COMMENTS ON THE MOFFAT'S NONSYMMETRIC THEORY OF GRAVITATION*

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The metricity condition and Lagrangian of Moffat's pure nonsymmetric gravity are shown to be incompatible with the assumptions and definitions used. The correct metricity condition and Lagrangian are derived and some conclusions are drawn.

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1. Introduction

General Relativity (GR) is the simplest theory of gravity in agreement with all the present day experimental data. One may wonder why then one should try to formulate alternative or generalized versions of this theory, without the need of explaining some new predictions. The motivations have a theoretical character, and they arise essentially if one compares general relativity with the standard model. The latter consists of the strong, weak and electromagnetic forces based on the gauge group SU(3)⊗SU(2)⊗U(1) and described by quantum relativistic fields interacting in a flat Minkowski space-time. On the contrary, the gravitational interactions modify the geometrical structure of space-time. In fact, they are represented by a new field, associated with the deformation of the geometry itself. Therefore, while three quarters of modern physics (standard model) acting at a microscopical level are successfully described at present in the framework of a flat and rigid space-time structure, the remaining quarter (the macroscopic physics of gravity) needs the introduction of a curved geometrical and dynamical background.

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To overcome this unsatisfactory situation it seems appropriate to try to extend the geometrical principles of GR in order to establish a possible connection between gravity and the other interactions.

One alternative, initiated first by Einstein, is to consider the metric $g_{\mu\nu}$ to be nonsymmetric [1]. The goal of this attempt was to unify gravitation and electromagnetism. In spite of the failure of this idea, Einstein successors were convinced that there is a part of truth in it. In particular, Moffat revived the problem but as a purely gravitational one [2–8]. Moffat's classical nonsymmetric gravitation theory (NGT) has been developed in a series of papers showing the consistency of the theory with the present day observational data [7].

In Section 2, we present briefly Moffat's formalism of NGT. In Section 3 we give our comments on the theory and show the non compatibility of the metricity condition and Lagrangian (given by Moffat) with the assumptions and definitions used. The ambiguities are shown explicitly, and the correct metricity condition and Lagrangian are derived. In Section 4, we draw our conclusions.

2. Formalism

In the Moffat's NGT, formulated successfully in a hyperbolic complex space [4, 6–8], the fundamental metric tensor $g_{\mu\nu}$ consist of symmetric and antisymmetric parts $g_{(\mu\nu)}$ and $g_{[\mu\nu]}$ respectively, and takes its values in the ring of hyperbolic complex numbers

$$g_{\mu\nu} = g_{(\mu\nu)} + \varepsilon g_{[\mu\nu]} \qquad (\varepsilon^2 = 1)\,. \tag{2.1}$$

The tensor $g_{\mu\nu}$ is hyperbolic complex Hermitian $\bar{g}_{\mu\nu} = g_{\nu\mu}$ ($\bar{g}_{\mu\nu}$ is the complex conjugate of the hyperbolic complex tensor $g_{\mu\nu}$) and its inverse $g^{\mu\nu}$ is defined by

$$g^{\mu\nu} g_{\rho\nu} = g^{\nu\mu} g_{\nu\rho} = \delta^{\mu}_{\rho}\,. \tag{2.2}$$

A displacement Hermitian field $\Gamma^{\lambda}_{\mu\nu}$ ($\bar{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$) was introduced, defining the variation of a vector components A^{λ} in an infinitesimal parallel transport

$$\delta A^{\lambda} = -\Gamma^{\lambda}_{\mu\nu} A^{\nu} dx^{\mu}\,. \tag{2.3}$$

The affine connection $W^{\lambda}_{\mu\nu}$ is defined by

$$W^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \tfrac{2}{3}\delta^{\lambda}_{\mu} W_{\nu}\,, \tag{2.4}$$

where

$$W_{\nu} \equiv W^{\alpha}_{[\nu\alpha]} = \tfrac{1}{2}(W^{\alpha}_{\nu\alpha} - W^{\alpha}_{\alpha\nu})\,. \tag{2.5}$$

The action in the case of the vacuum takes the form (given in Refs [4, 7])

$$S = -\frac{1}{16\pi G}\int(|g|)^{1/2} g^{\mu\nu} R_{\mu\nu}(W) d^4x\,, \tag{2.6}$$

where G is the Newton's gravitation constant, $g = \det(g_{\mu\nu})$ and $R_{\mu\nu}(W)$ is given by

$$R_{\mu\nu}(W) = W^{\beta}_{\mu\nu,\beta} - \tfrac{1}{2}(W^{\beta}_{\mu\beta,\nu} + W^{\beta}_{\nu\beta,\mu}) - W^{\beta}_{\alpha\nu}W^{\alpha}_{\mu\beta} + W^{\beta}_{\alpha\beta}W^{\alpha}_{\mu\nu}\,. \tag{2.7}$$

A hyperbolic complex vierbein e^a_μ obeying the following orthogonality condition was also introduced [8]

$$e^{\mu}_{a} e^{b}_{\mu} = \delta^{b}_{a}\,, \qquad e^{a}_{\mu} e^{\nu}_{a} = \delta^{\nu}_{\mu}\,. \tag{2.8}$$

Thus, the sesquilinear form of the tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = e^{a}_{\mu}\bar{e}^{b}_{\nu}\eta_{ab}\,, \tag{2.9}$$

where η_{ab} is the Minkowskian flat space metric and $\tilde{e}^b_\nu$ is the complex hyperbolic conjugate of the vierbein e^b_ν. The vierbein satisfies the compatibility condition [8]

$$e^{a}_{\mu,\sigma} + (\omega_\sigma)^{a}_{c} e^{c}_{\mu} - W^{\rho}_{\sigma\mu} e^{a}_{\rho} = 0\,, \tag{2.10}$$

where ω_σ is the NGT spin connection, defining the covariant derivative D_σ,

$$D_\sigma e^{a}_{\mu} = \partial_\sigma e^{a}_{\mu} + (\omega_\sigma)^{a}_{b} e^{b}_{\mu} \tag{2.11}$$

and satisfying the following condition:

$$(\omega_\sigma)_{ca} = -(\tilde{\omega}_\sigma)_{ac}\,. \tag{2.12}$$

The assumption of the hermiticity of the affine connection $W^{\lambda}_{\mu\nu}$ ($\tilde{W}^{\lambda}_{\mu\nu} = W^{\lambda}_{\nu\mu}$, as it is the case in Ref. [8]) implies that the compatibility condition (2.10) is simplified to the form

$$g_{\mu\nu,\sigma} - g_{\rho\nu}W^{\rho}_{\mu\sigma} - g_{\mu\rho}W^{\rho}_{\sigma\nu} = 0\,. \tag{2.13}$$

Moreover, the curvature tensor and scalar curvature expressed in the holonomic coordinates are given respectively by

$$R^{\lambda}{}_{\sigma\mu\nu} = (R_{\mu\nu})^{a}_{b} e^{\lambda}_{a} e^{b}_{\sigma} \tag{2.14}$$

and

$$R = e^{\mu a}\tilde{e}^{\nu b}(R_{\mu\nu})_{ab}\,, \tag{2.15}$$

where

$$(R_{\mu\nu})^a_b = \left([D_\mu, D_\nu]\right)^a_b = (\omega_\nu)^a_{b,\mu} - (\omega_\mu)^a_{b,\nu} + [\omega_\mu, \omega_\nu]^a_b \,. \tag{2.16}$$

Finally, the action of NGT, in the absence of matter, is given in Ref. [8] by

$$S_{\rm NGT} = -\frac{1}{16\pi G}\int |e|(R_{\mu\nu})^a_b e^\mu_a \bar{e}^{\nu b} d^4x \,, \tag{2.17}$$

where $|e| = (e\bar{e})^{1/2}$, with $e = \det(e^a_\mu)$.

3. Comments

Taking the definitions (2.4) and (2.5) of the affine connection $W^\lambda_{\mu\nu}$ together with the hypothesis $\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ [4, 7], one deduces immediately that

$$\tilde{W}^\lambda_{\mu\nu} + \tfrac{2}{3}\delta^\lambda_\mu \tilde{W}_\nu = W^\lambda_{\nu\mu} + \tfrac{2}{3}\delta^\lambda_\nu W_\mu \,. \tag{3.1}$$

Now, if one takes into account the hermiticity of $W^\lambda_{\mu\nu}$ (as it is the case in Ref. [8]), we end up with vanishing W_μ (defined in equation (2.5)), which implies automatically that

$$W^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} \,.$$

It is worth to mention that the latter relation contradicts all the fundamental relations in NGT. So, one cannot have the hermiticity of $W^\lambda_{\mu\nu}$ and $\Gamma^\lambda_{\mu\nu}$ simultaneously. Moreover, the Moffat's compatibility condition (2.10) (see Eq. (2.2) of Ref. [8]), is not only inconsistent with the given action (2.17) but with the following metricity condition (see Eq. (2.6) of Ref. [8]) as well

$$\partial_\sigma g_{\mu\nu} - g_{\rho\nu} W^\rho_{\mu\sigma} - g_{\mu\rho}\tilde{W}^\rho_{\nu\sigma} = 0 \,. \tag{3.2}$$

Consequently, the simplified metricity condition, Eq. (2.13) (obtained by assuming the hermiticity of $W^\lambda_{\mu\nu}$) is incompatible with Eq. (2.10) and the action (2.17).

In fact, to have the condition (3.2) and consistency with the proposed action (2.17), one has to start with the compatibility condition

$$e^a_{\mu,\sigma} + (\omega_\sigma)^a_c e^c_\mu - W^\rho_{\mu\sigma} e^a_\rho = 0 \qquad \text{(instead of } W^\rho_{\sigma\mu}\text{)} \,. \tag{3.3}$$

Then, using

$$\partial_\sigma g_{\mu\nu} = \partial_\sigma (e^a_\mu \bar{e}^b_\nu \eta_{ab}) \tag{3.4}$$

together with the assumed condition (2.12) and relation (3.1), one gets

$$\partial_\sigma g_{\mu\nu} - g_{\rho\nu}(W^\rho_{\mu\sigma} - \tfrac{2}{3}\delta^\rho_\mu \tilde{W}_\sigma) - g_{\mu\rho}(W^\rho_{\sigma\nu} + \tfrac{2}{3}\delta^\rho_\sigma W_\nu) = 0 \,. \tag{3.5}$$

Choosing W_σ to be pure imaginary (*i.e.* $\tilde{W}_\sigma = -W_\sigma$) leads to the following simplified form of Eq. (3.5)

$$\partial_\sigma g_{\mu\nu} - g_{\rho\nu}\Gamma^\rho_{\mu\sigma} - g_{\mu\rho}\Gamma^\rho_{\sigma\nu} = 0 \tag{3.6}$$

instead of Eq. (2.13). Here, $\Gamma^\rho_{\mu\sigma}$ is given by Eqs (2.3) and (2.4). It is important to mention that in all Moffat's papers about NGT, the chosen action yields equation (3.6) and not (2.13). Moreover, these two equations are incompatible except in the trivial case $W_\mu = 0$ or equivalently $W^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}$.

At present let us examine the form of the action. If one uses our compatibility condition (3.3) as well as the covariant derivative definition (2.11), the tensor $(R_{\mu\nu})^a_b$ can be written as

$$(R_{\mu\nu})^a_b = W^\rho_{\alpha\nu,\mu}e^a_\rho e^\alpha_b - W^\rho_{\alpha\mu,\nu}e^a_\rho e^\alpha_b + W^\rho_{\alpha\mu}W^\alpha_{\beta\nu}e^a_\rho e^\beta_b - W^\beta_{\rho\nu}W^\rho_{\alpha\mu}e^a_\beta e^\alpha_b \,. \tag{3.7}$$

Thus the Lagrangian of the action (2.17) will have the expression

$$\mathcal{L} = |e|e^\mu_a\tilde{e}^{\nu b}(R_{\mu\nu})^a_b \equiv |g|^{1/2}g^{\mu\nu}R_{\mu\nu}(W)\,, \tag{3.8}$$

where[1]

$$R_{\mu\nu}(W) = W^\alpha_{\mu\nu,\alpha} - W^\alpha_{\mu\alpha,\nu} + W^\beta_{\alpha\beta}W^\alpha_{\mu\nu} - W^\alpha_{\rho\nu}W^\rho_{\mu\alpha}\,. \tag{3.9}$$

In fact, the NGT Lagrangian (3.8) is the one proposed by Moffat in his first paper (see Eq. (2.15) of Ref. [2] without the source term and taking into account the definitions (2.7) and (2.16) of the same reference). Moreover, in the recent papers [4, 5, 7], Moffat has adopted a different expression for the Lagrangian, namely the one given in (2.6) with $R_{\mu\nu}(W)$ defined in (2.7).

It is to be noted that by using the constraint (3.3), it is possible to get the Lagrangian (2.6) instead of (3.8), in terms of the hyperbolic complex vierbeins e^a_μ and $(R_{\mu\nu})^a_b$. In fact, from Eq. (2.16), one has

$$(R_{\mu\nu})^a_a = (\omega_\nu)^a_{a,\mu} - (\omega_\mu)^a_{a,\nu}\,. \tag{3.10}$$

Moreover, Eq. (3.3) and the relation

$$e^\alpha_a e^a_{\alpha,\mu} = \mathrm{Tr}\,([e^{-1}]\partial_\mu[e]) = \partial_\mu \log e \tag{3.11}$$

give

$$\begin{aligned}(\omega_\mu)^a_a &= W^\alpha_{\alpha\mu} - \partial_\mu \log e\,,\\ (\tilde{\omega}_\mu)^a_a &= \tilde{W}^\alpha_{\alpha\mu} - \partial_\mu \log \tilde{e}\,,\end{aligned} \tag{3.12}$$

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[1] The expression (3.9) was denoted by $A_{\mu\nu}$ in Ref. [4] and $B_{\mu\nu}$ in Ref. [7].

where $[e]$ is the matrix e^a_μ and $e = \det[e]$. Now the relation (3.1), together with the choice $\tilde{W}_\sigma = -W_\sigma$ leads to

$$\tilde{W}^\lambda_{\mu\nu} = W^\lambda_{\nu\mu} + \tfrac{2}{3}\delta^\lambda_\nu W_\mu + \tfrac{2}{3}\delta^\lambda_\mu W_\nu \,. \tag{3.13}$$

After a simple contraction of the indices μ and λ, we obtain

$$\tilde{W}^\alpha_{\alpha\nu} = W^\alpha_{\nu\alpha} + \tfrac{10}{3} W_\nu \,. \tag{3.14}$$

Now, equating $(\omega_\sigma)^a_a$ and $(-\tilde{\omega}_\sigma)^a_a$, (given by Eqs (3.12)), one ends up with the expression

$$W^\alpha_{\alpha\mu} = 4W^\alpha_{\mu\alpha} - \tfrac{3}{2}\partial_\mu \log(e\bar{e}) \,. \tag{3.15}$$

With the help of (3.15) and if one substitutes $(\omega_\mu)^a_a$ (as given by Eq. (3.12)) in (3.10), one finally obtains

$$-\tfrac{1}{8}(R_{\mu\nu}) = -\tfrac{1}{2}(W^\alpha_{\nu\alpha,\mu} - W^\alpha_{\mu\alpha,\nu}) \,. \tag{3.16}$$

Thus, expressed in nonholonomic coordinates, the Lagrangian used by Moffat in Refs [4, 5, 7] has to have the following expression

$$\mathcal{L} = |e|\left((R_{\mu\nu})^a_b e^\mu_a \bar{e}^{\nu b} - \tfrac{1}{8}(R_{\mu\nu})^a_a e^\mu_b \bar{e}^{\nu b}\right) . \tag{3.17}$$

It is worth to stress that the compatibility condition (2.10) given by Moffat (in Ref. [8]) is neither consistent with the Lagrangian (3.8) (with (3.9)) nor with (3.17), and thus has to be replaced by Eqs (3.3) and (3.6).

Now, one may wonder if the hypothesis $\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ and the choice $\tilde{W}_\mu = -W_\mu$ will yield the hermiticity of the Lagrangian (3.17). To get the answer, let us write the Lagrangian (3.17) in holonomic coordinates as

$$\mathcal{L} = |g|^{1/2} g^{\mu\nu}\left[R^{(1)}_{\mu\nu} - \tfrac{1}{8}R^{(2)}_{\mu\nu}\right] , \tag{3.18}$$

where

$$R^{(1)}_{\mu\nu} = R^\lambda_{\mu\lambda\nu} = W^\lambda_{\mu\nu,\lambda} - W^\lambda_{\mu\lambda,\nu} + W^\lambda_{\alpha\lambda} W^\alpha_{\mu\nu} - W^\lambda_{\rho\nu} W^\rho_{\mu\lambda} \tag{3.19}$$

and

$$R^{(2)}_{\mu\nu} = R^\lambda{}_{\lambda\mu\nu} = W^\lambda_{\lambda\nu,\mu} - W^\lambda_{\lambda\mu,\nu} \,. \tag{3.20}$$

From Eqs (3.15) and (2.4) together with the relation $\Gamma^\alpha_{\alpha\mu} = \Gamma^\alpha_{\mu\alpha}$ (see the Appendix), one obtains

$$\Gamma^\alpha_{\mu\alpha} = \tfrac{1}{2}\partial_\mu \log(e\bar{e}) \tag{3.21}$$

which implies that

$$\Gamma^\alpha_{\mu\alpha,\nu} = \Gamma^\alpha_{\nu\alpha,\mu} \,. \tag{3.22}$$

The substitution of $W^{\lambda}_{\mu\nu}$, given by (2.4), in (3.19) yields

$$R^{(1)}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\lambda}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu} - \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} + \tfrac{2}{3}(W_{\mu,\nu} - W_{\nu,\mu})\,. \tag{3.23}$$

A straightforward calculation using the hermiticity of $\Gamma^{\lambda}_{\mu\nu}$ and the choice $\tilde{W}_{\mu} = -W_{\mu}$, together with the relation (3.22), gives

$$\tilde{R}^{(1)}_{\mu\nu} = R^{(1)}_{\nu\mu}\,. \tag{3.24}$$

On the other hand, and in order to show hermiticity of $R^{(2)}_{\mu\nu}$, it is preferable to write this tensor in terms of W_{μ}. In fact, equation (3.15) leads to the relation

$$W^{\alpha}_{\alpha\mu} = -\tfrac{8}{3}W_{\mu} + \tfrac{1}{2}\partial_{\mu}\log(e\tilde{e})\,, \tag{3.25}$$

which implies (together with expression (3.20)) that

$$R^{(2)}_{\mu\nu} = -\tfrac{8}{3}(W_{\nu,\mu} - W_{\mu,\nu})\,. \tag{3.26}$$

Now, thanks to the property $\tilde{W}_{\mu} = -W_{\mu}$, it is obvious from Eq. (3.26) that

$$\tilde{R}^{(3)}_{\mu\nu} = R^{(3)}_{\nu\mu}\,. \tag{3.27}$$

Thus, the tensors $R^{(1)}_{\mu\nu}$ and $R^{(2)}_{\mu\nu}$ are Hermitian, and consequently, the reality condition of the Lagrangian (3.17) is verified.

4. Conclusion

Basing on the above analysis and comments, one can conclude the following:

(i) It is not possible to have the hermiticity of $W^{\lambda}_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$ simultaneously, but rather one can apply the hermiticity condition to one of them (*e.g.* $\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$) together with another choice like $\tilde{W}_{\mu} = -W_{\mu}$.

(ii) Both Lagrangians (2.6) and (2.7) or (3.8) and (3.9) chosen by Moffat in papers [2, 4, 5, 7] are incompatible with Moffat's own constraint (2.10). Moreover, this constraint is inconsistent with both Moffat's equations (2.13) and (3.2): The correct compatibility condition is given by Eq. (3.3) which yields the correct metricity condition (3.6) with the Γ's rather than the W's.

(iii) The compatibility condition (2.10) proposed by Moffat does not allow one to write neither the Lagrangian (3.8) introduced by the author in his first papers (Refs [2] and [3]) nor the one used recently (Refs [4], [5] and [7]) as functions of the complex hyperbolic vierbein e^a_μ and spin connections $(\omega_\mu)_{ab}$. However, our corrected compatibility condition does.

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APPENDIX

Let us start with the definition (2.4). The latter implies that $\Gamma_\nu = 0$. In fact, assuming just the hermiticity condition of $\Gamma^\lambda_{\mu\nu}$, that is

$$\begin{aligned}\tilde{\Gamma}^\lambda_{\mu\nu} &= \mathrm{Re}(\Gamma^\lambda_{\mu\nu}) - \varepsilon\,\mathrm{Im}(\Gamma^\lambda_{\mu\nu}) \\ &= \Gamma^\lambda_{\nu\mu} = \mathrm{Re}(\Gamma^\lambda_{\nu\mu}) + \varepsilon\,\mathrm{Im}(\Gamma^\lambda_{\nu\mu}) \qquad (\varepsilon^2 = 1)\,, \end{aligned} \tag{A1}$$

one obtaines

$$\begin{aligned}\mathrm{Re}(\Gamma^\lambda_{\mu\nu}) &= \Gamma^\lambda_{(\mu\nu)}\,, \\ \mathrm{Im}(\Gamma^\lambda_{\mu\nu}) &= \varepsilon\Gamma^\lambda_{[\mu\nu]} \qquad (\varepsilon^2 = 1)\,,\end{aligned} \tag{A2}$$

where ( ) (respectively [ ]) means the symmetric (respectively antisymmetric) part. Now it is straightforward to show that $\Gamma_\nu = 0$ implies

$$\mathrm{Im}\,\Gamma^\alpha_{\nu\alpha} = \mathrm{Im}\,\Gamma^\alpha_{\alpha\nu} = 0\,. \tag{A3}$$

Thus

$$\tilde{\Gamma}^\alpha_{\alpha\nu} = \Gamma^\alpha_{\alpha\nu} = \Gamma^\alpha_{(\alpha\nu)} \in \mathbb{R}\,. \tag{A4}$$

($\mathbb{R}$ is the field of real numbers).

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