsymbol{x}, t)$. In the meson theory this is accomplished by defining the parity operator as

$$P: \ \boldsymbol{x} \to -\boldsymbol{x}; \quad U \to U^\dagger. \tag{10}$$

The Lagrangian (4), however, is invariant under $\boldsymbol{x} \to -\boldsymbol{x}$ and $U \to U^\dagger$ separately. This unwanted extra symmetry can be eliminated by adding the so-called Wess–Zumino term $\mathcal{L}_{\mathrm{WZ}}$, as Witten suggested. At the same time, this term reproduces the non-abelian anomaly structure of QCD. The

baryon current $\mathcal{B}_\mu$, can be derived from it via Noether's theorem. Witten showed that for consistency the Wess–Zumino term must be proportional to an integer and that this integer is to be identified with the number of colors N_C [13]. This term is present in any realistic extension of the non-linear sigma model that includes heavier mesons and/or strangeness.

We are now in a position to write down a complete Lagrangian which respects chiral symmetry and is in accord with vector dominance. For two flavors the gauge bosons are the ω, ρ and A_1 mesons and the vector fields are given by $V_\mu = \omega_\mu + \tau \cdot \rho_\mu$ and $A_\mu = \tau \cdot A_\mu$. Including the pion via the non-linear σ-model, the full $\pi\omega\rho A_1$ Lagrangian is then

$$\mathcal{L}_{\pi\omega\rho A_1} = \frac{f_\pi^2}{4} \operatorname{Tr}(\mathcal{D}_\mu U \mathcal{D}^\mu U^\dagger) + \tfrac{1}{2} m_\pi^2 f_\pi^2 \operatorname{Tr}(U-1)$$
$$-\tfrac{1}{8} \operatorname{Tr}(\omega_{\mu\nu}^2 + \rho_{\mu\nu}^2 + A_{\mu\nu}^2) + \frac{m_V^2}{4} \operatorname{Tr}(\omega_\mu^2 + \rho_\mu^2 + A_\mu^2) + N_C \mathcal{L}_{WZ} . \quad (11)$$

where the kinetic energy terms are defined as

$$\begin{aligned} \omega_{\mu\nu} &= \partial_\mu \omega_\nu - \partial_\nu \omega_\mu , \\ \rho_{\mu\nu} &= \partial_\mu \rho_\nu - \partial_\nu \rho_\mu + g \rho_\mu \times \rho_\nu , \\ A_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + g A_\mu \times A_\nu . \end{aligned} \quad (12)$$

Substituting the covarinat derivative (7) one finds a cross term whereby the pion and the A_1 fields mix in the vacuum. A proper diagonalization yields the physical fields

$$\tilde{A}_\mu = A_\mu - \frac{g \tilde{f}_\pi}{m_V^2} \partial_\mu \pi, \quad \tilde{\pi} = Z\pi , \quad (13)$$

where

$$\tilde{f}_\pi = Z f_\pi, \quad Z^{-1} = \left(1 + \frac{g^2 f_\pi^2}{m_V^2}\right)^{1/2} , \quad (14)$$

and generates the A_1 mass difference $m_A = Z^{-1} m_V$, that is,

$$m_A^2 - m_V^2 = g^2 \tilde{f}_\pi \left(\frac{m_A^2}{m_V^2}\right) . \quad (15)$$

This Lagrangian fits the mesonic sector quite well.

2.3. Derivation of the "Skyrme term"

One can derive the so-called "Skyrme Lagrangian" [5] under certain approximations. This Lagrangian consits of the $\mathcal{L}_\sigma$ supplemented by a

term $\mathcal{L}_4$ which is of fourth order in $\partial_\mu U$ and has historically played an important role. It is still used as a somewhat crude model. The strategy is to eliminate the A_1 field by imposing a chiral constraint on the gauge fields. This in turn allows the elimination of the ρ and ω mesons by neglecting their kinetic energies. In order to eliminate the axial meson $A_\mu = A_\mu^{\mathrm{L}} - A_\mu^{\mathrm{R}}$ as an independent field, one can assume that the fields U and $A_\mu^{\mathrm{L,R}}$ are related through a gauge transformation to a configuration with no pseudoscalar or axial vector mesons present. This amounts to a gauge fixing or equivalently to imposing the chiral constraint

$$A_\mu^{\mathrm{L}} = U A_\mu^{\mathrm{R}} U^\dagger + \frac{i}{g} U \partial_\mu U^\dagger . \tag{16}$$

The effective chiral Lagrangian for π, ρ and ω mesons is now obtained by the substitution of (16) into the Lagrangian (11), which yields

$$\begin{aligned}\mathcal{L}_{\pi\rho\omega} = {} & \frac{f_\pi^2}{4} \operatorname{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \tfrac{1}{2} f_\pi^2 m_\pi^2 \operatorname{Tr}(U-1) - \tfrac{1}{8} \operatorname{Tr}(\omega_{\mu\nu}^2 + \rho_{\mu\nu}^2) \\ & + \tfrac{1}{2} m_V^2 \omega_\mu^2 + \tfrac{1}{4} m_V^2 \operatorname{Tr}[\tau \cdot \rho_\mu - \frac{i}{g}(\xi^\dagger \partial_\mu \xi - \partial_\mu \xi \xi^\dagger)]^2 + N_{\mathrm{C}} \mathcal{L}_{\mathrm{WZ}} , \end{aligned} \tag{17}$$

where $\xi = U^{1/2}$. One can extract the $\rho\pi\pi$ interaction term by expanding around the vacuum $U = 1$, which yields the coupling term

$$\mathcal{L}_{\rho\pi\pi} = \frac{m_V^2}{4 g f_\pi^2} \rho_\mu \cdot (\pi \times \partial^\mu \pi) = \tfrac{1}{2} g_{\rho\pi\pi} \rho_\mu \cdot (\pi \times \partial^\mu \pi) . \tag{18}$$

The choice $g_{\rho\pi\pi} = g$ leads to the KSFR relation [14], $g^2 = m_V^2 / 2 f_\pi^2$, which is well fulfilled in nature.

For energies small compared to the vector meson mass, the vector fields can be eliminated as dynamical fields as well. By taking the limit $m_V \to \infty$, one can neglect the kinetic terms in the Lagrangian since

$$\begin{aligned} \tau \cdot \rho_\mu^{(0)} &= \frac{i}{g}(\xi^\dagger \partial_\mu \xi - \partial_\mu \xi \xi^\dagger) , \\ \omega_\mu^{(0)} &= -\frac{g N_{\mathrm{C}}}{m_V^2} \mathcal{B}_\mu , \end{aligned} \tag{19}$$

where the last equation follows from the explicit form of the Wess–Zumino term [15]. When substituted back into $\mathcal{L}_{\pi\rho\omega}$, the vector fields (19) yield terms of higher derivatives in U. The lowest-order one is

$$\mathcal{L}_4 = -\frac{1}{8} \operatorname{Tr}(\rho_{\mu\nu}^{(0)} \cdot \rho_{(0)}^{\mu\nu}) = \frac{1}{32 g^2} \operatorname{Tr}[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger]^2 , \tag{20}$$

which is precisely the fourth-oder term in the Skyrme Lagrangian. It can be seen that it arises from $\rho\pi\pi$ coupling.

2.4. *Gluonic degrees of freedom*

The mesonic Lagrangians discussed above respect chiral symmetry and the non-abelian anomaly structure of QCD, but they do not correctly reflect the behavior under scale transformations. Classically, the QCD Lagrangian (with massless quarks) is invariant under scale transformations $x^\mu \to e^\lambda x^\mu$, with λ a real dimensionless constant. The corresponding conserved current is the dilaton current $S_\mu = T_{\mu\nu}x^\nu$, where $T_{\mu\nu}$ is the energy-momentum tensor. At the quantum level, however, S_μ has a non-vanishing divergence [16, 17]

$$\partial^\mu S_\mu = \left(\frac{\beta(g)}{2g}\right) F^a_{\mu\nu}F^{\mu\nu}_a\,, \tag{21}$$

where β is the QCD beta function and g is the coupling constant. Since $\partial^\mu S_\mu = T^\mu_\mu$, the trace of the energy-momentum tensor, Eq. (21) is called the "trace anomaly". The anomalous scaling behavior of QCD can be restored at the effective Lagrangian level. One introduces a singlet scalar field $\chi(x)$ [18] such that $\partial^\mu S_\mu = -B\chi^4$, where B is a constant. According to Eq. (21) its vacuum value χ_0 directly relates to the gluon condensate in the non-perturbative vacuum. Fluctuations in χ around χ_0 may be interpreted as a scalar gluonium (or glueball) field.

A minimal model starts from the Skyrme Lagrangian and is given by [19]

$$\mathcal{L} = \partial_\mu\chi\partial^\mu\chi + \frac{f_\pi^2}{4}\left(\frac{\chi}{\chi_0}\right)^2 \mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger) + \frac{1}{32g^2}\,\mathrm{Tr}[\partial_\mu UU^\dagger, \partial_\nu UU^\dagger]^2 + \frac{m_\pi^2 f_\pi^2}{2}\left(\frac{\chi}{\chi_0}\right)^3 \mathrm{Tr}(U-1) - V(\chi)\,. \tag{22}$$

Apart from the small pion mass term the interaction between the gluonium field and the pion is solely due to the $(\chi/\chi_0)^2\,\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger)$ term, which describes the restoration of the spontaneously broken chiral symmetry in regions where χ deviates from χ_0. The effective potential $V(\chi)$ is uniquely determined by the vacuum energy density $B\chi_0/4$ ($U = 1$) and the condition that the minimum should occur at $\chi = \chi_0$ [19]

$$V(\chi) = B\frac{\chi_0}{4}[1 + 4\chi^4\ln(\chi/e^{1/4})]\,. \tag{23}$$

The two parameters B and χ_0 are obtained from the gluon condensate

$$\langle 0|\left(\frac{\beta}{2g}\right) F^a_{\mu\nu}F^{\mu\nu}_a|0\rangle = -B\chi_0^4 \tag{24}$$

and the glueball mass [20]

$$m_\chi^2 = \left(\frac{\partial^2 V}{\partial \chi^2}\right)\Big|_{\chi=\chi_0} = 4B\chi_0^2\,, \tag{25}$$

which, together with f_π and g, specify the parameters of the Lagrangian.

3. Finite energy configurations

As was first noted by Skyrme [5, 6], effective chiral Lagrangians have a much richer structure than suggested by the meson phenomenology. He realized that there exist classical finite-energy field configurations with non-trivial topology. The components of the ϕ-field entering in (3) play the role of angular variables which label points on the unit 3-sphere S^3 embedded in the four-dimensional space of internal symmetry. For finite energy, the field configurations must be such that $U(\boldsymbol{x})$ approaches the identity matrix at spatial infinity. This is most easily seen for the non-linear σ model but also holds for the other Lagrangians. The static energy

$$E = -\int d^3x \mathcal{L}_\sigma = \frac{f_\pi^2}{2}\,\mathrm{Tr}\int d^3x (\nabla U)^2 \tag{26}$$

is obviously finite for $U(\boldsymbol{x}) \to 1$ as $|\boldsymbol{x}| \to \infty$. Thus all points at spatial infinity are identified with a single point, and any finite-energy configuration defines a map of the 3-sphere S^3_∞ onto the internal 3-sphere S^3. The set of all such maps falls into different sectors of equivalent maps which are characterized by a topological index n. This index determines how often S^3 is covered by a given mapping and is called the "winding number". It is a constant of the motion. One may define a"topological current" $\mathcal{B}^\mu$ such that n is equal to the space integral of its zeroth component. This current has the form

$$\mathcal{B}^\mu = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\mathrm{Tr}[(U^\dagger\partial_\nu U)(U^\dagger\partial_\rho U)(U^\dagger\partial_\sigma U)]\,, \tag{27}$$

and the conservation law $\partial_\mu \mathcal{B}^\mu = 0$ is trivially fulfilled due to the antisymmetric character of the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$. It was Skyrme's inspired suggestion to identify the winding number with the baryon number

$$B = \int d^3x\, \mathcal{B}_0(x) = n\,, \tag{28}$$

an identification which was later put on firmer ground by Witten [12, 13].

For $B = 1$, the static field $U = \cos(\phi/f_\pi) + i\tau\cdot\hat{\phi}\sin(\phi/f_\pi)$ constitutes a map which covers S^3 once. Then the unit vector $\hat{\phi} = \phi/|\phi|$ as a function of x must cover the unit sphere S^2 in isospace as x takes on all values in 3-space. The most simple choice to achieve this is

$$\hat{\phi}(x) = \hat{x}\,. \tag{29}$$

Furthermore, a purely radial dependence $\phi(x) = f_\pi F(r)$ will result in minimal total energy. This then leads to the ansatz

$$U_{\rm H} = \exp(i\tau\cdot\hat{x}F(r))\,. \tag{30}$$

Because of its peculiar geometrical structure it has been termed the (defensive) "hedgehog". By explicit construction of the baryon current one immediately finds that

$$B = \frac{1}{\pi}[F(0) - F(\infty)] \tag{31}$$

which implies $F(0) = \pi$ and $F(\infty) = 0$ for $B = 1$.

The full profile function $F(r)$ is determined by minimizing the static energy with respect to F. For the Skyrme Lagrangian one has

$$M[F] = 4\pi\int dr r^2\left[\frac{f_\pi^2}{2}\left(F'^2 + \frac{2\sin^2 F}{r^2}\right) + \frac{1}{2g^2}\frac{\sin^2 F}{r^2}\left(\frac{\sin^2 F}{r^2} + 2F'^2\right)\right], \tag{32}$$

which is to be interpreted as the mass of the hedgehog, $M_{\rm H}$. In the language of large-$N_{\rm C}$ QCD, f_π and $1/g$ are of order $\sqrt{N_{\rm C}}$, so the hedgehog mass is of order $N_{\rm C}$. Minimization with respect to F then yields a non-linear differential equation

$$\left(\tilde{r}^2 + 2\sin^2 F\right)F'' + 2\tilde{r}F' + \sin 2F F'^2 - \sin 2F\left(1 + \frac{\sin^2 F}{\tilde{r}^2}\right) = 0 \tag{33}$$

in terms of the dimensionless variable $\tilde{r} = g f_\pi r$. Numerical solution with the proper boundary conditions results in a monotonically decreasing function of $\tilde{r}$ (Fig. 1).

When a finite pion mass is included, its large distance behavior is of the familiar Yukawa form

$$F(r) \to \frac{K}{r^2}(1 + m_\pi r)\exp(-m_\pi r) \qquad \text{as} \quad r \to \infty \tag{34}$$

for a point field from a baryonic source. Given $F(r)$ it is then straightforward to calculate the static energy. With the empirical values $f_\pi = 93$ MeV

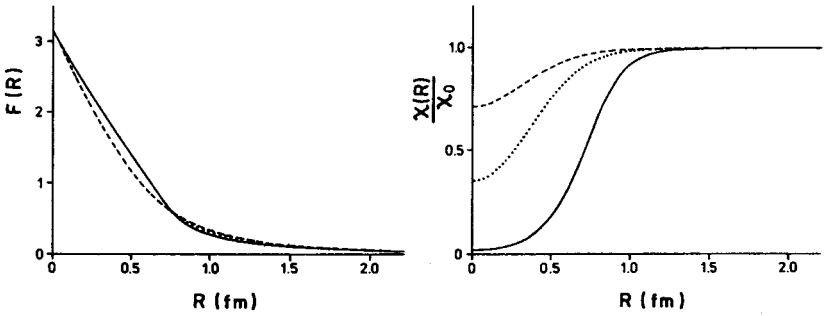

Fig. 1. Bag formation when gluonic degrees of freedom are included in the effective Lagrangian via the trace anomaly. The $B = 1$ profile function $F(r)$ and the normalized glueball field $\chi(r)$ are shown for several values of the gluon condensate and for a glueball mass $m_\chi = 1.5$ GeV. The largest local depression of χ is found for small condensate values as derived from mass fits in the MIT bag model. Lattice gauge simulations and QCD sum rules predict smaller gluon effects (dotted and dashed lines).

and $g \sim 6$, one obtains $M_H \sim 1.1$ GeV. The more sophisticated Lagrangians give rather simialr results. One interesting feature emerges as meson-gluon interactions are included so as to satisfy the trace anomaly. Then the formation of a $B = 1$ soliton is accompanied by a local distortion of the gluon condensate, a "bag". The depth of this bag strongly depends on the value of the gluon condensate as seen from Fig. 1. It has important consequences for the nucleon-nucleon potential to be discussed below.

3.1. Time-dependence and quantization for $B = 1$

The static $B = 1$ soliton, the "skyrmion", should not be identified with the nucleon or the Δ-isobar since it does not have a well-defined spin or isospin. Effects due to angular momentum require a quantum mechanical treatment [21]. For quantization we must consider time-dependent solutions. Only then can non-vanishing canonical momenta be constructed. In general it is difficult to obtain time-dependent solutions in a non-linear field theory. Some cases are obvious, however. One such case is the linearization around $U = 1$, which leads to weakly interacting pions in the $B = 0$ sector, as has been mentioned above. Similarly a linearization around the static hedgehog $U_H = \exp(i\tau \cdot \phi_H/f_\pi)$ for $B = 1$ can be performed which will lead to small oscillations describing πN-scattering [22]. Due to the symmetries involved, there occur modes that are not confined to small amplitudes since they have no restoring forces. Leaving the classical energy unchanged there are six such "zero modes": three corresponding to uniform translation

and three to uniform rotations. The rotations are connected to spin and isospin. Explicitly, the $B = 1$ zero modes are characterized by the following transformation

$$U(\boldsymbol{x}, t) = A(t) U_{\mathrm{H}}(\boldsymbol{x} - \boldsymbol{R}(t)) A^{\dagger}(t), \tag{35}$$

where A is a SU(2)-matrix which represents a global rotation in isospace. Such a rotation causes an orthogonal transformation on the isovector $\boldsymbol{\phi}$-field. For the hedgehog configuration (30), the same can be achieved by an $\boldsymbol{x}$-space rotation of the coordinate system. Thus we expect a correlation between spin and isospin for the skyrmion.

Inserting (35) into the Lagrangian, one obtains kinetic energy contributions due to the time-dependence. For the rotational term it is convenient to write $A(t)$ as

$$A(t) = a_0(t) + i\boldsymbol{\tau} \cdot \boldsymbol{a}(t); \qquad (a_0, \boldsymbol{a} \quad \text{real functions}) \tag{36}$$

with the constraint $\sum_{i=0}^{3} a_i^2 = 1$. Up to second-order in the time-derivatives one then finds

$$L = -M_{\mathrm{H}} + \tfrac{1}{2} M_{\mathrm{H}} \sum_{i=1}^{3} \dot{R}_i^2 + 2\lambda_{\mathrm{H}} \sum_{i=0}^{3} \dot{a}_i^2 . \tag{37}$$

The translational kinetic energy involves the classical soliton mass, and λ_{H} denotes the moment of inertia in the rotational energy.

To construct the hamiltonian one now treats the functions R_i and a_i as canonical variables. The conjugate momenta P_i and p_i follow immediately as

$$P_i = \frac{\partial L}{\partial \dot{R}_i} = M_{\mathrm{H}} \dot{R}_i, \qquad p_i = \frac{\partial L}{\partial \dot{a}_i} = 4\lambda_{\mathrm{H}} \dot{a}_i , \tag{38}$$

and hence

$$H = P_i \dot{R}_i + p_i \dot{a}_i - L = M_{\mathrm{H}} + \frac{1}{2M_{\mathrm{H}}} \sum_{i=1}^{3} P_i^2 + \frac{1}{8\lambda_{\mathrm{H}}} \sum_{i=0}^{3} p_i^2 . \tag{39}$$

Imposing the usual commutation relations $[P_i, R_j] = i\delta_{ij}$ and $[p_i, a_j] = i\delta_{ij}$ the quantum hamiltonian is

$$H = M_{\mathrm{H}} + \frac{1}{2M_{\mathrm{H}}} \sum_{i=1}^{3} \left(-\frac{\partial^2}{\partial R_i^2} \right) + \frac{1}{8\lambda_{\mathrm{H}}} \sum_{i=0}^{3} \left(-\frac{\partial^2}{\partial a_i^2} \right) . \tag{40}$$

The wavefunctions for translations are plane-wave states. The construction of the rotational wavefunctions is more involved. One should notice that

due to the constraint $\sum_{i=0}^{3} a_i^2 = 1$, the rotational energy involves a four-dimensional Laplacian constrained to S^3. The corresponding rotation group is $O(4)$, which has six generators L_{ij}. In analogy with ordinary rotations, these may be written as

$$L_{ij} = -i\left(a_i \frac{\partial}{\partial a_j} - a_j \frac{\partial}{\partial a_i}\right). \tag{41}$$

Then

$$\sum_{i=0}^{3}\left(-\frac{\partial^2}{\partial a_i^2}\right) = (L_{23}^2 + L_{31}^2 + L_{12}^2) + (L_{01}^2 + L_{02}^2 + L_{03}^2) = \boldsymbol{L}^2 + \boldsymbol{K}^2 . \tag{42}$$

The combinations $\frac{1}{2}(\boldsymbol{L} + \boldsymbol{K})$ and $\frac{1}{2}(\boldsymbol{L} - \boldsymbol{K})$ form two independent SU(2)--representations which we identify as the angular momentum $\boldsymbol{J}$ and the isospin $\boldsymbol{T}$

$$\boldsymbol{J} = \tfrac{1}{2}(\boldsymbol{L} + \boldsymbol{K}); \qquad \boldsymbol{T} = \tfrac{1}{2}(\boldsymbol{L} - \boldsymbol{K}). \tag{43}$$

Finally, in the center of momentum frame of the skyrmion

$$H = M_{\mathrm{H}} + \frac{1}{4\lambda_{\mathrm{H}}}(\boldsymbol{J}^2 + \boldsymbol{T}^2). \tag{44}$$

The eigenfuntions are labelled by $|JM_JTM_T\rangle$ with energies

$$M_{JT} = M_{\mathrm{H}} + \frac{1}{4\lambda_{\mathrm{H}}}(J(J+1) + T(T+1)). \tag{45}$$

As expected, $\boldsymbol{J}$ and $\boldsymbol{T}$ are not independent. It can be shown from the definition of these operators in terms of the generators L_{ij} that $\boldsymbol{J}^2 - \boldsymbol{T}^2 = 0$. Hence

$$M_{JT} = M_{\mathrm{H}} + \frac{1}{2\lambda_{\mathrm{H}}}(J(J+1)) = M_{\mathrm{H}} + \frac{1}{2\lambda_{\mathrm{H}}}(T(T+1)). \tag{46}$$

A convenient parameterization of the eigenfunctions which will be useful below is provided by the Euler angles of isorotations $\alpha\beta\gamma$ as Wigner $\mathcal{D}$ functions

$$\psi_{\mathcal{N}}(A) = \langle\alpha\beta\gamma|J = T, M_T, M_J\rangle = (-1)^{J+M_T}\left(\frac{2J+1}{8\pi^2}\right)^{1/2}\mathcal{D}^{(J)}_{-M_T M_J}(\alpha\beta\gamma), \tag{47}$$

where $J = T$ can take any integer or half-integer value. The angles $\alpha\beta\gamma$ are defined here as

$$A = \exp\left(-i\frac{\tau_3}{2}\alpha\right)\exp\left(-i\frac{\tau_2}{2}\beta\right)\exp\left(-i\frac{\tau_3}{2}\gamma\right). \tag{48}$$

Observed baryons have half-integer spin; for example, the nucleon has $J = T = {}^1/_2$ and the Δ-isobar has $J = T = {}^3/_2$. It can be shown in the context of the $SU(3)_L \times SU(3)_R$ extension of the model — where the strangeness flavor is included — that the fermionic character of the quantized baryon is indeed compulsory [13]. Given the wavefunctions, static baryon properties like charge radii, magnetic moments, and so on, can be calculated. Such observables involve vector and axial-vector currents which are constructed as Noether currents of the $SU(2)_L \times SU(2)_R$ symmetry. In general, the agreement with experimet is quite satisfactory considering the small number of parameters [21].

4. The two-soliton system

The basic problem is to extract from a description of interacting solitons a nucleon-nucleon potential which depends on only a few coordinates — namely, the positions, spins, and isospins of the two nucleons. In the single baryon case I have indicated how semiclassical quantization allows a truncation in the number of degrees of freedom of an isolated skyrmion to six collective modes which generate the nucleon's coordinates. It becomes a much more difficult procedure to identify the relevant collective coordinates when two solitons are interacting since interactions entail a distortion of each soliton's structure.

4.1. Collective coordinates

For two solitons there are three known unconstrained, time-independent solutions to the equations of motion. First, there is a $B = 2$ hedgehog solution

$$U_{H2}(\boldsymbol{x}) = \cos F_2(r) + i\boldsymbol{\tau}\cdot\hat{\boldsymbol{x}} \sin F_2(r)\,, \tag{49}$$

where the label 2 distinguishes this from the $B = 1$ solution. The boundary conditions $F_2(0) = 2\pi$ and $F_2(\infty) = 0$ guarantee the baryon number is two. The energy of this solution is $M_{H2} \sim 3M_H$. Evidently, this hedgehog is unstable. It has an "onion" structure, consisting of one spherical $B = 1$ object wrapped by another [23]. It represents an extreme in the two-skyrmion interaction, and should be of little consequence for low energies.

Second, there is the minimum-energy solution, which may be obtained from numerical relaxation of the full static equations of motion on a discrete spatial lattice. This stable solution was discovered independently in Refs. [24, 25]. In accord with a conjecture by Manton [27], which was motivated by studies of a two monopole system, the minimum-energy $B = 2$ solution has the shape of a torus. It will surely be significant for any semiclassical study of the NN interaction. In particular, it is the leading-order $1/N_C$ contribution to the deuteron [26].

The third static $B = 2$ solution is the trivial solution

$$U_\infty(\boldsymbol{x}) = A_1 U_{\rm H}(\boldsymbol{x} - \boldsymbol{R}_1) A_1^\dagger A_2 U_{\rm H}(\boldsymbol{x} - \boldsymbol{R}_2) A_2^\dagger , \tag{50}$$

where $U_{\rm H}(\boldsymbol{x})$ is the $B = 1$ hedgehog solution, the $A_{1,2}$ are SU(2) matrices giving the orientation of the first and second hedgehog in isospace, and the relative separation $|\boldsymbol{R}_1 - \boldsymbol{R}_2|$ is infinite. This is an asymptotically stable solution. Its energy is that of two hedgehogs.

Clearly, one may quantize the asymptotic solution by promoting the collective degrees of freedom $\{A_{1,2}, \boldsymbol{R}_{1,2}\}$ to dynamical variables — for these twelve collective coordinates here describe zero modes of non-interacting hedgehogs. In the limit of infinite separation, then, the quantized two-skyrmion system consists of products of $B = 1$ wavefunctions

$$\psi(\mathcal{N}_1, \mathcal{N}_2) = \psi_{\mathcal{N}_1 \mathcal{N}_2}(A_1, A_2) \exp(i\boldsymbol{P}_1 \cdot \boldsymbol{R}_1) \exp(i\boldsymbol{P}_2 \cdot \boldsymbol{R}_2) , \tag{51}$$

$$\psi_{\mathcal{N}_1 \mathcal{N}_2}(A_1, A_2) = \frac{\sqrt{(2J_1 + 1)(2J_2 + 1)}}{8\pi^2} (-1)^{J_1 + J_2 + i_1 + i_2} \mathcal{D}^{(J_1)}_{-i_1, s_1}(A_1) \mathcal{D}^{(J_2)}_{-i_2, s_2}(A_2) . \tag{52}$$

As the skyrmions move together, however, the system will depend on the variables $C \equiv A_1^\dagger A_2$ — the relative isospatial orientation of the skyrmions, and $R \equiv |\boldsymbol{R}_1 - \boldsymbol{R}_2|$ — the spatial separation between skyrmion centers. It is natural to select the twelve asymptotic zero modes as the collective coordinates [28–30]. Such a description is problematic, however, because it cannot be based upon simple static solutions to the equations of motion. At finite separation, $R = |\boldsymbol{R}_1 - \boldsymbol{R}_2|$ and $C = A_1^\dagger A_2$ become dynamical variables which describe non-zero modes.

Instead of attempting a quantization in the twelve asymtotic zero modes it is more useful to identify the true zero modes of a general $B = 2$ configuration. It is clear that for all seperations the system is invariant with respect to spatial translations $\boldsymbol{R}_T$, spatial rotations B, and isospatial rotations A of the field U:

$$U(\boldsymbol{x}, t) \to A U(\boldsymbol{D}(B) \cdot (\boldsymbol{x} - \boldsymbol{R}_T), t) A^\dagger . \tag{53}$$

which yields nine global collective variables. These describe zero modes. They move the system from the body-fixed to the lab frame. When promoted to dynamical variables, the canonical momenta corresponding to these global variables will be conserved. These are the total momentum $\boldsymbol{P}$, the total angular momentum $\boldsymbol{J}$, and the total isospin $\boldsymbol{T}$. In general, spatial and isospatial rotations of the field describe independent zero modes,

and there will therefore be no relation between $\boldsymbol{J}$ and $\boldsymbol{T}$ as there is for a single hedgehog. One can now move the body-fixed frame which may be chosen so that the two skyrmions are separated along the z-axis and so that the skyrmion in the $z > 0$ half of space (call it skyrmion 1) is unrotated. In the body-fixed frame the pion field depends on only the relative coordinates as $U_{2B}(\boldsymbol{x}; R, C)$. While the relative isospatial orientation C is easily identified in this frame the separation R is somewhat problematic. Intuitively, one would like to define R as the distance between the soliton centers, and this is certainly simple enough when the skyrmions are far apart. As they approach each other, however, the solitons distort; and it becomes difficult to tell where each center is. Indeed, for the minimum-energy toroidal solution, one cannot even identify the individual solitons. One choice is to take the separation to be the distance between the topological centers — that is, the points where the field $U(\boldsymbol{x}) = -1$. Another choice is to define the separation in terms of the baryon number density as twice the rms radius

$$R = 2\Big(\tfrac{1}{2}\int d^3x \boldsymbol{x}^2 \mathcal{B}_0(\boldsymbol{x})\Big)^{1/2}. \tag{54}$$

At large separations, where (50) is valid, these two definitions are equivalent. At small separations, however, they are quite different. For instance, the torus corresponds to coincident topological centers — that is, zero separation according to the first definition. According to the rms radius definition, however, the torus corresponds to a separation of $\sim$ 1fm. Clearly, if one defines the separation as in Eq. (54), it will require an infinite energy to push two skyrmions with any relative orientation on top of each other.

Suppose now that we know the two-soliton field in the body-fixed frame $U_2(\boldsymbol{x}; R(t), \boldsymbol{\gamma}(t))$, where $C = \exp(i\boldsymbol{\tau}\cdot\boldsymbol{\gamma})$. If the field is only slowly changing in time through the collective coordinates R and $\boldsymbol{\gamma}$ — that is, the motion is adiabatic. It is then possible to derive the interaction Hamiltonian (see Ref. [31]) for details as

$$H = \tfrac{1}{2} P_i M^{-1}_{ij}(R, C) P_j + \mathcal{U}(R, C), \tag{55}$$

where

$$P_i = \frac{\delta L}{Q_i} = M_{ij}\dot{Q}_j \tag{56}$$

are the canonical momenta of the nine zero-mode coordinates of global motion plus the body-fixed frame coordinates. The two-skyrmion potential is simply the static energy in the body frame

$$\mathcal{U} = \int d^3x \mathcal{V}(U_{2B}). \tag{57}$$

According to the $1/N_C$ expansion, the mass tensor $M(R,C)$ and the potential $U(R,C)$ are of order N_C, while the momenta P are of order 1. Thus the nucleon-nucleon potential will have $\mathcal{O}(N_C)$ "static" contributions from U and $\mathcal{O}(1/N_C)$ "kinetic" contributions from $PM^{-1}P$. The rotational kinetic energy is characterized by $1/\lambda_H$, and the translational kinetic energy is characterized by the smaller quantity $1/M_H$; so C may be considered a "fast" variable with respect to R [29, 30]. This amounts to calculating a generalized nucleon-nucleon potential for R under the Born–Oppenheimer approximation by diagonalizing H in a set of asymptotic basis functions (52). Because the Hamiltonian depends upon the relative orientation $C = A_1^\dagger A_2$, however, the state $(j_1, j_2) = (1/2, 1/2)$ with the lowest asymptotic energy will not be a good eigenstate of the system; there will be an admixture of $(1/2, 3/2)$, $(3/2, 1/2)$, and $(3/2, 3/2)$ states. In the language of perturbation theory, these are intermediate delta states in the nucleon-nucleon interaction.

In the next section I shall describe such calculations in the Skyrme model where, for convenience, the product ansatz is chosen. This ansatz is not a solution to the equations of motion, but its use can nevertheless be quite instructive. In fact, only recently have calculations appeared which go beyond the product ansatz.

4.2. The "product ansatz"

Because of the nonlinearity, the equations of motion are difficult to solve. For an exact treatment one has to rely on numerical techniques which, the $B = 2$ sector, become quite tedious. It would thus be a great advantage if one could somehow guess an accurate analytic form for the field U_{2B}. In particular, if we knew how the field depended on the collective coordinates — and therefore on the derivatives of U_{2B} with respect to R and C — the evaluation of the Hamiltonian (55) would be greatly facilitated. There has therefore been a long history of ansatz use in skyrmion-skyrmion calculations. The most widely used is the so-called "product ansatz" [32, 33, 35, 49]. Already in the early 1960's Skyrme suggested employing this ansatz in which U_{2B} is the product of two $B = 1$ hegdehog fields

$$U_P(\boldsymbol{x}) = A_1 U_H(\boldsymbol{x} - \boldsymbol{R}_1) A_1^\dagger A_2 U_H(\boldsymbol{x} - \boldsymbol{R}_2) A_2^\dagger \tag{58}$$

for all separations $R = |\boldsymbol{R}| = |\boldsymbol{R}_1 - \boldsymbol{R}_2|$. This is clearly a good approximation at large R, for then one soliton should not feel the presence of the other. A main advantage is its compact form, which greatly reduces the numerical difficulty in extracting a nucleon-nucleon potential from skyrmions.

After the initial work by Skyrme who first calculated the static potential between two hedgehogs [6] Yabu and Ando [32] repeated the calculation

in more detail, showing how to project onto states of good spin and isospin. One finds that the static potential $\mathcal{U}$ in (55) gives rise to a one-pion-exchange potential in the limit in which the two skyrmions are well-separated. Since the product ansatz must be accurate in precisely this limit, one is guaranteed that the long-range part of the nucleon-nucleon potential is correct. Of course, this is to be expected since the model has been designed to reproduce low-energy pion phenomenology. Substituting the product ansatz into the Lagrange density and integrating over space, one gets a Lagrangian for the twelve collective coordinates $A_1, A_2, \boldsymbol{R}_1, \boldsymbol{R}_2$. Under the adiabatic assumption, time-dependence will enter only through these coordinates which leads to the Hamiltonian (55) where $M_{i,j}$ and $\mathcal{U}$ are evaluated from the product field U_p. Taking then matrix elements of $\mathcal{U}$ between asymptotic two-nucleon states and defining

$$\mathcal{U}_{NN}(\boldsymbol{R}) = \langle N_1' N_2' | \mathcal{U}(\boldsymbol{R}, A_1^\dagger A_2) | N_1 N_2 \rangle - 2M_{\mathrm{H}}\,, \tag{59}$$

where from Eq. (52)

$$\langle A_1 A_2 | N_1 N_2 \rangle = -\frac{(-1)^{i_1+i_2}}{4\pi^2} \mathcal{D}^{(1/2)}_{-i_1,s_1}(A_1) \mathcal{D}^{(1/2)}_{-i_2,s_2}(A_2)\,, \tag{60}$$

there are only a small number of relative isospin orientations C which can contribute to the matrix elements. After some algebra it is found that

$$\mathcal{U}_{NN}(R) = \mathcal{U}^{(0)}(R) + \tfrac{1}{9}(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)\left[(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\mathcal{U}_{\tau\sigma}(R) + S_{12}\mathcal{U}_{\tau T}(R)\right], \tag{61}$$

where $S_{12} = 3(\boldsymbol{\sigma}_1 \cdot \hat{\mathbf{R}})(\boldsymbol{\sigma}_2 \cdot \hat{\boldsymbol{R}}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ is the usual tensor operator. Interestingly enough, these three terms are known from phenomenological potentials to give the strongest contributions to the nucleon-nucleon interaction. From a computational standpoint it is useful to note here that the potentials $\mathcal{U}^{(0)}(R)$, $\mathcal{U}_{\tau\sigma}(R)$, and $\mathcal{U}_{\tau T}(R)$ can be evaluated easily by fixing C at three different values and computing the classical potential $\mathcal{U}(\boldsymbol{R}, C)$ from these. A convenient choice of these three orientations is given in Fig. 2.

With relatively small numerical effort, the product ansatz can also be used for any separation. Jackson, Jackson, and Pasquier [34] and Vinh Mau, *et al.* [36] were the first to do this, finding the central, spin-spin, and tensor channels from Eq. (61) by direct computation. A general agreement was obtained with conventional phenomenological potentials such as the Paris potential. A striking difference is that no intermediate range attraction was found in the central channel. Since in conventional potentials it is this attraction which is responsible for the binding of nuclei, this could be a major failure of the Skyrme model. In fact, much work has gone into correcting for this supposed failure.

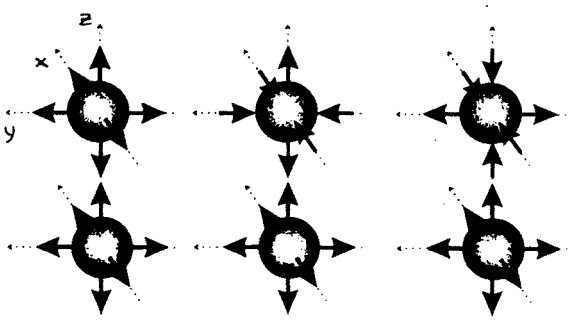

Fig. 2. The three relative isospin orientations which enter the evaluation of the NN-potential in the product ansatz to leading order. The left most configuration corresponds to $C = 1$, the next to $C = i\tau_3$ and the right most to $C = i\tau_2$. In an exact treatment the latter leads to formation of a torus at small separation.

One improvement is to consider also the kinetic term in Eq. (55). Oka [37] and Odawara, Morimatsu, and Yazaki [38] have considered the R-dependence of the inertial mass resulting from the product ansatz, which implies a momentum dependence in the two-nucleon potential. Independently, Nyman and Riska [39, 40] have considered the R- and C-dependence of those terms in the mass matrix which give rise to central and spin-orbit interactions. As described in Ref. [39], for example, when the two skyrmions approach there is an enhancement of their moments of inertia; and this translates into an attraction in the central channel. However, these terms are in general small, and we cannot expect an $\mathcal{O}(1/N_C)$ attraction to overcome an $\mathcal{O}(N_C)$ repulsion.

Another obvious improvement is to expand the basis used for the isorotational variables. As mentioned above, the nucleon-nucleon states (60) are not good eigenstates of the Hamiltonian (55) because $\mathcal{U}(\boldsymbol{R}, C)$ mixes states with different spin and isospin. Then the NN-potential becomes

$$V_{NN} = \langle NN'|H'|NN'\rangle - \sum_{\mathcal{N}\mathcal{N}' \neq NN'} \frac{\langle NN'|H'|\mathcal{N}\mathcal{N}'\rangle\langle\mathcal{N}\mathcal{N}'|H'|NN'\rangle}{E_{\mathcal{N}\mathcal{N}'} - 2M_N} + \ldots , \tag{62}$$

where $\mathcal{N}$ denotes the spin-isospin states as given by Eq. (52) and H' is the "intrinsic" (body-fixed) part of H. The energy differences $E_{\mathcal{N}\mathcal{N}'} - 2M_N \propto 1/\lambda_H$ are of the order $1/N_C$, however, so one cannot expect a perturbative expansion to work very well when the mixing becomes appreciable. Considering only nucleons and deltas,[1] it does not require a great deal of numerical

[1] Witten [12, 13] has argued that the only allowed generalized nucleon states are those with isospin $t \leq N_C/2$, which for N_C=3 means only the nucleon and the delta.

effort to simply diagonalize the matrix $\langle \mathcal{N}_1\mathcal{N}_2|H|\mathcal{N}_3\mathcal{N}_4\rangle$. Such a calculation has been performed by Saito, *et al.* [33] and DePace, *et al.* [41]. One important result is that all six allowed local terms in the nucleon-nucleon interaction are generated, so that the potential is now

$$V_{NN} = V_c(R) + V_\sigma(R)\sigma_1\cdot\sigma_2 + V_T(R)S_{12} + \tau_1\cdot\tau_2\Big(V_\tau(R) + V_{\tau\sigma}(R)\sigma_1\cdot\sigma_2 + V_{\tau T}(R)S_{12}\Big). \quad (63)$$

State mixing thus produces three new terms and corrections to the three terms which already exist from first approximation. Considering Eq. (62), one can expect that the inclusion of intermediate states will produce an attractive contribution to the central channel; and in fact a significant attractive contribution was found in Refs. [33, 41]. It is not enough, however, to overcome the repulsion found from the lowest-order term $\langle N_1' N_2'|\mathcal{U}|N_1N_2\rangle$.

There have also been many other suggested improvements. Kälbermann and Eisenberg [42] considered coupling to the Roper resonance — the breathing mode of the hedgehog — as a possible source of attraction. Jackson and Jackson [35, 43] have considered one-loop corrections from pion fluctuations about the product ansatz. Others have argued that one should consider extensions of the Skyrme model, either by including new higher derivative terms in the Lagrangian [44–46], or by explicitly adding vector mesons [47, 48] or a scalar "gluon" field [50] as described in Section 3. Of course, all this precludes the possibility that the product ansatz simply gives inaccurate results at short and intermediate ranges and that the nucleon-nucleon potential obtained from exact solutions might, in fact, be different. This is indeed the case.

5. Exact calculations

From the discussion in the previous section, it seems clear that one must analyze the equations of motion numerically. At the very least, this is necessary to determine the reliability of the various ansätze. There have been two sets of approaches in this direction. One method is to solve the full time-dependent problem. This is quite a difficult task if one wants to include all the important degrees of freedom. Nevertheless, some progress has been made in this direction. A second approach is to introduce constraints so as to fix the collective coordinates, and then to solve the resulting static equations of motion. The numerical effort involved in this second approach is considerably less. Its validity relies on the assumption of adiabaticity; that is the time-dependent motion can be parameterized in terms of a few collective variables.

5.1. Time-dependent numerical simulations

Time-dependent solutions of the classical field equations are useful in visualizing the dynamics as well as guiding suitable approximations to represent it. In a non-linear field theory, however, it is quite difficult to obtain such solutions. In most cases one has to rely on numerical methods. To see how these methods work I shall discuss the Skyrme Lagrangian for simplicity. It is convenient to rewrite the Lagrangian by introducing a four-vector

$$\Psi = \frac{1}{f_\pi}\begin{pmatrix}\sigma\\ \boldsymbol{\pi}\end{pmatrix}, \tag{64}$$

involving the chiral σ field and the three isospin components of the pion field. The fields Ψ_α, are subject to the constraint

$$\Psi_a^2 = \frac{1}{f_\pi^2}(\sigma^2 + \boldsymbol{\pi}^2) = 1\,, \tag{65}$$

at all space-time points and the boundary conditions $\Psi_0 = 1$, $\Psi_{1,2,3} = 0$ at spatial infinity. The Lagrangian density then takes the form [24, 52, 53]

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \Psi_a)^2 + \tfrac{1}{4}(\partial_\mu \Psi_a)^2(\partial_\nu \Psi_b)^2 - \tfrac{1}{4}(\partial_\mu \Psi_a \partial_\mu \Psi_b)^2 + \tfrac{1}{2}\mu(\Psi_a^2 - 1)\,. \tag{66}$$

The constraint (65) has been added via a Lagrange multiplier μ. Furthermore, dimensionless variables have been defined through the transformation

$$x_\mu \to \alpha x_\mu\,, \qquad \mathcal{L} \to \epsilon\alpha^{-3}\mathcal{L}\,, \tag{67}$$

where α, the characteristic length, and ϵ, the characteristic energy, are written in terms of the two parameters g and f_π as $\alpha = (f_\pi g)^{-1}$ and $\epsilon = f_\pi/g$. With $f_\pi = 93$ MeV and $g = 6.0$, for example, one finds $\alpha = 0.35$ fm^{-1} and $\epsilon = 15.5$ MeV. Introducing conjugate momenta $\Pi_a \equiv \delta\mathcal{L}/\delta(\partial_t \Psi_a)$, we obtain the Hamiltonian density as

$$\mathcal{H} = \tfrac{1}{2}\Pi_a \mathcal{M}^{-1}_{ab}\Pi_b + \tfrac{1}{4}(\partial_i \Psi_a)\mathcal{C}_{ab}(\partial_i \Psi_b) + \tfrac{1}{4}(\partial_i \Psi_a)^2\,. \tag{68}$$

From this follow the first-order equations of motion

$$\begin{aligned}\partial_t \Psi_a &= \mathcal{M}^{-1}_{ab}\Pi_b\,,\\ \partial_t \Pi_a &= \partial_i(\mathcal{C}_{ab}\partial_i \Psi_b) + \mu\Psi_b\,,\end{aligned} \tag{69}$$

which are well-suited for numerical time evolution. The symmetric 4×4 matrices $\mathcal{M}$ and $\mathcal{C}$ are defined as

$$\begin{aligned}\mathcal{M}_{ab} &= \left[\tfrac{1}{4} + (\partial_i \Psi_c)^2\right]\delta_{ab} - (\partial_i \Psi_a)(\partial_i \Psi_b)\,,\\ \mathcal{C}_{ab} &= \left[\tfrac{1}{4} - (\partial_\mu \Psi_c)^2\right]\delta_{ab} + (\partial_\mu \Psi_a)(\partial_\mu \Psi_b)\,.\end{aligned} \tag{70}$$

There are several methods for numerical solution of the field equations (69). A common way to proceed is to discretize them on a equidistant spatial lattice and express spatial derivatives by a finite difference scheme. Imposing the boundary conditions at the outer edges of the lattice and specifying initial conditions, the system is advanced in time — usually via leapfrog methods [54]. In terms of the known fields at previous times $t - \Delta t$ and t one obtains them at $t + \Delta t$. The leapfrog method may also be used as a relaxation method for finding static solutions. Then the time variable is interpreted as pseudo-time, and the momenta are set to zero after each time-step. The system thus relaxes to the minimum energy field configuration.

A way to study the short-distance behavior in the $B = 2$ sector and thereby test the validity of the "product ansatz" is to collide two solitons at high relative velocity and small impact parameter. From the solutions of the field equations quantities such as the deflection function $\theta(b)$, energy and momentum flows, soliton deformations, and so on, can be obtained. Explicit visualization of the non-linear processes exhibited by such "observables" may then guide systematic improvements of the product ansatz.

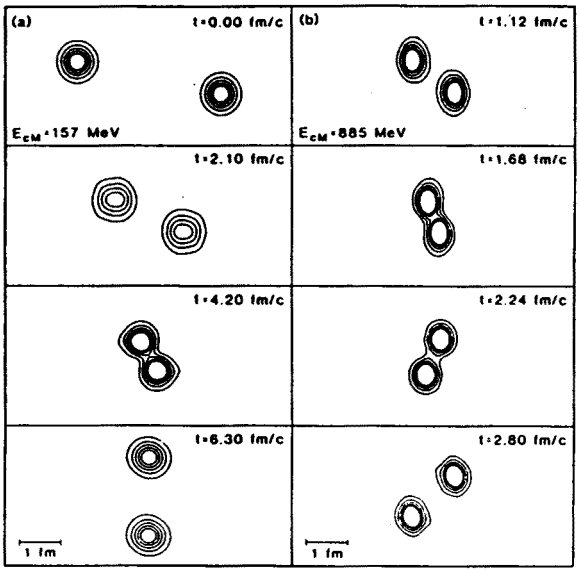

Fig. 3. The time-evolution of the baryon density $\mathcal{B}_0(x)$ in a soliton-soliton collision at an impact parameter of 0.8 fm and two initial velocities $v_0/c = 0.4$ and $v_0/c = 0.75$.

After initial work by Verbaarschot, *et al.* [24] in two space dimensions, collision simulations have been extended to three spatial dimensions by Allder, *et al.* [53] and more recently by Crutchfield, *et al.*[55, 56].

Two events from Ref. [53] for "defensive" hedgehogs (relative orientation $C = 1$) at impact parameter $b = 0.8$ fm and initial velocities $v_0/c = 0.40$ ($E_{cm} = 157$ MeV) and $v_0/c = 0.75$ ($E_{cm} = 885$ MeV) are shown in Fig. 3. At the lower energy, after the collision the solitons recede in a direction perpendicular to the incident directions. This "90° scattering" has also been found for zero impact parameter in the calculation of Crutchfield, *et al.* [55, 56] for the most attractive initial isospin configuration ($C = i\tau_2$). Here the scattering goes through the torus as an intermediate configuration: the skyrmions approach along the z-axis, form the torus at closest approach (with the y-axis the axis of symmetry), and recede along the x-axis. Precisely the same result is found for collisions in the S^2 σ model in $2 + 1$ dimensions studied by the Durham group [57]. It seems to be a generic feature of topological solitons. Indeed, studies of monopole scattering first led Manton [58] to suggest such a behavior for skyrmions as well.

At higher-energies the Lorentz contraction of the baryon density is clearly visible and the inelasticity increases. Systematic studies of collision trajectories as a function of impact parameter allow extraction of the deflection angle $\theta(b)$ and thereby of the classical cross section. The energy dependence of $\theta(b)$ is found to be very pronounced [53]. One can also infer the inelasticity as measured by the ratio $v_f(b)/v_i$ of final and initial velocity. As expected it depends strongly on b, being smallest at low energy and intermediate b, where both rotational and vibrational degrees of freedom are excited. At higher energies v_f/v_i rapidly decreases as the impact parameter decreases, indicating that most of the relative energy then goes into intrinsic excitations of the individual skyrmions.

While these types of numerical studies of the scattering problem are very useful in building one's intuition it should be kept in mind that the simulations are entirely classical and therefore render results of limited validity. For example, there are no thresholds for pion radiation and shape oscillations of scattered solitons for the classical system. A semiclassical treatment, on the other hand, is computationally non-trivial and not without conceptual difficulty — as is well known, for example, from time-dependent Hartree–Fock calculations for heavy-ion reactions.

5.2. Constrained static solutions

Static calculations are numerically more stable and easier to employ than time-dependent simulations. Several relaxation algorithms have been used for static $B = 2$ skyrmion calculations [52, 26, 30, 59]. To employ techniques such as these which find minimum-energy static configurations, one must introduce constraints that fix the collective coordinates of the two skyrmions. In the body-fixed frame, the relative ions-orientation can be fixed via boundary conditions at the $z = 0$ plane; and the separation

can be fixed via Lagrange multipliers [52] or by "pinning down" the points where $U_{2B} = -1$, the topological centers [25, 59]. In this way an explicit dependence of the two-skyrmion system upon the collective coordinates may be obtained.

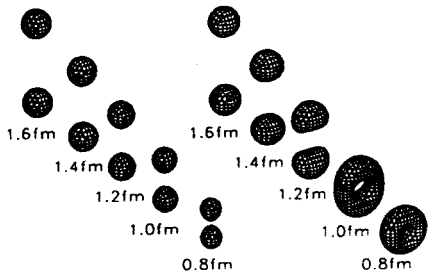

Fig. 4. The baryon density $B_0(\boldsymbol{x})$ for different separations R of two interacting skyrmions. The two relative isospatial orientations $C = 1$ (left) and $C = i\tau_2$ (right) are shown.

Now, as discussed in Section 4.1, the truncated Hamiltonian (55) for the interacting $B = 2$ system contains a potential $\mathcal{U}(R, C)$ which depends on only the separation R, taken here to be along the z-axis, and the relative isospatial orientation C. The latter can be expressed as Euler angles via the SU(2) matrix $C = A_1^\dagger A_2$ (see Eq. (48)). The product ansatz, as was shown in Section 4.2, generates a potential $\mathcal{U}$ which is essentially determined by three functions of R — one function corresponding to an isospin-exchange of zero and the other two to an isospin-exchange of one between the skyrmions. It seems reasonable — and is certainly numerically expedient — to also assume for the exact calculations that the potential contains only these terms. They are generated from the three relative iso-rotations $C = 1, i\tau_2, i\tau_3$ depicted in Fig. 2. The potential for these orientations may be found from the exact field solution obtained through constrained minimization of the static Lagrangian which holds the distance R fixed [51]. Such a calculation has been performed by discretizing the equations of motion (69) on a spatial lattice and relaxing from an initial B=2 configuration via the pseuto-time algorithm described in Section 5.1. Fig. 4 displays the evolution of the baryon number density as the two solitons approach in a given isospatial orientation [51, 60]. The cases $C = 1$ and $C = i\tau_2$ are shown; the situation for $C = i\tau_3$ looks much like that for $C = 1$. As the solitons come close together significant distortions in shape are observed. Consistent with time-dependent simulations they shrink on approach for $C = 1$. As expected, formation of the torus, the lowest-energy $B = 2$ configuration, is found for $C = i\tau_2$. This leads to an attractive interaction energy and provides medium-range attraction already in first approximation, and may resolve a long-standing

problem [51]. Indeed, the central, tensor, and spin-spin potentials obtained via Eq. (61) from the exact calculations compare much better with the Paris potential [61], for example, than do the corresponding potentials calculated using the product ansatz [51]. As indicated earlier these calculation can be improved by allowing for mixing through intermediate states [33, 41, 62, 60], which corrects the potential (61) still to leading order in $1/N_C$. If we consider the separation R as a "slow" variable such that the isospatial orientations C adjust instantaneously (see Section 4.1), this implies diagonalization of the intrinsic part of the Hamiltonian (55) in the space of asymptotic ($R \to \infty$) NN, $N\Delta$ and $\Delta\Delta$ states. This Born–Oppenheimer approach is a familiar procedure in molecular and nuclear physics. Most conveniently, the states are written in an $LSJT$-representation labeled by the conserved (total) isospin T, the total angular momentum J as well as the orbital angular momentum L and the spin S. The model space splits into subspaces that are either symmetric or antisymmetric under the exchange of identical particles. In accordance with the Pauli principle we must select the antisymmetric ones, where the states satisfy the standard selection rule $L + S + T =$ odd. Indeed, insistence on antisymmetry under exchange of two skyrmions [52] leads to the so-called Finkelstein–Rubinstein constraints [63, 64], which correspond to the above selection rule.

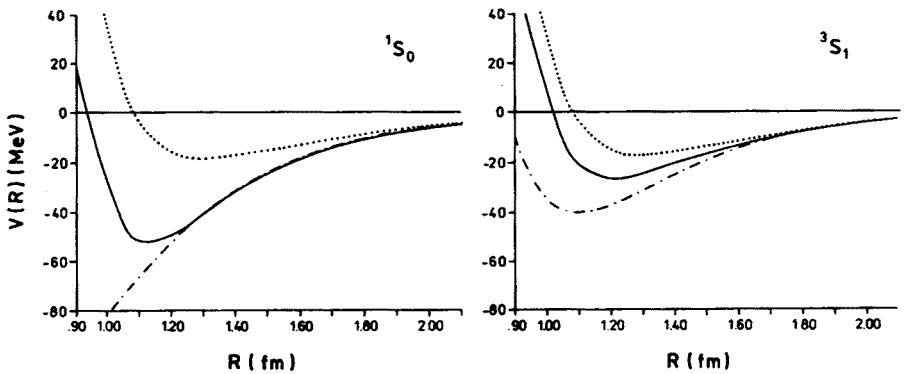

Fig. 5. The nucleon-nucleon potential in the $^1S_0$ ($T = 1$) and $^3S_1$ ($T = 0$) channels. The dotted lines indicate the lowest-order results, while the full lines include mixing through intermediate NN, $N\Delta$ and $\Delta\Delta$ states. The RSC potential is indicated by the dashed-dotted lines.

One may follow a state at large R, where it is made up purely of nucleons, to small R, where there is mixing. The energy eigenvalue of this state as a function of R then defines the adiabatic nucleon-nucleon potential. At $R \sim$ 1fm, the mixing amplitudes are typically 2-4%, which is similar to results from conventional meson-exchange potentials with intermediate iso-

bars. As with the product ansatz calculation this inclusion of intermediate states generates the three remaining allowed local terms in the nucleon-nucleon potential — namely, $V_\tau(R)\tau_1 \cdot \tau_2$, $V_\sigma(R)\sigma_2 \cdot \sigma_2$, and $V_T(R)S_{12}$ — as well as corrections to the three other local terms in Eq. (61). Results [62, 60] are shown in Fig. 5 for the $^1S_0$ channel (left part) and the $^3S_1$ channel (right part). They indicate that mixing leads to a considerable increase in the attraction, particularly for $^1S_0$, and hence an improvement when compared with phenomenological interactions (here the Reid soft core (RSC) [65] potential). Similar agreement is achieved in other channels.

As discussed in Section 4.1 there are also $\mathcal{O}(1/N_C)$ contributions from the translational kinetic term. These give rise to non-local spin-orbit and momentum-dependent terms, as well as to higher order corrections to the local potential. Evaluation of the mass matrix M, however, is not a simple task since it involves derivatives with respect to the relative coordinates R and C. This means one must find field solutions for many values of these coordinates, which increases the computational effort needed. For R, this increase is not too great; and it has been found that the inertial mass of an interacting skyrmion does not differ much from its asymptotic value M_H until it nears the separation where the torus forms. This results in a short-range momentum-dependent contribution. Determination of the C-dependence of M — and therefore the spin-orbit interaction — from exact numerical calculations, however, remains a problem for future study.

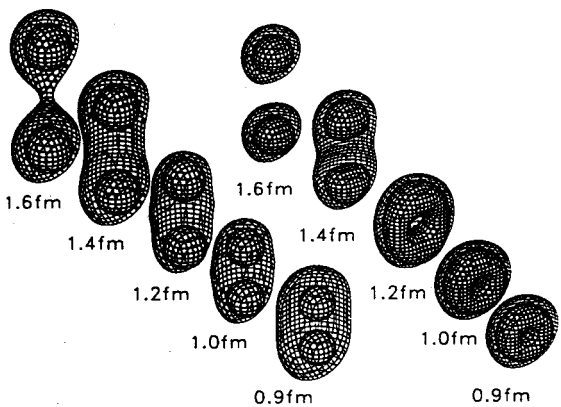

Fig. 6. Contours of constant baryon density $\mathcal{B}_0(\boldsymbol{x})$ and constant gluonium field $\chi(\boldsymbol{x})$ for different values of R, the separation of two interacting solitons. The two relative isospatial orientations $C = 1$ (left) and $C = i\tau_2$ (right) are shown.

Since after inclusion of intermediate states the skyrmion results become quantitatively comparable to empirical nucleon-nucleon potentials, one may ask what the role of gluons is — especially since the scalar gluonium field

χ is expected to give additional attraction [50]. This question can be studied within the minimal model of Schechter [19] (Section 2.3), where the Skyrme Lagrangian is supplemented by gluon degrees of freedom to ensure anomalous scaling behavior. In the $B = 1$ sector this leads to more or less pronounced bag formation (Fig. 1). With the numerical machinery for solving the equations of motion in hand, it is possible to include gluons also in the $B = 2$ sector; and some results have recently been obtained [60]. Fig. 6 displays the baryon density together with contours of constant χ field at half of its smallest value for the two isospatial orientations $C = 1, i\tau_2$. At large separation each soliton is embedded in its own "bag". These bags begin to merge, however, at distances of $\sim$ 1.5 fm; and the two solitons then reside in a common bag. This leads to considerable increase in the attraction, especially for V_C [60], and it then becomes a quantitative question which values of the glueball mass and the gluon condensate are compatible with phenomenological interactions. It seems that shallow bags with large values of the gluon condensate are to be preferred over deep bags, in agreement with constraints from glueball decay into two pions [67]. After inclusion of intermediate state mixing, such shallow bags further improve the agreement with empirical potentials such as RSC or the Argonne potential [66]. In general, the deviations are of the order of the differences in the phenomenological potentials themselves.

6. Concluding remarks

I have dicussed how a realization of baryons as topological solitons of an effective chiral theory leads to a nucleon-nucleon interaction which compares well with realistic potentials. Such theories are modeled on QCD and incorporate $1/N_C$ as the natural expansion parameter which characterizes the semiclassical quantization of their solitons. Calculations have been presented under the assumption that only twelve zero and nearly-zero collective modes of the two-skyrmion system need be considered for quantization. From consideration of the product ansatz, it was shown that the theory produces the correct long-range OPEP form for the nucleon-nucleon interaction. At closer range one must resort to numerical calculations. Then an intermediate-range attraction in the central channel and a short-range repulsion was found. Depending on the definition of separation, the core is of similar range to that of the Paris potential. When intermediate states were included, all six allowed local terms in the nucleon-nucleon interaction are present, with relative strengths which agree with phenomenological potentials. This is all at order N_C. It was shown that when the order-$1/N_C$ kinetic terms which contain the inertial masses and moments of inertia of the two interacting skyrmions are considered, spin-orbit and momentum-dependent terms are also generated. The picture of nucleons as solitons

thus yields a natural derivation of a qualitatively accurate nucleon-nucleon potential, and the use of extended Skyrme models seems likely to produce quantitative agreement with phenomenology.

The author have benefitted greatly from discussions and collaboration with J. Verbaarschot, T. Walhout and H.W. Wyld. I would like to especially thank T. Waindzoch for help with some of the calculations and figures as well as for providing his most recent results on gluonic effects.

REFERENCES

[1] H. Yukawa, *Proc. Phys. Math. Soc. Japan* **17**, 48 (1935).
[2] R. Machleidt, K. Holinde, C. Elster, *Phys. Rep.* **149**, 1 (1987).
[3] G. t'Hooft, *Nucl. Phys.* **B72**, 461 (1974); *Nucl. Phys.* **B75**, 461 (1974).
[4] E. Witten, *Nucl. Phys.* **B160**, 57 (1979).
[5] T.H.R. Skyrme, *Proc. Roy. Soc.* **260**, 127 (1961).
[6] T.H.R. Skyrme, *Nucl. Phys.* **31**, 556 (1962).
[7] A. Bohr, B.R. Mottelson, *Nuclear Structure*, Vol. 1 ed. Benjamin, New York, 1969.
[8] R. Rajaraman, *Solitons and Instantons*, North-Holland, Amsterdam, 1987.
[9] B.W. Lee, *Chiral Dynamics*, eds Gordon & Breach, London, 1972.
[10] S. Weinberg, *Phys. Rev. Lett.* **166**, 1568 (1968).
[11] J. Sakurai, *Ann. Phys.* **11**, 1 (1960); *Currents and Mesons*, University of Chicago Press, 1969.
[12] E. Witten, *Nucl. Phys.* **B223**, 422 (1983).
[13] E. Witten, *Nucl. Phys.* **B223**, 433 (1983).
[14] K. Kawarabayashi, M. Suzuki, *Phys. Rev. Lett.* **16**, 255 (1966); Riazuddin, Fayyazuddin, *Phys. Rev.* **147**, 1071 (1966).
[15] U.G. Meißner, *Phys. Rep.* **161**, 213 (1988).
[16] J.C. Collins, A. Duncan, S.D. Joglekar, *Phys. Rev.* **D16**, 438 (1977).
[17] N.K. Nielsen, *Nucl. Phys.* **B210**, 212 (1977).
[18] J. Ellis, *Nucl. Phys.* **B22**, 478 (1970).
[19] J. Schechter, *Phys. Rev.* **D21**, 3393 (1981).
[20] H. Gomm, J. Schechter, *Phys. Lett.* **158B**, 449 (1985).
[21] G.S. Adkins, C.R. Nappi, E. Witten, *Nucl. Phys.* **B228**, 552 (1983).
[22] G. Holzwarth, B. Schwesinger, *Rep. Prog. Phys.* **49**, 825 (1986).
[23] M. Kutschera, C.J. Pethick, *Nucl. Phys.* **A440**, 670 (1985).
[24] J.J.M. Verbaarschot, T.S. Walhout, J. Wambach, H.W. Wyld, *Nucl. Phys.* **A461**, 603 (1986).
[25] V.B. Kopeliovich, B.E. Shtern, *JETP Lett.* **45**, 203 (1987); S.V. Zenkin, V.B. Kopeliovich, B.E. Shtern, *Sov. J. Nucl. Phys.* **45**, 106 (1987).
[26] E. Braaten, L. Carson, *Phys. Rev.* **D38**, 3525 (1988); *Phys. Rev.* **D39**, 838 (1989).
[27] N.S. Manton, *Phys. Lett.* **B192**, 177 (1987).

[28] D.I. Dyakonov, *Sov. J. Nucl. Phys.* **45**, 987 (1987); D.I. Dyakonov, A.D. Mirlin, *Sov. J. Nucl. Phys.* **47**, 421 (1988).
[29] N.S. Manton, *Phys. Rev. Lett.* **60**, 1916 (1988).
[30] T.S. Walhout, *Nucl. Phys.* **A531**, 596 (1991).
[31] T.S. Walhout, J. Wambach. *From Skyrmions to the Nucleon-Nucleon Potential, IJMPE Reports in Nuclear Physics* (1992), in press.
[32] H. Yabu, K. Ando, *Prog. Theor. Phys.* **74**, 750 (1985).
[33] T. Otofuji, S. Saito, M. Yasuno, T. Kurihara, H. Kanada, *Phys. Rev.* **C34**, 1559 (1986); S. Saito, *Nucl. Phys.* **A463**, 169c (1987).
[34] A. Jackson, A.D. Jackson, V. Pasquier, *Nucl. Phys.* **A432**, 567 (1985).
[35] A.D. Jackson, A.Jackson, *Nucl. Phys.* **A446**, 89c (1985).
[36] R. Vinh Mau, M. Lacombe, B. Loiseau, W.N. Cottingham, P. Lisboa, *Phys. Lett.* **150B**, 259 (1985).
[37] M. Oka, *Phys. Lett.* **175B**, 15 (1986).
[38] H. Odawara, O. Morimatsu, K. Yazaki, *Phys. Lett.* **175B**, 115 (1986).
[39] E.M. Nyman, D.O. Riska, *Phys. Lett.* **175B**, 392 (1986).
[40] D.O. Riska, E.M. Nyman, *Phys. Lett.* **183B**, 7 (1987).
[41] H. DePace, H. Müther, A. Faessler,*Nucl. Phys.* **A457**, 541 (1986); *Z. Phys.* **A326**, 33 (1987); *Phys. Lett.* **188B**, 307 (1987).
[42] G. Kälbermann, J.M. Eisenberg, *Nucl. Phys.* **A500**, 589 (1989).
[43] A. Jackson, A.D. Jackson, *Nucl. Phys.* **A457**, 687 (1986).
[44] A. Jackson, A.D. Jackson, A.S. Goldhaber, G.E. Brown, L.C. Castillejo, *Phys. Lett.* **154B**, 101 (1985).
[45] G. Kälbermann, J.M. Eisenberg, R.R. Silbar, M.M. Sternheim, *Phys. Lett.* **179B**, 4 (1986).
[46] M. Oka, *Phys. Rev.* **C36**, 720 (1987).
[47] M. Lacombe, B. Loiseau, R. Vinh Mau, W.N. Cottingham, *Phys. Lett.* **161B**, 31 (1985); *Phys. Lett.* **169B**, 121 (1986).
[48] M. Lacombe, B. Loiseau, R. Vinh Mau, W.N. Cottingham, *Phys. Rev. Lett.* **57**, 170 (1985).
[49] E.M. Nyman, D.O. Riska, *Phys. Scr.* **34**, 533 (1986).
[50] H. Yabu, B. Schwesinger, G. Holzwarth, *Phys. Lett.* **224B**, 25 (1989).
[51] T.S. Walhout, J. Wambach, *Phys. Rev. Lett.* **67**, 314 (1991).
[52] J.J.M. Verbaarschot, T.S. Walhout, J. Wambach, H.W. Wyld, *Nucl. Phys.* **A468**, 520 (1987).
[53] A.E. Allder, S.E. Koonin, R. Seki, H.M. Sommermann, *Phys. Rev. Lett.* **59**, 2836 (1987).
[54] D. Potter, *Computational Physics*, Wiley Press, New York, 1973.
[55] W.Y. Crutchfield, J.B. Bell, LLNL Report UCRL-JC-107427 (1991).
[56] W.Y. Crutchfield, N. Snyderman, V.R. Brown, *Phys. Rev. Lett.* **68**, 1660 (1992).
[57] R.A. Leese, M. Peyard, W. Zakrewski, *Nonlinearity* **3**, 773 (1990).
[58] N.S. Manton, *Phys. Lett.* **B198**, 226 (1987).
[59] B.E. Shtern, *Sov. J. Nucl. Phys.* **49**, 332 (1989). (1958) 1178 .
[60] T. Waindzoch, J. Wambach, *Phys. Lett.* **B**, in press.

[61] M. Lacombe, B. Loiseau, J. M. Richard, R. Vinh Mau, J. Cote, D. Pires, R. DeToureil, *Phys. Rev.* **C21**, 861 (1980).
[62] N.R. Walet, R.D. Amado, A. Hosaka, preprint UPR-00123MT, submitted to *Phys. Rev. Lett.*
[63] D. Finkelstein, J. Rubinstein, *J. Math. Phys.* **9**, 1762 (1968).
[64] E. Braaten, L. Carson, *Phys. Rev.* **D38**, 3525 (1988).
[65] R.V. Reid, *Ann. Phys.* **50**, 411 (1968).
[66] R.B. Wiringa, R.A. Smith, T.L. Ainsworth, *Phys. Rev.* **C29**, 1207 (1984).
[67] P. Jain, R. Johnson, J. Schechter, *Phys. Rev.* **D35**, 2230 (1987).