THE KEPLER PROBLEM IN THE LOBACHEVSKY SPACE AND ITS SOLUTION

N.A. Chernikov

Joint Institute for Nuclear Research
141980, Dubna, Russia

(Received January 3, 1992)

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PACS numbers: 04.90.+e

The first of December 1992 is the 200-th anniversary of the birthday of Nikolai Ivanovich Lobachevsky. We present this paper to celebrate this anniversary.

Upon solving the longstanding problem Lobachevsky discovered an extraordinary geometry. He delivered a report on it in 1826 at the Kazan University and wrote a paper [1] published in "The Kazan Bulletin edited by the Emperor's University of Kazan" (1829–30). In this paper, for the first time in the mathematical literature, a geometric theory was presented based on all Euclidean postulates except for the fifth one which is referred to as the Euclidean postulate of parallels.

It is useful to illustrate the difference between the two geometric theories, the Lobachevskyan and Euclidean ones, by using a sphere with radius ρ as an example.

According to the third Euclidean postulate "from any center it is possible to draw a circle of arbitrary radius" [2]. This means that both in Euclidean and Lobachevskyan geometries the value of ρ may be arbitrary. Lobachevsky has shown that the internal geometry of a sphere is independent of the fifth Euclidean postulate. Therefore, both in the Euclidean and Lobachevskyan geometries we can draw on the sphere "parallels and meridians" and we may introduce polar coordinates θ and φ. We denote the length of the "equator" by $2\pi r$. In the usual way we get the metric form $r^2(d\theta^2 + \sin^2\theta d\varphi^2)$ of the sphere and its surface element $r^2 \sin\theta d\theta d\varphi$. The surface of the whole sphere is equal to $4\pi r^2$. The difference between the two geometries consists in the dependence of r on ρ: in the Euclidean geometry $r = \rho$ while in the Lobachevskyan one

$$r = k \sinh \frac{\rho}{k} . \tag{1}$$

Here k is some characteristic constant called the Lobachevsky constant. For $k \to \infty$ the Lobachevskyan geometry transforms into the Euclidean geometry. If $k < \infty$, then the sphere of radius ρ for small values of ρ/k has approximately the Euclidean geometry.

In both geometrical theories the radius is perpendicular to the sphere. Therefore, the metric form of the space is equal to

$$ds^2 = d\rho^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) . \tag{2}$$

In the equatorial plane $\theta = \pi/2$ and the metric form equals

$$ds^2 = d\rho^2 + r^2 d\varphi^2 . \tag{3}$$

The constant k enters into all the formulae of the Lobachevskyan geometry in that part where it differs from the Euclidean geometry. For example, the sum of angles A, B, C of an arbitrary triangle in the Euclidean planimetry is equal to π, while in the Lobachevsky planimetry it is smaller than π and the area of the triangle is equal to

$$F = k^2(\pi - A - B - C) . \tag{4}$$

The volume of publication restricts our review, but we have to mention the surfaces orthogonal to a bundle of parallel lines. Lobachevsky called them limiting spheres (orispheres) and showed that the internal geometry of an orisphere coincides with the Euclidean planimetry. Let us emphasize: rejecting the fifth Euclidean postulate for the plane, Lobachevsky has proved it for the orisphere!

We also mention that Lobachevsky formulated two completely new problems: on the astronomical verification of the geometry of our visible world and on the "kinds of changes which will occur in Mechanics after introducing in it the imaginary Geometry" [1], p. 261.

Both these problems are actual now. In paper [3] published in the "Scientific Notes of the Emperor's University of Kazan" Lobachevsky wrote in 1835:

" ... there should be no contradiction in our brain when we admit that some forces in Nature follow a certain Geometry while other forces follow their particular Geometry. To clarify this idea, we assume, many others believe in this also, that the attractive forces weaken due to the propagation of their action through the sphere. In the usual Geometry the surface of the sphere is $4\pi r^2$ if its radius is r and, therefore, the decrease of the force is quadratic in distance. In the imaginary Geometry I found the area of the sphere

$$\pi(e^r - e^{-r})^2$$

and perhaps the molecular forces follow this Geometry and their manifold should then depend on the number e always being very large. By the way let this be a pure hypothesis which should be proved by more convenient arguments; but we should not doubt that the forces produce all by themselves: the motion, velocity, time, mass and even distances and angles" [3], p. 159.

It is possible to get agreement of the formula presented in this part of the text with the previous one describing the area of the sphere of radius ρ, if we abandon notation (1) and put

$$k = 1, \quad r = \rho, \quad e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty}\frac{1}{k!}\,.$$

However, in the subsequent considerations it is more convenient to keep notation (1).

We shall obtain the Newtonian theory of gravitation in the Lobachevsky space provided we can find the fundamental solution of the Poisson equation

$$\Delta\Phi = 4\pi\alpha\delta(x)\delta(y)\delta(z) \tag{5}$$

in the space with metric (2). This equation contains the following ingredients:

the gravitation potential Φ acting on a test body and produced by the point mass m;
the constant $\alpha = \gamma m$, where γ is the Newton constant;
the Laplace operator in the Lobachevsky space

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial\rho}r^2\frac{\partial}{\partial\rho} + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right);$$

the Dirac δ-function with arguments

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta\,. \tag{6}$$

According to the remark of Lobachevsky we assume that the "attractive force" (a covector of the force) has the following components:

$$F_1 = -\frac{\partial\Phi}{\partial\rho} = -\frac{\alpha}{r^2}, \quad F_2 = -\frac{\partial\Phi}{\partial\theta} = 0, \quad F_3 = -\frac{\partial\Phi}{\partial\varphi} = 0\,.$$

Consequently

$$\Phi = -\frac{\alpha}{k \tanh \frac{\rho}{k}} . \tag{7}$$

Let us consider the motion of a test body. Denoting the time by t, we obtain from (2) and (7) the Lagrangian of a test body

$$L = \tfrac{1}{2}(\dot{\rho}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2) - \Phi \tag{8}$$

as well as the equations of motion

$$\begin{aligned}
&\frac{d}{dt}\dot{\rho} - rr'(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) + \frac{\alpha}{r^2} = 0 ,\\
&\frac{d}{dt}(r^2\dot{\theta}) - r^2 \sin\theta\cos\theta\dot{\varphi}^2 = 0 ,\\
&\frac{d}{dt}(r^2 \sin^2\theta\dot{\varphi}) = 0 ,
\end{aligned} \tag{9}$$

where

$$r' = \cosh\frac{\rho}{k}, \qquad \frac{\alpha}{r^2} = \Phi' ; \tag{10}$$

the dot denotes the time derivative.

Since the Lagrangian (8) does not depend on time, the energy

$$E = \tfrac{1}{2}(\dot{\rho}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\dot{\varphi}^2) + \Phi \tag{11}$$

is conserved. Due to the spherical symmetry of the Lagrangian, all the components of the angular momentum

$$\begin{aligned}
M_1 &= y\dot{z} - z\dot{y} = -r^2(\sin\theta\cos\theta\cos\varphi\dot{\varphi} + \sin\varphi\dot{\theta}) ,\\
M_2 &= z\dot{x} - x\dot{z} = -r^2(\sin\theta\cos\theta\sin\varphi\dot{\varphi} - \cos\varphi\dot{\theta}) ,\\
M_3 &= x\dot{y} - y\dot{x} = r^2 \sin^2\theta\dot{\varphi} ,
\end{aligned} \tag{12}$$

are also conserved; it is easy to check this directly by differentiating Eqs (11) and (12).

The conservation of the angular momentum means in this case that the test body is moving on the Lobachevskyan plane that passes through the origin of attraction. Without loss of generality we can take this plane as the equatorial plane with $\theta = \pi/2$. In this case $M_1 = 0$, $M_2 = 0$, $M_3 = M$, where

$$M = r^2\dot{\varphi} , \tag{13}$$

and the energy integral takes the form

$$E = \tfrac{1}{2}(\dot{\rho}^2 + r^2\dot{\varphi}^2) + \Phi . \tag{14}$$

Substituting (13) into (14) we get the differential equation for the trajectory $\rho = \rho(\varphi)$ in the form

$$E = \frac{M^2}{2r^4}\left[\left(\frac{d\rho}{d\varphi}\right)^2 + r^2\right] + \Phi . \tag{15}$$

Now, taking into account the form of Φ given by (7) we introduce the notation

$$k \tanh\frac{\rho}{k} = \frac{1}{u} . \tag{16}$$

Since

$$du = \frac{1}{r^2}d\rho, \qquad u^2 = \frac{1}{r^2} + \frac{1}{k^2} \tag{17}$$

and equation (15) takes the form

$$E = \frac{M^2}{2}\left[\left(\frac{du}{d\varphi}\right)^2 + u^2 - \frac{1}{k^2}\right] - \alpha u . \tag{18}$$

The solution of this equation is

$$u = \frac{\alpha}{M^2} + \sqrt{\frac{\alpha^2}{M^4} + \frac{2E}{M^2} + \frac{1}{k^2}}\cos\varphi . \tag{19}$$

The integration constant is chosen here so that the maximal value of u corresponds to $\varphi = 0$. If we denote

$$p = \frac{M^2}{\alpha}, \qquad \epsilon = \sqrt{1 + \frac{2EM^2}{\alpha^2} + \frac{M^4}{\alpha^2 k^2}} , \tag{20}$$

the equation for the trajectory of the test body may be written in the form

$$k \tanh\frac{\rho}{k} = \frac{p}{1 + \epsilon\cos\varphi} . \tag{21}$$

In accordance with our choice of the integration constant for equation (18) the point of the orbit (21) nearest to the attraction center corresponds to the angle $\varphi = 0$. The distance from the attraction center to the nearest point of the orbit equals ρ_1 given by

$$k\frac{\rho_1}{k} = \frac{p}{1 + \epsilon} . \tag{22}$$

The most remote point of the orbit exists only for a finite motion. In this case the following condition

$$k > \frac{p}{1 - \epsilon}, \qquad i.e. \quad \epsilon < 1 - \frac{p}{k} , \tag{23}$$

should be satisfied; this is equivalent to the condition

$$E < -\frac{\alpha}{k}. \tag{24}$$

We shall restrict ourselves to this case. We shall then get the solution of the Kepler problem of motion of a planet around the Sun in the Lobachevsky space.

In the case of a finite motion, the most remote point of the orbit corresponds to the angle $\varphi = \pi$ while the maximal distance ρ_2 is given by

$$k \tanh \frac{\rho_2}{k} = \frac{p}{1-\epsilon}. \tag{25}$$

Let us denote the size of the orbit by $2a$. Clearly it is equal to $\rho_1 + \rho_2$. From (22) and (25) we find

$$k \tanh \frac{2a}{k} = \frac{2p}{1-\epsilon^2+p^2/k^2} = -\frac{\alpha}{E}. \tag{26}$$

Now we find the period T of the motion of the planet. According to (13) we have

$$MT = \int_0^{2\pi} r^2(\varphi)d\varphi. \tag{27}$$

The integrand may be found from (1) and (21)

$$\begin{aligned} r^2(\varphi) &= \frac{p^2}{(1+\epsilon\cos\varphi)^2 - p^2/k^2} \\ &= \frac{pk}{2}\left(\frac{1}{1+\epsilon\cos\varphi - p/k} - \frac{1}{1+\epsilon\cos\varphi + p/k}\right). \end{aligned} \tag{28}$$

The integral is evaluated by the substitution $\xi = \tan\frac{\varphi}{2}$. Then we get

$$\frac{d\varphi}{m+n\cos\varphi} = \frac{2}{\sqrt{m^2-n^2}} d\arctan\left(\sqrt{\frac{m-n}{m+n}}\tan\frac{\varphi}{2}\right),$$

$$\int_0^{2\pi} \frac{d\varphi}{m+n\cos\varphi} = \frac{2\pi}{\sqrt{m^2-n^2}} \qquad \text{for } m > |n|.$$

Therefore,

$$MT = \pi pk\left(\frac{1}{\sqrt{(1-p/k)^2-\epsilon^2}} - \frac{1}{\sqrt{(1+p/k)^2-\epsilon^2}}\right). \tag{29}$$

Substituting (20) into (29) we get

$$T = \frac{\pi k}{\sqrt{2}}\left(\frac{1}{\sqrt{-E-\alpha/k}} - \frac{1}{\sqrt{-E+\alpha/k}}\right). \tag{30}$$

The period T depends on the energy E, but it does not depend on the angular momentum M.

The substitution of (26) into (30) gives the following expression for the square of the period:

$$T^2 = \frac{4\pi^2}{\alpha}\left(k\sinh\frac{a}{k}\right)^3 \cosh\frac{a}{k}. \tag{31}$$

It is interesting that the planet's orbit on the Lobachevskyan plane may be defined in the same way as on the Euclidean one: the orbit is a set of points having a given sum of distances from two given points. This set we shall call the ellipse. The given points are called foci and denoted by F and $\tilde{F}$. The given sum of distances is denoted by $2a$ and called the length of the major axis. We denote the distances from F and $\tilde{F}$ to a point M laying on the ellipse by ρ and $\tilde{\rho}$. Thus $\rho + \tilde{\rho} = 2a$. Let the middle point of the segment $F\tilde{F}$ be O, while its length is equal $2c$. We shall call the point O the centre of the ellipse. The major axis passes through the foci while the minor one passes through the centre orthogonally to the major axis. On the minor axis $\rho = \tilde{\rho} = a$. The length of the minor axis is denoted by $2b$.

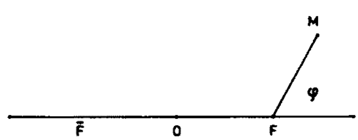

Fig. 1. In this figure the major axis, both foci and the centre lying on the major axis, are shown. The point M lies on the ellipse.

From the triangle $M\tilde{F}F$ it follows that

$$\cosh\frac{2a-\rho}{k} = \cosh\frac{2c}{k}\cosh\frac{\rho}{k} - \sinh\frac{2c}{k}\sinh\frac{\rho}{k}\cos(\pi-\varphi). \tag{32}$$

After a number of transformations we arrive at Eq. (21), where

$$p\sinh\frac{2a}{k} = k(\cosh\frac{2a}{k} - \cosh\frac{2c}{k}), \quad \epsilon\sinh\frac{2a}{k} = \sinh\frac{2c}{k}. \tag{33}$$

When the point M lies on the minor axis, this triangle becomes isosceles, while the triangle MOF becomes rectangular. From this we find

$$\cosh\frac{a}{k} = \cosh\frac{b}{k}\cosh\frac{c}{k}. \tag{34}$$

Substituting this into the first of Eqs (33) we get

$$p \tanh \frac{a}{k} = k \tanh^2 \frac{b}{k} . \tag{35}$$

It is interesting also that it is possible to obtain the same problem in the nonrelativistic limit ($c \to \infty$) in the gravity theory with two connections [4], choosing the second one in the form of the Christoffel symbols for the metric

$$d\check{s}^2 = c^2 dt^2 - d\rho^2 - (k \sinh \frac{\rho}{k})^2 (d\theta^2 + \sin^2 \theta d\varphi^2) .$$

It seems that this problem is important from the point of view of the relativistic theory of gravitation [5].

The velocity of light c plays the role of the Lobachevsky constant in the velocity space [6]. In the invisible world of velocities the role of the distance ρ plays the rapidity s, and the role of the quantity represented by the left-hand side of (25) is taken by the velocity v so that

$$\frac{v}{c} = \tanh \frac{s}{c} .$$

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