ECONOMY MASS FORMULA FOR LEPTONS AND QUARKS *

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On the base of the model of three families of algebraically composite leptons and quarks a semiempirical mass spectral formula is found for charged leptons e^-, μ^-, τ^-. With two phenomenological parameters the formula gives successfully $m_\tau = 1776.80$ MeV, where the experimental m_e and m_μ are used. In an economical way introducing only three extra parameters, the formula is extended phenomenologically to quarks u, c, t and d, s, b. Then, with the use of experimental m_e/m_μ and m_c, m_b it determines quark masses, as well as the Cabibbo–Kobayashi–Maskawa quark mixing matrix with the use of experimental m_e/m_μ and $|V_{us}|$. The agreement with existing data is promising.

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The concept of leptons and quarks composed of algebraic partons was recently introduced to propose an explanation for the puzzling phenomenon of three replicas of these fundamental fermions [1].

The concept is based on the remarkable sequence $N = 1, 2, 3, \ldots$ of composite representations

$$\Gamma^\mu = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_i^\mu \tag{1}$$

for the Dirac algebra

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}, \tag{2}$$

where the matrices γ_i^μ $(i = 1, 2, \ldots, N)$ are defined by the sequence $N = 1, 2, 3, \ldots$ of Clifford algebras

$$\{\gamma_i^\mu, \gamma_j^\nu\} = 2\delta_{ij} g^{\mu\nu} \qquad (i, j = 1, 2, \ldots, N). \tag{3}$$

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Then, in the framework of the standard-model coupling, the sequence $N = 1, 2, 3, \ldots$ of Dirac-type equations

$$[\Gamma \cdot (p - gA) - M]\,\psi = 0 \tag{4}$$

appears. Here, A_μ symbolize the standard-model gauge fields including the coupling matrices of the SU(3)⊗SU(2)⊗U(1) group *i.e.*, λ's, τ's, Y and $\Gamma^5 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3$.

Except for $N = 1$, the representations (1) are reducible. In fact, for any $N > 1$ one can construct, beside the combination $\Gamma_1^\mu \equiv \Gamma^\mu$ given in Eq. (1), other Jacobi-type independent combinations $\Gamma_2^\mu, \ldots, \Gamma_N^\mu$, *viz.*

$$\Gamma_2^\mu = \frac{1}{\sqrt{2}}\left(\gamma_1^\mu - \gamma_2^\mu\right)\,, \quad \Gamma_3^\mu = \frac{1}{\sqrt{6}}\left(\gamma_1^\mu + \gamma_2^\mu - 2\gamma_3^\mu\right)\,, \ldots\,, \tag{5}$$

such that

$$\left\{\Gamma_i^\mu, \Gamma_j^\nu\right\} = 2\delta_{ij} g^{\mu\nu} \qquad (i, j = 1, 2, \ldots, N)\,. \tag{6}$$

Thus, one may always represent Γ_1^μ in the form

$$\Gamma_1^\mu = \gamma^\mu \otimes \underbrace{\mathbf{1} \otimes \cdots \otimes \mathbf{1}}_{(N-1)\text{times}}\,. \tag{7}$$

For instance, when $N = 3$, one may write

$$\Gamma_1^\mu = \gamma^\mu \otimes \mathbf{1} \otimes \mathbf{1}\,, \quad \Gamma_2^\mu = \gamma^5 \otimes i\gamma^5\gamma^\mu \otimes \mathbf{1}\,, \quad \Gamma_3^\mu = \gamma^5 \otimes \gamma^5 \otimes \gamma^\mu\,. \tag{8}$$

Here, $\gamma^\mu, \mathbf{1}$ and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ are the usual 4×4 Dirac matrices.

In the representation (7), the Dirac type equations (4) take the forms

$$[\gamma \cdot (p - g\,A) - M]_{\alpha_1\beta_1}\; \psi_{\beta_1\alpha_2\ldots\alpha_N} = 0\,, \tag{9}$$

where $M_{\alpha_1\beta_1} = \delta_{\alpha_1\beta_1} M$. For $N = 1$ and $N = 2$ they are the usual Dirac equation and the Dirac form [2] of the Kähler equation [3], respectively. For $N = 3, 4, 5, \ldots$ one gets new equations. In general, the wave function (or dynamical field) $\psi = (\psi_{\alpha_1\alpha_2\ldots\alpha_N})$ carries N Dirac bispinor indices $\alpha_i = 1, 2, 3, 4$ $(i = 1, 2, \ldots, N)$ of which only the first, α_1, is affected by the Dirac matrices $\gamma^\mu = (\gamma^\mu_{\alpha_1\beta_1})$ and so coupled to the standard-model fields. The rest of them, $\alpha_2, \ldots, \alpha_N$, are free. Thus, only α_1 may be "visible", say, in the magnetic field, while $\alpha_2, \ldots, \alpha_N$ are "hidden". Hence, an algebraically composite particle defined by Eq. (4) or (9) may display only a "visible" spin 1/2, although it possesses also $N - 1$ "hidden" spins 1/2 .

One may say that the particle's degrees of freedom corresponding to the matrices γ_i^μ $(i = 1, 2, \ldots, N)$ define its "algebraic partons". Then, the visible and hidden indices describe their "centre-of-mass" and "relative" degrees of freedom, respectively (*cf.* the form of Γ_1^μ and $\Gamma_2^\mu, \ldots, \Gamma_N^\mu$). So, one may speak as well of the "visible" and "hidden" algebraic partons, whose degrees of freedom correspond to the matrices Γ_i^μ $(i = 1, 2, \ldots, N)$ (or, more precisely, to the pairs of matrices $\Sigma_j^{\mu\nu} = \frac{i}{2}\left[\Gamma_j^\mu, \Gamma_j^\nu\right]$ and $\Gamma_j^5 = i\Gamma_j^0\Gamma_j^1\Gamma_j^2\Gamma_j^3$ $(j = 1, 2, \ldots, N)$ which commute for different j's and determine the bispinor indices α_j $(j = 1, 2, \ldots, N)$ through the pairs of eigenvalues of Σ_j^{12} and Γ_j^5).

In Ref. [1], we have assumed that: *(i)* the theory of relativity applies both to the visible index α_1 and to the hidden indices $\alpha_2, \ldots, \alpha_N$, and *(ii)* the particle's degrees of freedom described by the hidden indices are physically undistinguishable and so obey the Fermi statistics along with the Pauli exclusion principle (*i.e.*, $\psi_{\alpha_1\alpha_2\ldots\alpha_N}$ is fully antisymmetric with respect to $\alpha_2, \ldots, \alpha_N$). These assumptions, jointly with the probability interpretation of the wave function $\psi = (\psi_{\alpha_1\alpha_2\ldots\alpha_N})$, have led us in Ref. [1] to the conclusion that all solutions for ψ appearing in the sequence $N = 1, 2, 3, \ldots$ of Dirac-type equations (4) or (9) reduce solely to three spin-1/2 states with $N = 1, 3, 5$, giving three and only three replicas of the Dirac particle of any familiar standard-model signature. In Ref. [1], these three replicas have been interpreted as responsible for the phenomenon of three families of leptons and quarks.

The explicit form of the three spin-1/2 states with $N = 1, 3, 5$ is

$$\begin{aligned}
\psi_{\alpha_1}^{(1)} &\equiv \psi_{\alpha_1}, \\
\psi_{\alpha_1}^{(3)} &\equiv \frac{1}{4}\left(C^{-1}\gamma^5\right)_{\alpha_2\alpha_3}\psi_{\alpha_1\alpha_2\alpha_3} = \psi_{\alpha_1\,12} = \psi_{\alpha_1\,34}, \\
\psi_{\alpha_1}^{(5)} &\equiv \frac{1}{24}\varepsilon_{\alpha_2\alpha_3\alpha_4\alpha_5}\psi_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5} = \psi_{\alpha_1\,1234},
\end{aligned} \tag{10}$$

where the chiral representation $\gamma^5 = \mathrm{diag}(1, 1, -1, -1)$ is used. It can be seen that in the sector wave functions $\psi_{\alpha_1}, \psi_{\alpha_1\alpha_2\alpha_3}, \psi_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5}$ the bispinors (10) are repeated (up to the sign) 1, 4, 24 times, respectively. Thus, the following overall wave function comprises three sectors $N = 1, 3, 5$ (or three fundamental-fermion families):

$$\Psi = \hat{\rho}\begin{pmatrix} \psi_{\alpha_1}^{(1)} \\ \psi_{\alpha_1}^{(3)} \\ \psi_{\alpha_1}^{(5)} \end{pmatrix}, \tag{11}$$

where the sector-weighting (or family-weighting) matrix

$$\hat{\rho} = \frac{1}{\sqrt{29}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{24} \end{pmatrix} \tag{12}$$

is introduced.

The formula (11) implies that the mass matrix for any triplet of fundamental fermions of a given standard-model signature has the form

$$\hat{M} = \hat{\rho}\,\hat{h}\,\hat{\rho} \tag{13}$$

with $\hat{h}$ being a Higgs coupling strength matrix. While it is an open question how the matrix $\hat{h}$ should look like, it seems certain that within the unknown form of $\hat{h}$ a lot of new physics should be coded.

In the present note, first, we try to guess a phenomenological form of $\hat{h}$ in the case of the triplet of charged leptons e^-, μ^-, τ^-. In this case, $\hat{h}$ is a diagonal matrix

$$\hat{h} = \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_3 & 0 \\ 0 & 0 & h_5 \end{pmatrix}, \tag{14}$$

since e^-, μ^-, τ^- do not mix, and the respective masses m_e, m_μ, m_τ are well defined and precisely known.

On the base of a numerical experience, we can propose the following ansatz for the matrix (14):

$$h_N = \mu\left(N^2 - \frac{1-\varepsilon^2}{N^2}\right) \qquad (N = 1,3,5), \tag{15}$$

where $\mu > 0$ and $1 - \varepsilon^2$ are two real parameters (*cf.* Ref. [4]). Then, the eigenvalues $M_N = \rho_N{}^2 h_N$ of the mass matrix (13) take the forms

$$\begin{aligned} m_e \equiv M_1 &= \frac{\mu}{29}\varepsilon^2, \\ m_\mu \equiv M_3 &= \frac{4}{9}\frac{\mu}{29}(80+\varepsilon^2), \\ m_\tau \equiv M_5 &= \frac{24}{25}\frac{\mu}{29}(624+\varepsilon^2). \end{aligned} \tag{16}$$

Hence, using the experimental values for m_e and m_μ, we predict :

$$m_\tau = \frac{6}{125}(351 m_\mu - 136 m_e) = 1776.80\,\text{MeV}, \tag{17}$$

as well as

$$\mu = \frac{29}{320}(9m_\mu - 4m_e) = 85.9924\,\text{MeV} \tag{18}$$

and

$$\varepsilon^2 = \frac{320 m_e}{9m_\mu - 4m_e} = 0.172329. \tag{19}$$

The agreement with the new experimental values for m_τ ,

$$m_\tau = (1776.9^{+0.4}_{-0.5} \pm 0.2)\text{MeV}\,, \quad m_\tau = (1776.3 \pm 2.4 \pm 1.4)\,\text{MeV}\,, \tag{20}$$

reported recently by the Beijing Electron-Positron Collider Group [5] and the ARGUS Collaboration [6], respectively, is excellent.

Concluding our argument for charged leptons, some comments are due. From the methodological point of view the successful simple form (15) of h_N is a purely phenomenological ansatz (note that a significant theoretical contribution to the formula $M_N = \rho_N{}^2 h_N$ is the weighting factor $\rho_N{}^2$ determined by the matrix (12) squared). Nevertheless, the physical interpretation of the form (15) would be important, especially, if it introduced new physical contents.

As given in Eq. (15), the Higgs coupling strength h_N is the sum of two terms proportional, respectively, to N^2 and $1/N^2$. The form of the first term, μN^2, has its natural counterpart in the conventional, spatial N-body problem, corresponding to some equal-strength two-body interactions (and selfinteractions) of N particles. In contrast, there seems to be no such counterpart for the form of the second term, $-\mu(1-\varepsilon^2)/N^2$, which grows nonconventionally to zero with $N \to \infty$. Perhaps, this term, playing the role of a nonconventional "parton correlation term", reflects phenomenologically a surprizing piece of new physics.

The influence of the moderately small parameter ε^2 on the mass m_τ is rather small (though measurable). In fact, if in the second term it is neglected in comparison with 1, one obtains $m_\tau = 1780.13$ MeV where the experimental m_μ is used. But then $m_e = 0$, what is a significant change for the mass m_e. (However, if the whole second term is put equal to zero, one gets a larger change, $m_\tau = 1760.97$ MeV and $m_e = 2.93496$ MeV with the use of experimental m_μ.) So, this remark expresses an essential correctness of the ansatz of the type $h_N = \mu(N^2 - 1/N^2)$, at least, for heavy leptons μ^- and τ^-, though, still, the leading effect for heavy leptons is due to the first term (whose form happens to have a conventional counterpart in the spatial N-body problem).

Since finding the successful ansatz (15) for charged leptons we made a lot of numerical studies in order to extend it to up and down quarks (and, possibly, also to neutrinos). In the present paper we report on a form of the extended ansatz which seems to be most adequate of all forms considered by us hitherto.

We write the mass matrices for four triplets $f = \nu,\, e,\, u,\, d$ of neutrinos $(\nu_e,\, \nu_\mu,\, \nu_\tau)$, charged leptons $(e^-,\, \mu^-,\, \tau^-)$, up quarks $(u,\, c,\, t)$ and down quarks $(d,\, s,\, b)$, respectively, as in Eq. (13):

$$\hat{M}^{(f)} = \hat{\rho}\,\hat{h}^{(f)}\,\hat{\rho}\,, \tag{21}$$

where the family-weighting matrix $\hat{\rho}$ is given in Eq. (12). Then, we propose the following ansatz for the Higgs strength matrices:

$$\hat{h}^{(f)} = \mu^{(f)} \left[A^{(f)2} \hat{N}^{(f)2} - \frac{A^{(f)2} + \varepsilon^2 (B^{(f)2} - L^{(f)2})}{\hat{N}^{(f)2}} + \varepsilon C^{(f)} \left(\hat{a}\, e^{i\varphi^{(f)}} + \hat{a}^{+} e^{-i\varphi^{(f)}} \right) \right] , \tag{22}$$

where

$$\hat{N}^{(f)} = \hat{N} + C^{(f)} \hat{n}(\hat{n}-1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5+2C^{(f)} \end{pmatrix} \tag{23}$$

with

$$\hat{N} = \hat{1} + 2\hat{n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} , \hat{n} = \hat{a}^{+}\hat{a} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} , \tag{24}$$

while

$$\hat{a} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} , \hat{a}^{+} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{25}$$

are (restricted) annihilation and creation operators of pairs of hidden partons within leptons and quarks (note that $\hat{a}^3 = 0$ and $\hat{a}^{+3} = 0$). Here, we put (without any *a priori* justification):

$$A^{(f)} = N_C^{(f)} Q^{(f)2} = \begin{cases} 0 \\ 1 \\ 4/3 \\ 1/3 \end{cases} , \quad B^{(f)} = \begin{cases} 0 \\ 0 \\ 1/3 \\ 1/3 \end{cases} ,$$

$$L^{(f)} = \begin{cases} 1 \\ 1 \\ 0 \\ 0 \end{cases} , C^{(f)} = A^{(f)} \frac{1}{2} N_C^{(f)} (N_C^{(f)} - 1) = \begin{cases} 0 \\ 0 \\ 4 \\ 1 \end{cases} \tag{26}$$

for $f = \nu,\, e,\, u,\, d$, respectively, with

$$Q^{(f)} = \begin{cases} 0 \\ -1 \\ 2/3 \\ -1/3 \end{cases} , \quad N_C^{(f)} = \begin{cases} 1 \\ 1 \\ 3 \\ 3 \end{cases} , \tag{27}$$

while the mass scales $\mu^{(\nu)}, \mu^{(e)} (\equiv \mu), \mu^{(u)}, \mu^{(d)}$, the phase difference $\varphi^{(u)} - \varphi^{(d)}$ and the coupling coefficient ε are free parameters to be fitted to some of the masses and mixing angles (μ and ε^2 are already determined by experimental m_e and m_μ and so are given as in Eqs. (18) and (19), since $\hat{h}^{(e)}(\equiv \hat{h}$) is just the diagonal matrix (14) with its elements defined by ansatz (15)]. The last term in Eq. (22) appears only for quarks and causes their mixing. It gives also dominant contributions to m_u and m_d .

For neutrinos, Eqs (21) and (22) with (23) and (26) yield

$$m_{\nu_e} = \frac{9}{4} m_{\nu_\mu} = \frac{25}{24} m_{\nu_\tau} = \frac{\mu^{(\nu)}}{29}, \tag{28}$$

thus $m_{\nu_\mu} \leq m_{\nu_\tau} \leq m_{\nu_e}$. In particular,

$$m_{\nu_e} = m_{\nu_\mu} = m_{\nu_\tau} = 0 \tag{29}$$

if $\mu^{(\nu)} = 0$.

In contrast to neutrinos and charged leptons, for up quarks and down quarks the mass matrix (21) with the ansatz (22) requires diagonalization. The results of our numerical calculation are as follows.

Taking as the input

$$m_c = (1.3 \text{ to } 1.5)\,\text{GeV}, \quad m_b = (4.5 \text{ to } 5.0)\,\text{GeV} \tag{30}$$

(and using the lepton value (19) of ε^2), one obtains the predictions:

$$\begin{aligned} m_u &= (4.0 \text{ to } 4.6)\,\text{MeV}, \quad m_d = (6.8 \text{ to } 7.5)\,\text{MeV}, \\ m_t &= (150 \text{ to } 170)\,\text{GeV}, \quad m_s = (130 \text{ to } 150)\,\text{MeV} \end{aligned} \tag{31}$$

(here, the lowest masses m_u and m_d are taken to be nonnegative, as always for Dirac masses). Hence, $m_u/m_d = 0.59$ to 0.61 , while experimental estimates for this ratio are 0.3 to 0.7. The mass scales in Eqs. (22) for $f = u, d$ turn out to be

$$\mu^{(u)} = (0.60 \text{ to } 0.69)\,\text{GeV}\ , \ \mu^{(d)} = (1.0 \text{ to } 1.1)\,\text{GeV}, \tag{32}$$

when fitted to the input (30). The predicted mass ratios within triplets u, c, t and d, s, b are independent of the input (30), being determined by ε^2 only (*i.e.*, by the experimental ratio m_e/m_μ):

$$\begin{aligned} m_u/m_c &= 0.00305405, \quad m_d/m_s = 0.0509344, \\ m_c/m_t &= 0.00877909, \quad m_s/m_b = 0.0294873, \end{aligned} \tag{33}$$

whereas

$$m_c/\mu^{(u)} = 2.18281, \quad m_b/\mu^{(d)} = 4.51254. \tag{34}$$

Some popular experimental estimates [7] for the first, second and fourth of the ratios (33) are

$$0.0038 \pm 0.0012\,, \quad 0.051 \pm 0.004\,, \quad 0.033 \pm 0.011\,, \tag{35}$$

respectively.

The elements of the Cabibbo–Kobayashi–Maskawa quark mixing matrix $\hat{V} = \hat{U}^{(u)+}\hat{U}^{(d)}$ with $\hat{U}^{(u)}$ and $\hat{U}^{(d)}$ diagonalizing $\hat{M}^{(u)}$ and $\hat{M}^{(d)}$, respectively, are also independent of the input (30) fixing the mass scales $\mu^{(u)}$ and $\mu^{(d)}$. In fact, taking in this case as the input [8]

$$|\, V_{us}\, | = 0.218 \text{ to } 0.224 \tag{36}$$

(and making use of the lepton-value (19) of ε^2) one gets the following predictions (after a conventional rephasing of up and down quark states is made):

$$\hat{V} = \begin{pmatrix} 0.976 & 0.218 & 0.00240e^{-i\,66.7^\circ} \\ -0.218 & 0.975 & 0.0455 \\ 0.00925e^{-i13.5^\circ} & -0.0446e^{i0.7^\circ} & 0.999 \end{pmatrix}$$
$$\text{to} \begin{pmatrix} 0.975 & 0.224 & 0.00246e^{-i\,60.3^\circ} \\ -0.224 & 0.974 & 0.0459 \\ 0.00933e^{-i12.9^\circ} & -0.0450e^{i0.7^\circ} & 0.999 \end{pmatrix}. \tag{37}$$

These are consistent with the actual experimental limits [8]:

$$|\, \hat{V}_{\rm exp}\, | = \begin{pmatrix} 0.9747 \text{ to } 0.9759 & 0.218 \text{ to } 0.224 & 0.002 \text{ to } 0.007 \\ 0.218 \text{ to } 0.224 & 0.9735 \text{ to } 0.9751 & 0.032 \text{ to } 0.054 \\ 0.003 \text{ to } 0.018 & 0.030 \text{ to } 0.054 & 0.9985 \text{ to } 0.9995 \end{pmatrix}. \tag{38}$$

The predicted CP–violating phase is here

$$\delta = 66.7^\circ \text{ to } 60.3^\circ\,. \tag{39}$$

This is a rephasing–invariant phase equal to the combination $-\alpha_{ub} - \alpha_{cs} + \alpha_{us} + \alpha_{cb}$ of the phases $\alpha_{kl} = \arg V_{kl}$ $(k = u, c, t\,,\ l = d, s, b)$ calculated originally (before the conventional rephasing $\alpha_{kk} \to 0$ and $\alpha_{us} \to 0\,, \alpha_{cb} \to 0\,, \alpha_{ub} \to -\delta$ is made). The phase difference in Eqs (22) for $f = u$, d turns out to be

$$\varphi^{(u)} - \varphi^{(d)} = 93.4^\circ \text{ to } 101^\circ\,, \tag{40}$$

when fitted to the input (36).

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