LEADING LOGARITHMIC CALCULATIONS OF QED CORRECTIONS AT LEP* †

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(Received February 12, 1992)

We present the leading logarithmic third order, $\mathcal{O}(\beta^3)$, corrections to the electron structure functions in QED. For the non singlet component we present exact Monte Carlo algorithm and compare several approximate (exponentiated) solutions. Using structure functions we discuss the QED *initial state* photonic (*i.e.* without additional fermion pairs) corrections to *total cross-section* at LEP. We find the size of $\mathcal{O}(\beta^3)$ corrections, in the range $|\sqrt{s}-M_Z| \leq 2.5$GeV, to be $\Delta\sigma^{(3)}/\sigma \leq 0.12\%$ for Kuraev–Fadin and $\leq 0.01\%$ for Jadach–Ward *ad hoc* exponentiated formulas, respectively. We show that in the case of the exponentiated formulas the stronger cuts more efficiently eliminate higher order corrections. Finally, we propose a new, compact, "pragmatic", *i.e.* without numerically unimportant subleading terms, $\mathcal{O}(\alpha^3)$ formula for the initial state QED photonic part of the total cross-section at LEP, accurate to $\delta\sigma/\sigma \leq 0.015\%$ for $|\sqrt{s}-M_Z| \leq 2.5$GeV and 0.035% for $|\sqrt{s}-M_Z| \leq 7.5$GeV.

PACS numbers: 12.20.Ds

1. Introduction

In 1983 the W and Z bosons were discovered in the proton-antiproton collider at CERN. It was a great success of the unified theory of electromagnetic and weak interactions. The theory, called the Electroweak Standard

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* Thesis submitted to the Jagellonian University in partial fulfillment of the requirements for the Ph. D. degree, performed under the supervision of dr. S. Jadach.

† Work partly supported by the MEN grant DNS-P/04/215/90-2 and KBN grant 20-38-09101.

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Model (EWSM), is based on the local gauge symmetry group $SU_2 \times U_1$. The EWSM predicted intermediate heavy bosons, charged W^+, W^- and neutral Z in addition to the massless photon of the conventional QED. The experimental discovery of all these massive resonances strongly confirmed the EWSM. The masses of the bosons were measured at the time of discovery within a few per cent accuracy and well agreed with the predictions of EWSM.

Later, more accurate measurements have been possible in the LEP collider at CERN. This 27 km long circular collider of the e^+ and e^- beams, of the energy up to 55 GeV per beam, was completed in 1989. Its aim is to produce a high statistics $\sim 10^7$ of the Z events. It is able to perform new tests of EWSM and precisely determine its basic parameters. There are two main groups of observables measured at LEP. First, connected with Z line shape are: peak total cross-section, peak position, width and branching ratios. The second includes forward-backward and spin asymmetries. Within the first group the typical LEP experimental accuracy is below one per cent whereas in the second of order of a few per cent. One of the spectacular results, coming from the total width measurements (first group of observables) is determination of the number of fermion families to be three (2.90 ± 0.10, see Ref. [1]).

On the other hand, any deviation from the Standard Model would be a signal of a "new" physics. Until now, however, no unexpected signals have been found. Present LEP results are used to set better mass limits for new particles (top, Higgs, supersymmetric particles etc.), see *e.g.* Ref. [1].

To analyse the data one has to have theoretical formulas including all necessary corrections up to a given accuracy level. Let us concentrate on one of the observables, the peak total cross-section. The corrections come from three sectors of the theory: QED, weak and QCD. The bulk of corrections comes from QED (in total cross-section up to 50% and more) covering all the subtle details of EWSM. In this respect the detailed control of the QED sector is crucial for the exact measurements.

In the paper we discuss only the QED part. Such a selective treatment is possible because the electromagnetic corrections are separated (factorized in the scattering matrix element) in a natural way from the rest. The dominant QED soft radiation is a long-range phenomenon, taking place long time before and after the hard scattering process, and it is therefore well separated in time and space from hard process, see *e.g.* Refs [2] or [3].

The leading $\mathcal{O}(\alpha^2)$ initial state QED contribution to the total cross-section was first computed with the help of leading logarithmic structure functions technique in Refs [2,4]. Later, the complete initial state $\mathcal{O}(\alpha^2)$ QED formula was presented in Ref. [5]. In Ref. [2] the $\mathcal{O}(\alpha^2)$ formula was also supplied with all-orders soft photonic corrections. This was an example

of the so called *exponentiation* procedure. Loosely speaking, the exponentiation is a smooth interpolation between finite order analytical expression and the multiple soft photons emission formula, see later in this Section for more detailed discussion on exponentiation. A collection of various exponentiations can be found in the proceedings of the 1989 LEP Workshop Ref. [6]. However, the estimation of the size of $\mathcal{O}(\alpha^3)$ corrections was not included and selective comparison of the exponentiation prescriptions has not been done at the time. This kind of analysis has been initiated in Refs [7,8].

In the presented work we repeat and complete the discussion of Refs [7,8]. As before, we discuss the QED photonic corrections to the total cross-section in $e^+e^- \to Z+\gamma \to f\bar{f}$. In Ref. [7] the calculations were done within the leading logarithmic (LL) framework and two exact solutions to the LL non singlet evolution equation were presented. To this purpose the Monte Carlo algorithm and the numerical inversion of the Mellin transforms were used. The analytical result for the $\mathcal{O}(\beta^3)$, $\beta = 2(\alpha/\pi)L$, $L = \ln(s/m_e^2)$ correction to the exponentiated formulas for non singlet function was also presented. Finally, the comparison of various exponentiations at the level of structure functions was performed. In Ref. [8] the new, compact, analytical $\mathcal{O}(\alpha^3)$ formula for the total cross-section was presented and the comparison of various exponentiations was carried out. In order to get the exact LL result the M.C. algorithm for the convolution of non singlet structure functions was used. In this paper we repeat major results of the above works and add the following new ones: We present the *complete* $\mathcal{O}(\beta^3)$ corrections to the non singlet function (virtual and real terms) and the singlet function as well. We find the exact solution of the singlet function by numerical inversion of the Mellin transforms, calculate the size of $\mathcal{O}(\beta^3)$ corrections to formulas for the total cross-section of Ref. [6] and discuss the cut-off dependence. Finally, we discuss the next-to-leading corrections in the total cross-section.

The basis of our work is the leading logarithmic approximation method, also called the structure functions method. This formalism, widely used in QCD, was in fact born in the QED framework. It was introduced in the Gribov and Lipatov works [9,10]. Authors of Ref. [9] have summed up to infinite order all the LL contributions to the deep inelastic ep scattering and to the e^+e^- annihilation. They found the dominant LL contributions to be of the form of the so called "ladder" graphs. The calculation was done in the Feynman gauge (unphysical), with the help of Sudakov phase space parameterization and Ward identities. The results were later rewritten in the form of evolution equation in Ref. [10]. The excellent guide to the resummation technique, discussion of the physical gauges and other details can be found in Ref. [11].

The structure functions were introduced in QCD by Altarelli and Parisi in Ref. [12]. Starting from the results of Operator Product Expansion (OPE) they rewrote and reinterpreted the formal OPE result in terms of parton distributions and their evolution in energy scale Q^2. Various calculations at the LL and the next-to-leading levels were performed, including processes like Drell–Yan, deep inelastic leptoproduction, e^+e^- into hadrons *etc.*, for a review of corresponding results see for example Ref. [13]. A key point of all these results was the factorization theorem, see *e.g.* Refs [13-16,11,3]. It states that the cross-section to a given process can be split up into two parts: a short distance hard cross-section and a structure function containing long distance effects. The hard cross-section is calculated perturbatively and does not contain any mass singularities. The remaining part, called the structure function, consists of the experimental input (structure of the hadron) at the fixed energy scale $s_0 = Q_0^2$. They may be evolved to higher energy scale by the evolution equations. All the spurious mass singularities are factored out at the scale s_0 and replaced by the experimental data. This is very different from the QED case where at the scale $s_0 \sim m_e^2$ there is no internal structure of the electron. This sets the unique analytical boundary condition for evolution equations. In other words the QED structure functions are *calculable* without any experimental input.

In the past few years, supported by the formalism developed in QCD, the structure functions revived in QED. It was due to Kuraev and Fadin [2], who, in analogy to the Drell–Yan process, described the initial state corrections to $e^+e^- \to f^+f^-$ with the help of electron (positron) structure functions, factored out from the cross-section. Then, the LL evolution equations were used to determine the structure functions. The NLL calculations for $e^+e^- \to f^+f^-$ were done in Ref. [5]. Some further developments can be found *e.g.* in Refs [17-22]. Let us also mention other areas of application of QED structure functions: LEP-II and HERA experiments, see Refs [23,24].

In the LL approach all the leading (collinear) contributions are simultaneously summed up to the infinity. The leading terms are defined as proportional to $\alpha^n L^n$ and the corresponding approximation is called the leading logarithmic (LL), the next-to-leading (NLL) approximation also sums terms proportional to $\alpha^n L^{n-1}$ and so on. In the case of the e^+e^- annihilation at LEP the NLL terms are already of order $\ln^{-1}(m_Z^2/m_e^2) \sim 4\%$ of the LL ones. The extension of this approach from LL to NLL is, however, difficult in practical calculations.

The other thing to be discussed is the concept of exponentiation. It may have several meanings. In the presented work we will often refer to this keyword. Therefore, we wish to clarify it here. Special attention should be payed to its relation to the LL approximation. *(i)* The rigorous way of exponentiation in perturbative QED is to follow the scheme by Yennie,

Frautschi and Suura in the 1961 paper Ref. [25]. The entire QED perturbative series of any process is rearranged in such a way that the infrared (IR) singularities are already from the beginning factored out, summed up to infinite order and properly cancelled. The remaining, not exponentiated, IR finite series can be calculated perturbatively. The YFS exponentiation is theoretically well founded and it forms a well defined order-by-order perturbative prescription for practical calculations. *(ii)* The other, common type of exponentiation is one or two dimensional "ad hoc" exponentiation of inclusive distributions. Inclusive distribution resulting from finite order calculation is improved by hand in the soft photons limit. The basic requirement in this procedure is that the IR behavior of the theory known from the YFS analysis is reproduced. *(iii)* Finally, the third possibility is the LL-exponentiation, compare Ref. [9]. It is, in fact, a well defined mathematical problem of finding the fastest convergent solution to the non singlet evolution equation. The LL approximation defines a closed and unique framework for QED calculations. The set of evolution equations for the structure functions is complete and the boundary conditions are unique (no internal structure of the electron). The evolution equations have the unique solution and can be solved approximately in a few different ways (a) iterative to the finite order or (b) in the soft limit to the infinite order. A certain interpolation between these two solutions, *i.e.* another approximate solution, we call the LL-exponentiation. Definite answers can be given both for the structure function itself and, after convolution, for the total cross-section as well.

All three ways of exponentiating are used in practical calculations, sometimes mixed together. For example, in Ref. [2] the $\mathcal{O}(\beta^2)$ solution for LL, non singlet structure function was derived, exponentiated (LL--exponentiation) and, then, supplied with subleading terms taken from the perturbative calculations, and improved in the soft limit as well (*ad hoc* exponentiation). In Ref. [5], see also Ref. [6], the finite order $\mathcal{O}(\alpha^1)$ and $\mathcal{O}(\alpha^2)$ perturbative inclusive distributions were exponentiated *ad hoc*. Finally, the exclusive YFS exponentiation was extensively used in practical Monte Carlo applications in Ref. [26] and related papers.[1]

Keeping in mind the above distinction the approach used in this paper is as follows. The results of our work will be obtained with the help of leading logarithmic approximation in QED. We stay within the LL framework as long as possible. We present and compare various solutions of LL evolution equations (LL-exponentiation). Next, still at the LL level, we discuss the total cross-section. Our LL $\mathcal{O}(\beta^3)$ result allows us to calculate the dominant

[1] In particular, in Ref. [27] the renormalization group method was applied to YFS scheme in order to improve its ultraviolet behaviour.

$\mathcal{O}(\beta^3)$ correction to total cross-section using various LL-exponentiation prescriptions. Finally, we discuss the formulas for total cross-section beyond the LL approximation. The exponentiation of NLL terms, although with close correspondence to the LL one, is the *ad hoc* one. We discuss nonleading effects and new "pragmatic" third order formula.

We summarize here the main *new* results of the paper.

- We calculate the complete $\mathcal{O}(\beta^3)$ corrections to electron non singlet and singlet structure functions. For the singlet contribution we also obtain the exact solution by numerical inversion of the Mellin transforms.
- We calculate the size of $\mathcal{O}(\alpha^3)$ corrections to commonly used QED photonic formulas for total cross-section at LEP. In the range $|\sqrt{s} - M_Z| \leq 2.5\text{GeV}$ it varies between $\Delta\sigma^{(3)}/\sigma \leq 0.12\%$ for Kuraev–Fadin exponentiation and $\Delta\sigma^{(3)}/\sigma \leq 0.01\%$ for the Jadach–Ward one. It also demonstrates that among *ad hoc* exponentiations discussed in Ref. [6] the best is by Jadach and Ward of Ref. [26].
- We show that the compact $\mathcal{O}(\beta^3)$ formula for initial state QED photonic corrections to total cross-section at LEP, proposed in Ref. [8], is accurate to $\delta\sigma/\sigma \leq 0.015\%$ in the range $|\sqrt{s} - M_Z| \leq 2.5\text{GeV}$ and to $\delta\sigma/\sigma \leq 0.035\%$ for $|\sqrt{s} - M_Z| \leq 7.5\text{GeV}$

The content of the paper is the following. In Section 2 notation and the master LL evolution equations are introduced. In Section 3 the analytical, Monte Carlo and numerical solutions of non singlet and singlet evolution equations are presented. In the non singlet case the discussion and comparison of exponentiation schemes is performed. Section 4 is devoted to application of structure functions to total cross-section at LEP. The higher order corrections are determined, various exponentiation schemes are compared and the compact formula for QED corrections is presented.

2. General framework of LL calculations

In this Section we introduce notation, collect definitions and present the basic set of evolution equations for the structure functions.

The LL electron structure functions $D_e^p(x,s)$, $p = e, \bar{e}, \gamma$ can be defined as density dn_p/dx of a given *virtual* particles p in the initial electron, where x is a fraction of the initial energy (longitudinal momentum) of the electron. The second argument $s = Q^2$ is the energy scale characteristic for the process (in our case of $e^+e^- \to \mu^+\mu^-$ it is the center-of-mass energy squared). The boundary condition for the evolution is the requirement that for $s = m_e^2$ electron does not have any internal structure, *i.e.* $D_e^e(x, m_e^2) = \delta(1-x)$.

Instead of D_e^e and $D_e^{\bar{e}}$ one often uses functions

$$D^{NS} = D_e^e - D_e^{\bar{e}} \quad \text{and} \quad D^S = D_e^{\bar{e}}. \tag{1}$$

$D^{\rm NS}$, the non singlet structure function corresponds to the diagrams where incoming electron takes part in the hard interaction ("valence"). In the case of non-running coupling constant used ($\alpha(s) = \alpha = 1/137$) the non singlet function describes corrections due to the multiple bremsstrahlung from an electron line. Inclusion of the running coupling constant also takes into account a similar mechanism of fermion pairs emission (some photons are replaced by fermion pairs). It can be seen *e.g.* in direct Gribov's and Lipatov's resummation of the leading ladder diagrams in Ref. [9]. The running coupling constant appears there effectively as a result of inclusion of the LL vacuum polarization in photonic lines. It is equivalent in turn to additional fermions in the final state. The singlet structure function describes the other possible mechanism of fermion pairs production. For the symmetry we introduce also function $D^+ = D_{\rm e}^{\rm e} + D_{\rm e}^{\bar{\rm e}}$.

The evolution equations read [10], [12], [2]

$$D_{\rm e}^{\rm e}(x,s) = \delta(1-x) + \int\limits_{m_{\rm e}^2}^{s} \frac{ds'}{s'}\frac{\alpha(s')}{2\pi}\Big(D_{\rm e}^{\rm e}(\cdot,s') \otimes P_{\rm e}^{\rm e}(\cdot)(x) + D_{\rm e}^{\gamma}(\cdot,s') \otimes P_{\gamma}^{\rm e}(\cdot)(x)\Big), \tag{2}$$

$$D_{\rm e}^{\bar{\rm e}}(x,s) = \int\limits_{m_{\rm e}^2}^{s} \frac{ds'}{s'}\frac{\alpha(s')}{2\pi}\Big(D_{\rm e}^{\bar{\rm e}}(\cdot,s') \otimes P_{\bar{\rm e}}^{\bar{\rm e}}(\cdot)(x) + D_{\rm e}^{\gamma}(\cdot,s') \otimes P_{\gamma}^{\bar{\rm e}}(\cdot)(x)\Big) \tag{3}$$

and

$$D_{\rm e}^{\gamma}(x,s) = \int\limits_{m_{\rm e}^2}^{s} \frac{ds'}{s'}\frac{\alpha(s')}{2\pi} \Big(-\frac{2}{3}D_{\rm e}^{\gamma}(x,s') + D_{\rm e}^{\rm e}(\cdot,s') \otimes P_{\rm e}^{\gamma}(\cdot)(x) + D_{\rm e}^{\bar{\rm e}}(\cdot,s') \otimes P_{\bar{\rm e}}^{\gamma}(\cdot)(x)\Big), \tag{4}$$

where

$$P_{\rm e(\bar{e})}^{\rm e(\bar{e})}(z) \equiv P(z) = \frac{1+z^2}{1-z} - \delta(1-z)\int\limits_0^1 dx \frac{1+x^2}{1-x}, \tag{5}$$

$$P_{\gamma}^{\rm e(\bar{e})}(z) = z^2 + (1-z)^2 \quad \text{and} \quad P_{\rm e(\bar{e})}^{\gamma}(z) = \frac{1+(1-z)^2}{z} \tag{6}$$

are the probabilities (splitting kernels) of transitions $\rm e(\bar{e}) \to e(\bar{e}), \gamma \to e(\bar{e})$ and $\rm e(\bar{e}) \to \gamma$ respectively. The convolution symbol $\otimes$ stands for

$$P_1(\cdot) \otimes P_2(\cdot)(x) = \int\limits_0^1 dx_1 dx_2 \delta(x - x_1 x_2) P_1(x_1) P_2(x_2). \tag{7}$$

Due to term $-{}^2/_3 D_e^\gamma$, Eq. (4) looks apparently different from its QCD analog of Ref. [12]. However, by rewriting $-{}^2/_3 D_e^\gamma(x,s') \equiv D_e^\gamma(\cdot,s') \otimes P_\gamma^\gamma(\cdot)(x)$ with $P_\gamma^\gamma(x) = -({}^2/_3)\delta(1-x)$ we recover the evolution equation for gluon structure function of Ref. [12] restricted to the QED case. The splitting kernel P_γ^γ corresponds to first order, purely virtual, vacuum polarization correction present also in QED.

At the LL level there is no difference between e and $\bar{e}$ at all. Certain differences might appear in the nonleading calculations, see Ref. [13]. The two terms in $P(z)$ correspond to the real and virtual photons. Cancellation of the IR singularities between them becomes more transparent if we use the explicit IR cut-off ϵ. The $P(z)$ is rewritten as

$$P(z) = \delta(1-z)\left(\frac{3}{2} + 2\ln\epsilon\right) + \Theta(1-\epsilon-z)\,\frac{1+z^2}{1-z}. \tag{8}$$

As before, the second term represents the real emission above the cut-off, whereas below the cut-off the singularities cancel resulting in a finite expression (first term).

Finally, $\alpha(s)$ is the running coupling constant. In the case of electrons only it is of the form $\alpha_e(s) = \alpha/\left(1-(\alpha/3\pi)\ln(s/m_e^2)\right)$. If also muons are allowed it becomes

$$\alpha_{e+\mu}(s) = \frac{\alpha}{\left(1 - \frac{\alpha}{3\pi}\ln\frac{s}{m_e^2} - \Theta(s-m_\mu^2)\frac{\alpha}{3\pi}\ln\frac{s}{m_\mu^2}\right)}.$$

At the end of this section we rewrite the master equations for D^{NS} and D^+ (*i.e.* D^{S}) in the form more convenient to the further analysis. We introduce new variable

$$\beta(s) = \int_{m_e^2}^{s} \frac{2\alpha(s')ds'}{\pi s'}. \tag{9}$$

Employing the various definitions of $\alpha(s)$ (see above) we find

$$\beta(s) = \begin{cases} \frac{2\alpha}{\pi}\ln\frac{s}{m_e^2} & \text{for } \alpha(s) = \alpha, \\ -6\ln(1 - \frac{\alpha}{3\pi}\ln\frac{s}{m_e^2}) & \text{for } \alpha(s) = \alpha_e(s), \\ -3\ln\left((1 - \frac{\alpha}{3\pi}\ln\frac{s}{m_e^2})^2 - (\frac{\alpha}{3\pi})^2\ln^2\frac{s}{m_\mu^2}\right) & \text{for } \alpha(s) = \alpha_{e+\mu}(s). \end{cases} \tag{10}$$

After differentiation with respect to β, the formal solution to Eq. (4) reads

$$D_e^\gamma(x,\beta) = \frac{1}{4}\exp\left(-\frac{\beta}{6}\right)\int_0^\beta d\eta \exp\left(\frac{\eta}{6}\right) D^+(\cdot,\eta)\otimes P_e^\gamma(\cdot)(x). \tag{11}$$

Substituting solution (11) to Eqs (2) and (3) we get

$$D^{\mathrm{NS}}(x,\beta) = \delta(1-x) + \tfrac{1}{4}\int_0^\beta d\eta D^{\mathrm{NS}}(\cdot,\eta)\otimes P(\cdot)(x) \tag{12}$$

$$\begin{aligned} D^{+}(x,\beta) =& \delta(1-x) \\ &+ \tfrac{1}{4}\int_0^\beta d\eta\Big(D^{+}(\cdot,\eta)\otimes P(x) \\ &+ \tfrac{1}{2}\exp\left(-\frac{\eta}{6}\right)\int_0^\eta dy \exp\left(\frac{y}{6}\right) D^{+}(\cdot,y)\otimes R(x)\Big) \end{aligned} \tag{13}$$

$$R(x) = P_e^{\gamma}\otimes P_{\gamma}^{e}(x) = \frac{1-x}{3x}(4+7x+4x^2) + 2(1+x)\ln x \tag{14}$$

Equations (12) and (13) together with (5) and (14) form the master set of LL evolution equations. Extensive discussion of their solutions will be presented in the next Section.

3. Solutions of the LL evolution equations

The exact analytical solutions of the equations (12) and (13) are not known. Here by the exact analytical solution we mean any one in the form of standard special functions and not the moments of the Mellin transforms. The strategy of solving Eqs (12) and (13) is, in general, twofold. One can construct solutions perturbatively expanding order by order in the effective parameter β. Until recently only the second order calculations were done, see Ref. [2][2]. They are in agreement with the full $\mathcal{O}(\alpha^2)$ results of Ref. [5]. However, to estimate the accuracy of $\mathcal{O}(\alpha^2)$ formulas it is necessary to know the $\mathcal{O}(\alpha^3)$ LL correction. In the following we will present new $\mathcal{O}(\beta^3)$ solutions of Eqs (12) and (13), see also Ref. [7] for more details.

The second way of solving convolution-like equations is to use the Mellin transforms converting the convolution into an ordinary multiplication. The inverse transformation is, unfortunately, difficult to carry out analytically. Only in the soft photons limit, $x \to 1$, of non singlet function (the singlet function vanishes for $x \to 1$) inversion can be done analytically. This way the old Gribov–Lipatov result of Ref. [9] is reproduced[3]. In order to get the

[2] See also note [32] in Ref. [6].

[3] In fact, at the last step of direct resummation of Feynman graphs, Gribov and Lipatov also used the Mellin transforms technique.

exact solution we will perform the complex integration in the inverse Mellin transforms numerically, both for non singlet and singlet structure functions.

Finally, for the non singlet function we propose also the exact Monte Carlo algorithm, see Refs [7,8], based on Ref. [28].

The further detailed analysis will be done separately for non singlet and singlet functions.

3.1. The non singlet function

3.1.1. Analytical perturbative $\mathcal{O}(\beta^3)$ result

The formal iterative solution to (12) is of the form

$$D^{\mathrm{NS}}(x,\beta) = \delta(1-x) + \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{\beta}{4}\right)^k P^{\otimes k}(x),$$
$$P^{\otimes k}(x) = \underbrace{P \otimes \cdots \otimes P}_{k}(x). \tag{15}$$

First term of this series is simply the $P(x)$ of Eq. (8), second term is already known in the literature, we quote it here for the sake of completeness, see Ref. [24],

$$\tfrac{1}{2} P \otimes P(x) = \delta(1-x)\Big(2\ln^2\epsilon + 3\ln\epsilon + \tfrac{9}{8} - 2\zeta(2)\Big) + \Theta(1-\epsilon-x)$$
$$\Big[\frac{1+x^2}{1-x}\big(2\ln(1-x) - \ln x + \tfrac{3}{2}\big) + \frac{1}{2}(1+x)\ln x - 1 + x\Big], \tag{16}$$

where $\epsilon \ll 1$ is an infrared regulator in the beam energy units, $\zeta(2) = \pi^2/6$ and $\zeta(3) = 1.2020569031\ldots$. We have calculated the third order correction to Eq. (15). It reads

$$\frac{1}{3!}\left(\frac{\beta}{4}\right)^3 P\otimes P\otimes P(x) = \left(\frac{\beta}{2}\right)^3 \delta(1-x)\delta_{\mathrm{term}} + \left(\frac{\beta}{2}\right)^3 \Theta(1-\epsilon-x)\theta_{\mathrm{term}} \tag{17}$$

with

$$\theta_{\mathrm{term}} = \Big[\frac{1}{2}\frac{1+x^2}{1-x}\Big(\frac{9}{32} - \frac{\pi^2}{12} + \frac{3}{4}\ln(1-x) - \frac{3}{8}\ln x + \frac{1}{2}\ln^2(1-x)$$
$$+ \frac{1}{12}\ln^2 x - \frac{1}{2}\ln x\ln(1-x)\Big) + \frac{1}{8}(1+x)\ln x\ln(1-x)$$
$$- \frac{1}{4}(1-x)\ln(1-x) + \frac{1}{32}(5-3x)\ln x - \frac{1}{16}(1-x)$$
$$- \frac{1}{32}(1+x)\ln^2 x + \frac{1}{8}(1+x)\mathrm{Li}_2(1-x)\Big], \tag{18}$$

$$\delta_{\mathrm{term}} = \frac{1}{3}\zeta(3) - \frac{1}{2}\zeta(2)\Big(\frac{3}{4} + \ln\epsilon\Big) + \frac{1}{6}\Big(\frac{3}{4} + \ln\epsilon\Big)^3. \tag{19}$$

Θ is the Heaviside step function, $\Theta(x > 0) = 1, \Theta(x \leq 0) = 0$.

Let us mention the interesting point on the iteration of (17) leading to the $\delta_{\rm term}$. Its nontrivial structure is a result of the real photons phase space rearrangement. For example in the series (15) the 3-photon phase space is given by $\Theta(1-\epsilon-x_1)\Theta(1-\epsilon-x_2)\Theta(1-\epsilon-x_3)$ whereas in (17) it is converted into $\Theta(1-\epsilon-x_1x_2x_3)$. This requires extension of the cut-offs, which, if carefully treated, leads to $\delta_{\rm term}$.

We may also use different method in calculation of the $\delta_{\rm term}$. The $D^{\rm NS}$ function satisfies the normalization condition (conservation of the lepton number)

$$\int\limits_0^1 dx D^{\rm NS}(x,\beta) = 1. \tag{20}$$

The same holds for every term of (15) separately

$$\int\limits_0^1 dx P^{\otimes k}(x) = \left[\int\limits_0^1 dx P(x)\right]^k = 0. \tag{21}$$

Hence,

$$\delta_{\rm term} = -\int\limits_0^{1-\epsilon} dx \theta_{\rm term}(x). \tag{22}$$

The advantage of the above method is that we can calculate the $\theta_{\rm term}$ careless of any finite cut-off ϵ effects and then restore the $\delta_{\rm term}$ from (22).

One important remark should be added. The $\delta_{\rm term}$ of the first three terms of (15) constitute the beginning of the expansion of the function

$$D^{\rm virt}(\epsilon,\beta) = \frac{\exp\left(-\frac{1}{2}\beta C_{\rm Euler}\right)}{\Gamma(1+\frac{1}{2}\beta)} \exp\left(\frac{1}{2}\beta\left(\frac{3}{4}+\ln\epsilon\right)\right),\ C_{\rm Euler} = 0.5772157\ldots. \tag{23}$$

It is up to the $e^{\beta/4}$ the infrared factor of Ref. [25]. It results from all-orders exponentiation and cancellation of virtual and real soft photons below the IR cut-off ϵ.[4] This way the artificial logarithmic singularities $\sim \ln^k \epsilon$ are summed up to the well behaved function $\epsilon^{\beta/2}$.

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[4] The extra $e^{\beta/4}$ corresponds to the remaining perturbative $\bar{\beta}_0$ term in YFS analysis of Ref. [25], see also Section 3.1.3, or directly to the Gribov–Lipatov soft solution (next Section).

3. 1. 2. Mellin transforms method

The Mellin transform of the function $f(x)$ is defined as

$$M[f](\zeta) \equiv F^{(\zeta)} = \int_0^\infty dx f(x) x^{\zeta-1} \tag{24}$$

and its inverse as

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\zeta x^{-\zeta} F^{(\zeta)}. \tag{25}$$

c is arbitrary constant such that $c > k$ for any k satisfying

$$\int_0^\infty dx\ x^{k-1} |f(x)| < \infty.$$

Since $M[P_1 \otimes P_2](\zeta) = M[P_1](\zeta) M[P_2](\zeta)$, hence, the Eq. (12) becomes

$$D^{\mathrm{NS}(\zeta)}(\beta) = 1 + \frac{1}{4} \int_0^\beta d\eta\, D^{NS(\zeta)}(\eta) P^{(\zeta)}, \tag{26}$$

$$P^{(\zeta)} = \psi(1) + \psi(3) - \psi(\zeta) - \psi(\zeta+2), \tag{27}$$

where $\psi(\zeta)$ is the Eulers ψ function. The formal solution to Eq. (26) is

$$D^{\mathrm{NS}(\zeta)}(\beta) = \exp\left(\frac{1}{4}\beta P^{(\zeta)}\right). \tag{28}$$

Inverse Mellin transform leads to the explicit exact solution

$$D^{\mathrm{NS}}(x,\beta) = \frac{1}{2\pi} \exp\left(\frac{1}{2}\beta\left(\frac{3}{4} - C_{\mathrm{Euler}}\right)\right) \int_{-\infty}^{\infty} dt\, x^{-it-c} \exp\left(\frac{\beta}{4}\left(-\psi(it+c) - \psi(it+c+2)\right)\right). \tag{29}$$

Analytical evaluation of the (29) cannot be done. The only known result is in the soft limit $x \to 1$ (*e.g.* Ref. [2]). One gets it keeping only the asymptotic form of the $\psi(z) \sim \ln z$. From the Hankel's representation of the Γ function we find

$$D^{\mathrm{Gribov}}(x,\beta) = \frac{\exp\left(\frac{1}{2}\beta\left(\frac{3}{4} - C_{\mathrm{Euler}}\right)\right)}{\Gamma\left(1+\frac{1}{2}\beta\right)} \frac{1}{2}\beta (1-x)^{\frac{1}{2}\beta-1}. \tag{30}$$

The same result was previously obtained by Gribov and Lipatov in Ref. [9] from the direct resummation of the Feynman graphs. Recently, some further refinements of the soft limit formula (30) has been proposed, see Ref. [29]. In the asymptotic expansion of $\psi(z)$ one has kept not only the leading term $\ln z$ but also some higher order terms.

The above result we can link to our previous perturbative calculations. Integrating (30) over soft photons domain up to some IR cut-off we recover the $D^{\rm virt}$ function (23)

$$\int\limits_{1-\epsilon}^{1} dx D^{\rm Gribov}(x,\beta) = D^{\rm virt}(\epsilon,\beta) \tag{31}$$

as conjectured previously in Section 3.1.1 from the perturbative expression for $\delta_{\rm term}$.

3. 1. 3. Exponentiation in LL

The final discussion of the previous paragraph introduced us to the subject of exponentiation. From the technical point of view the LL-exponentiation may be regarded as interpolation procedure between finite order solution (15) and the all orders soft solution of Eq. (30). Adding some higher order terms, one improves the convergence of the initial perturbative series (15). For the reader interested in detailed introduction into the subject of exponentiation we recommend Refs [8] and [30-32].

The most common exponentiation procedure is defined in Refs [2], [5]. According to Eq. (31) one replaces the $\delta_{\rm term}$ of Eq. (15) by the $D^{\rm Gribov}$ function. Next the $D^{\rm Gribov}$ function, expanded to a given order, is also extracted from the $\theta_{\rm term}$ of Eq. (15)

$$D^{\rm NS}_{\rm KF}(x,\beta) = D^{\rm Gribov}(x,\beta) + \Delta^{\rm KF}(x,\beta). \tag{32}$$

The explicit $\mathcal{O}(\beta^3)$ result for the $\Delta^{\rm KF}(x,\beta)$ reads

$$\begin{aligned}
\Delta^{\rm KF}(x,\beta) &= -\frac{1}{4}\beta(1+x) \\
&+ \frac{1}{16}\beta^2\left(-2(1+x)\ln(1-x) - 2\frac{\ln x}{1-x} + \frac{3}{2}(1+x)\ln x - \frac{5}{2} - \frac{1}{2}x\right) \\
&+ (\frac{\beta}{2})^3\left[-\frac{1}{2}(1+x)\left(\frac{9}{32} - \frac{\pi^2}{12} + \frac{3}{4}\ln(1-x) + \frac{1}{2}\ln^2(1-x)\right)\right. \\
&+ \frac{1}{2}\frac{1+x^2}{1-x}\left(-\frac{3}{8}\ln x + \frac{1}{12}\ln^2 x - \frac{1}{2}\ln x\ln(1-x)\right) \\
&+ \frac{1}{8}(1+x)\ln x\ln(1-x) - \frac{1}{4}(1-x)\ln(1-x) + \frac{1}{32}(5-3x)\ln x \\
&\left. - \frac{1}{16}(1-x) - \frac{1}{32}(1+x)\ln^2 x + \frac{1}{8}(1+x){\rm Li}_2(1-x)\right].
\end{aligned} \tag{33}$$

The above scheme was proposed by Kuraev and Fadin [2] and in the following is referred to as the KF solution[5].

The interesting question at this moment is: can one find another exponentiation prescription and which would be converging faster? In fact, one may invent many other $\mathcal{O}(\beta^3)$ interpolation prescriptions. All of them must reproduce the same finite $\mathcal{O}(\beta^3)$ result (17) but they may differ in $\mathcal{O}(\beta^4)$ and higher order terms. The only way to compare their accuracy can be a comparison with the exact solution. In the following sections we will discuss these questions with the help of the exact Monte Carlo solution.

In the following we would like to present another exponentiation prescription proposed by Jadach and Ward in Ref. [26]. It is motivated by the classical Yennie, Frautschi and Suura paper [25] on the all orders cancellation of IR singularities. For the convenience of the reader we review briefly the main ideas and results of YFS work. It will be useful for both the further discussion of the Jadach–Ward (JW) exponentiation and the discussion of the total cross-section beyond the LL approximation as well.

In the framework of QED calculations proposed by Yennie, Frautschi and Suura, the IR photonic singularities (both real and virtual) are explicitly factored out and then summed up to infinite order. The remaining non singular terms can be treated perturbatively with any required accuracy. The net result for any physical observable becomes manifestly IR finite. From the technical point of view it is certain rearrangement of the standard Feynman-like perturbative series. Applied to the initial state emission in $e^+e^- \to \mu^+\mu^-$ the rearrangement looks schematically as follows:

$$\frac{d\sigma}{d\epsilon} = \sum_{n=0} \alpha^n \frac{d\sigma_n}{d\epsilon} = \exp\bigl(2\alpha(\mathrm{Re}\, B_{\mathrm{virt}} + \overline{B}_{\mathrm{real}})\bigr) \frac{\beta_S}{\epsilon} F(\beta_S) \sum_{k=0} \bar{\beta}_k, \tag{34}$$

$$F(\beta_S) = \frac{\exp(-\beta_S C_{\mathrm{Euler}})}{\Gamma(1+\beta_S)}, \quad \beta_S = 2\frac{\alpha}{\pi}\left(\ln\frac{s}{m_e^2} - 1\right). \tag{35}$$

Here ϵ stands for the total energy carried away by the additional photons, in the beam energy units. The Feynman-like perturbative series of $d\sigma_n/d\epsilon$ is effectively replaced by the series of $\bar{\beta}_k$. Subscript k refers to the number of emitted real photons and each $\bar{\beta}_k$ separately is given by perturbative series. The difference is that in each $d\sigma_n/d\epsilon$ separately the IR cancellation must be done, resulting in a spurious IR cut-off singularity. The $\bar{\beta}_n$ from the construction are IR finite, and the cut-off singularities are factored out and

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[5] Note that for some time the soft singularities were exponentiated inconsequently (*e.g.* Ref. [33]). This was pointed out by Kuraev and Fadin [2].

summed properly to the infinite order. IR cancellation takes place in[6]

$$2\alpha(\mathrm{Re}\, B_{\mathrm{virt}} + \overline{B}_{\mathrm{real}}) = \beta_S \ln\epsilon + \frac{1}{4}\beta_S + \frac{\alpha}{\pi}\left(\frac{\pi^2}{3} - \frac{1}{2}\right). \tag{36}$$

The β_S corresponds to the LL β of Eq. (10). The important difference is in nonleading term -1 correctly included in β_S. For further details (*e.g.* the formal definitions of $\bar{\beta}_k$ terms) we refer to the original paper Ref. [25] (applications directly oriented on e^+e^- annihilation can be found *e.g.* in Refs [35,27,36,26]).

After trivial identification $\epsilon = 1 - x$, formula (34) and (36) with $\bar{\beta}_k = 0$ for $k \geq 1$ looks very much like D^{Gribov}, Eq. (30), the $\bar{\beta}_0$ term contains among others the missing $e^{\beta_S/2}$. One should remember however that Eqs (34)-(36) reproduce exactly the soft limit whereas (30) only within the LL approximation.

The YFS program only rearranges the entire QED perturbative series without any approximations. Both sides of the expansion (34) are exactly equivalent and can be worked out order by order. This is the general strategy used by Jadach and Ward in Ref. [26] and related papers. This way, the important soft limit of the cross-section is under control throughout all the calculations. The remaining $\bar{\beta}_k$ terms can be next determined with the help of standard perturbative technique without spoiling the IR behaviour.

Based on the formula (34) Jadach and Ward proposed another form of *ad hoc* exponentiation. The soft exponent is here extracted in the multiplicative form from the whole formula (also from the hard term). The corresponding LL-exponentiation reads

$$D^{\mathrm{NS}}_{\mathrm{JW}}(x,\beta) = D^{\mathrm{Gribov}}(x,\beta)\Delta^{\mathrm{JW}}(x,\beta), \tag{37}$$

$$\begin{aligned}\Delta^{\mathrm{JW}}(x,\beta) =& \frac{1}{2}(1+x^2) + \frac{1}{4}\frac{\beta}{2}\left(-\frac{1}{2}(1+3x^2)\ln x - (1-x)^2\right) \\ &+ \frac{1}{8}\left(\frac{\beta}{2}\right)^2\left((1-x)^2 + \frac{1}{2}(3x^2-4x+1)\ln x\right. \\ &\left. + \frac{1}{12}(1+7x^2)\ln^2 x + (1-x^2)\mathrm{Li}_2(1-x)\right).\end{aligned} \tag{38}$$

---

[6] More generally, the IR region corresponds to one of the mass singularities of the theory, the photonic one. Due to the Kinoshita–Lee-Nauenberg theorem Ref. [34] mass singularities disappear in the properly defined measurable quantity. This happens when one sums over all the states degenerated in mass. In our case over arbitrary number of the soft photons. Actually, this was done in the YFS analysis.

Up to the $\mathcal{O}(\beta^3)$ terms both formulas (37) and (32) are identical, the difference starts from $\mathcal{O}(\beta^4)$ terms. Compared to Δ^{KF}, Eq. (33), the Δ^{JW} is substantially simpler. In particular, it does not contain any singular $\ln(1-x)$ terms. This already indicates which formula is the better (more accurate) one. One more indication for the JW exponentiation can be found in Ref. [29]. Therein, in the soft limit, the approximate solution of the form as Eq. (37) is calculated with the help of Mellin transforms. The Δ^{JW} series of Eq. (38) is reproduced therein in the soft limit up to second order, *i.e.* up to $\mathcal{O}(\beta^1)$ hard terms. The multiplicative form (37) results there directly from the approximate calculations. The most convincing argument, coming from the comparison with the exact solution, will be given later on[7].

For the completeness we also quote the LL version of the third prescription, proposed in Ref. [6]

$$D^{\mathrm{NS}}_{\mathrm{LW}}(x,\beta) = \frac{1+x^2}{2} D^{\mathrm{Gribov}}(x,\beta) + \Delta^{\mathrm{LW}}(x,\beta). \tag{39}$$

$$\begin{aligned}
\Delta^{LW}(x,\beta) &= \frac{1}{16}\beta^2\left(-2\frac{\ln x}{1-x} + \frac{3}{2}(1+x)\ln x - 1 + x\right) \\
&+ (\frac{\beta}{2})^3\left[+\frac{1}{2}\frac{1+x^2}{1-x}\left(-\frac{3}{8}\ln x + \frac{1}{12}\ln^2 x - \frac{1}{2}\ln x \ln(1-x)\right)\right. \\
&+ \frac{1}{8}(1+x)\ln x \ln(1-x) - \frac{1}{4}(1-x)\ln(1-x) + \frac{1}{32}(5-3x)\ln x \\
&\left. - \frac{1}{16}(1-x) - \frac{1}{32}(1+x)\ln^2 x + \frac{1}{8}(1+x)\mathrm{Li}_2(1-x)\right].
\end{aligned} \tag{40}$$

Let us finally remark on the normalization of the exponentiated approximate solution. As we stated before the non singlet structure function satisfies the normalization condition $\mathcal{N} \equiv \int_0^1 dx D^{\mathrm{NS}}(x,\beta) = 1$. It is exactly fulfilled by any iterative order by order solution. After exponentiation, however, the normalization becomes only approximate. We have numerically checked that the solutions done in J-W prescription satisfy $\mathcal{N}[\mathcal{O}(\beta^1)] - 1 \leq 10^{-3}$, $\mathcal{N}[\mathcal{O}(\beta^2)] - 1 \leq 10^{-5}$ and $\mathcal{N}[\mathcal{O}(\beta^3)] - 1 \leq 10^{-6}$ relations. The discrepancies are due to incomplete higher orders (partially included by exponentiation).

3.1.4. Monte Carlo exact solution

So far, we have discussed various approximate solutions to (12). To estimate their actual accuracy, or to select the most efficient exponentiation we have at our disposal an exact solution of Eq. (12). In the following we

[7] See also Note Added at the end of the paper.

present such an exact solution obtained with the Monte Carlo technique. The M.C. algorithm was described in detail in Ref. [7]. Here we recall only its main ingredients.

The basis of the algorithm is the exact iterative solution of Eq. (12) rewritten in the form

$$D^{\mathrm{NS}}(x,\beta) = \lim_{\epsilon\to 0} \exp\left(\frac{1}{2}\beta\left(\ln\epsilon + \frac{3}{4}\right)\right) \sum_{n=0}^{\infty} \prod_{i=1}^{n} \int_{\epsilon x}^{1-x} \frac{dw_i}{w_i} \rho_n(w_1 \ldots w_n) W_n, \tag{41}$$

where

$$\rho_n = \frac{1}{n!}\left(\frac{\beta}{2}\right)^n \prod_{i=1}^{n} \theta(1-x-w_i)\theta(w_i-\epsilon x)\delta\left(1-x-\sum_{i=1}^{n} w_i\right),$$

$$W_n = \prod_{i=1}^{n} \frac{1}{2}\left(1+\left(1-\frac{w_i}{1-\sum_{j=1}^{i-1} w_j}\right)^2\right)\theta\left(w_i-\epsilon\left(1-\sum_{j=1}^{i} w_j\right)\right). \tag{42}$$

Formula (41) is nontrivial for M.C. integration since it involves an infinite series of integrals. Each of them represents a multiple photon emission. Photon emissions are almost uncorrelated. They get correlated each with other through the weights W_n.

The crucial observation now is that dropping the weights W_n the remaining simplified distributions ρ_n almost factorize (up to the δ-function) with respect to the w_i variables. The emission becomes almost independent *i.e.* Poissonian. The Monte Carlo algorithm for this class of distributions has been presented in Ref. [28].

The simplified integral (41) can be done analytically

$$\bar{D}^{\mathrm{NS}}(x,\beta) = \exp\left(\tfrac{1}{2}\beta\left(\ln\epsilon + \frac{3}{4}\right)\right) \sum_{n=0}^{\infty} \prod_{i=1}^{n} \int_{\epsilon x}^{1-x} \frac{dw_i}{w_i} \rho_n(w_1 \ldots w_n)$$

$$= \frac{\beta \exp\left(\frac{1}{2}\beta\left(\frac{3}{4} - C_{\mathrm{Euler}} + \ln\frac{1-x}{x}\right)\right)}{2(1-x)\Gamma\left(1+\frac{1}{2}\beta\right)}. \tag{43}$$

The entire M.C. algorithm is organized as follows:

- Generate variables n and $w_1, \ldots, w_n$ according to the simplified distributions ρ_n Ref. [28].
- Calculate the corrective weight $W_n(w_1, \ldots, w_n)$ for each M.C. point.

- Average weights over the entire M.C. series. The integral (41) is given by

$$D^{\mathrm{NS}}(x,\beta) = \bar{D}^{\mathrm{NS}}(x,\beta)\langle W(x,\beta)\rangle. \tag{44}$$

Few remarks are in order. *(i)* Let us notice that the IR cut-off ϵ is a dummy parameter in the algorithm. We have numerically checked that results do not depend on its choice in the wide range of ϵ from 10^{-4} to 10^{-12}. *(ii)* The M.C. algorithm, as presented above, provides us not only with the value of D^{NS} function, but also explicitly simulates the photonic cascade. The multiplicity n of each event is well defined, and the set of variables $(w_1,\ldots,w_n)$ can be interpreted as energies of the photons, or equivalently as their longitudinal momenta. *(iii)* At the LL level the transverse momenta are not well defined, they are integrated over. The only trace of them is in the effective parameter β which is given as

$$\beta = \int_{m_e^2}^{s} \frac{dk_T^2}{k_T^2}\frac{2\alpha(k_T^2)}{\pi}.$$

Transverse momenta correlation, conservation *etc.* are effects of subleading order. We are free to add to our algorithm distribution of transverse momenta dp_T^2/p_T^2 up to $p_T \sim \sqrt{s}$. These distributions however, are not controlled within the LL approximation and there is certain degree of arbitrariness in their choice. This holds true for any LL Monte Carlo algorithm in QED, compare Refs [20,19,37,38]. More sophisticated description of transverse momenta distributions is required in the NLL evolution equations *e.g.* Refs [5,13,14].

Since the series (41) is an exact solution to Eq. (12), the D^{NS} function resulting from our algorithm is automatically normalized to unity.

3. 1. 5. Numerical inversion of the Mellin transforms

As a second exact solution of Eq. (12) we numerically calculated the inverse Mellin transform, Eq. (29). In the case of non singlet function the direct integration is difficult to carry out. The complex, oscillating integrand does not vanish quickly enough with $|t| \to \infty$. The result would therefore either depend on the upper cut-off A used or would be unstable due to machine errors. We overcame this difficulty by subtracting from Eq. (29) the asymptotic form of the integrand and integrating it analytically, actually Eq. (30) represents result of integration. The remaining part is better convergent and can be integrated. As the integration contour we used line $z = 1 + it, t \in (-A, A)$.

3. 1. 6. Numerical comparison of solutions

Having in hand exact solution to Eq. (12) we estimate the accuracy of the various previously discussed approximate solutions. In particular we

compare different exponentiation prescriptions and find the best convergent one.

At first, we discuss and compare two exact results: of the M.C. and Mellin type. The accuracy of M.C. solution is under precise control. It is given by the standard deviation and depends only on the number N of generated events. The accuracy improves rather slowly with the increasing statistics ($\delta D/D \sim 1/\sqrt{N}$) but in principle can be arbitrarily high. The situation of Mellin integration is more complicated. To increase precision one should extend the integration domain. On the other hand it quickly makes the integration routines unstable and the results meaningless. The actual numerical error is therefore under poor control. Both method agree within the relative accuracy 10^{-4}. It is also the best accuracy reached with Mellin integration.

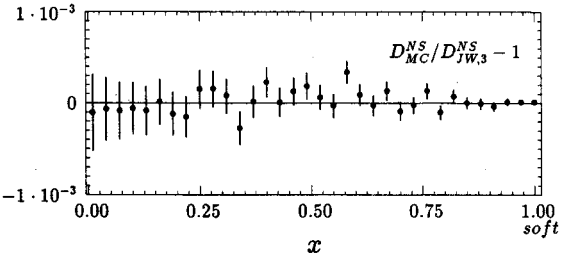

Fig. 1. The ratio of the exact Monte Carlo result to the third order exponentiated analytical solution of the JW type for the LL non singlet electron structure function. Calculation was done for $\sqrt{s} = 92$GeV. Samples of 10^6 events were used for each x-point. The M.C. statistical error is displayed.

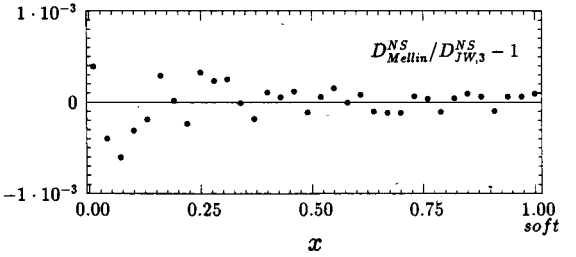

Fig. 2. The ratio of the exact Mellin transform result to the third order exponentiated analytical solution of the JW type for the LL non singlet electron structure function. Calculation was done for $\sqrt{s} = 92$GeV.

In Figs 1 and 2 we plotted respectively, the ratios of both M.C. and Mellin solutions with the best approximate third order result, exponentiated according to JW prescription, of Eq. (37). The error bars of Fig. 1 represent M.C. error for the statistics 10^6 events per point. The comparison can be summarized as follows:

- All three solutions agree within the accuracy $5 \cdot 10^{-4}$ in hard limit and $1 \cdot 10^{-4}$ in soft limit.
- The analytical solution (37) is accurate to about $1 \cdot 10^{-4}$ and can be treated as a numerical parameterization of the exact solution within this accuracy.

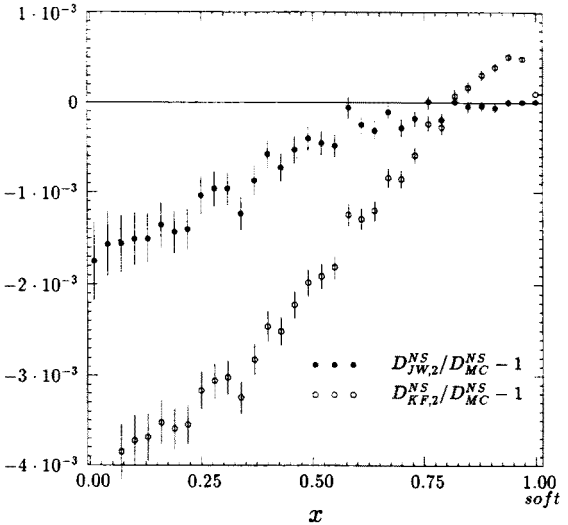

Fig. 3. The ratio of the approximate to exact solutions for the LL non singlet electron structure function. The approximate solutions are the exponentiated $\mathcal{O}(\beta^2)$ solution of either the KF type (open circles) or JW type (filled circles). The exact solution is obtained by means of the M.C. technique. The error bars represent the M.C. statistical error. Calculation was done for $\sqrt{s} = 92\text{GeV}$. Samples of 10^6 events were used for each x-point.

Having discussed the exact solutions we can compare various exponentiation procedures. In Fig. 3 we show the second order solutions exponentiated by KF and JW prescriptions, both normalized to the exact M.C. solution. The results of comparison are as follows:

- The JW prescription is definitely closer to the exact result than the KF one. It holds true not only for hard but for soft region as well (the latter one is especially important in the case of resonances or strong cut-offs).
- In the soft part ($x > 0.8$) the accuracy of the $\mathcal{O}(\beta^2)$ JW solution is almost the same as the $\mathcal{O}(\beta^3)$ JW one, compare Fig. 1. Differences are of order 10^{-5}.
- The third prescription (39), not shown in Fig. 3, places in between the JW and KF exponentiations.

The final conclusion is that the JW exponentiation prescription (37) is

better than both the commonly used KF one of Eq. (32), and the one of Eq. (39) as well.

3.2. The singlet function

3. 2. 1. Analytical perturbative $\mathcal{O}(\beta^3)$ solution

The general iterative series for D^S resulting from Eqs (13) and (15) we found to be of the form

$$D^S(x,\beta) = \frac{1}{2}\sum_{n=2}\frac{1}{n!}\left(-\frac{\beta}{6}\right)^n \sum_{m=0,r=1}^{m+2r\leq n}\binom{m+r}{r} \times \binom{n-m-r-1}{r-1}\left(-\frac{3}{2}P\right)^{\otimes m}\otimes\left(\frac{9}{2}R\right)^{\otimes r}(x). \tag{45}$$

$P(x)$ and $R(x)$ are defined in Eqs (8) and (14).

Up to the second (*i.e.* first nontrivial) order the solution (45) was presented *e.g.* in Ref. [2]. We have also calculated the *new* third order correction. In the $\mathcal{O}(\beta^3)$ we need to compute analytically the convolution $P\otimes R$ (one pair + one photon). Unlike the non singlet case, the $R(x)$ distribution is infrared regular and finite: $R(x=1)=0$. It reflects the fact that the probability of emitting a soft pair (through the "singlet" mechanism) tends to zero in the IR limit. We do not deal with any IR singularities during the calculations and the final result reads

$$P\otimes R(x) = \left(\frac{3}{2}+2\ln(1-x)\right)R(x) + (1+x)\left(-\ln^2 x + 4\mathrm{Li}_2(1-x)\right) + \frac{1}{3}(-9-3x+8x^2)\ln x + \frac{2}{3}\left(-\frac{3}{x}-8+8x+3x^2\right), \tag{46}$$

$$\begin{aligned} D^S(x,\beta) &= \frac{1}{2!}\beta^2\frac{1}{16}R(x) + \frac{1}{3!}\beta^3\left(\frac{1}{32}P\otimes R(x) - \frac{1}{96}R(x)\right) \\ &= \frac{1}{16}\left[\frac{1}{2!}\beta^2 R(x) + \frac{1}{3!}\beta^3\left(\left(\frac{3}{4}+\ln(1-x)-\frac{1}{6}\right)R(x)\right.\right. \\ &\quad + \frac{1}{3}\left(-\frac{3}{x}-8+8x+3x^2\right) + \frac{1}{6}(-9-3x+8x^2)\ln x \\ &\quad \left.\left. + \frac{1}{2}(1+x)\left(-\ln^2 x + 4\mathrm{Li}_2(1-x)\right)\right)\right]. \end{aligned} \tag{47}$$

3. 2. 2. Mellin transforms results

As in the non singlet case the use of the Mellin transforms, see Eqs (24)-(25), converts the Eq. (13) into ordinary linear differential equation for the

moments $D^{+(\zeta)}(\beta)$ of the $D^+(x,\beta)$ function

$$D^{+(\zeta)\prime\prime}(\beta) = \left(\frac{1}{4}P^{(\zeta)} - \frac{1}{6}\right)D^{+(\zeta)\prime}(\beta) + \left(\frac{1}{24}P^{(\zeta)} + \frac{1}{8}R^{(\zeta)}\right)D^{+(\zeta)}(\beta),$$
$$D^{+(\zeta)}(0) = 1, \quad D^{+(\zeta)\prime}(0) = \frac{1}{4}P^{(\zeta)}, \tag{48}$$

where prime denotes differentiation with respect to β. The solution to Eq. (48) is of the following form, compare Ref. [9],

$$D^{+(\zeta)}(\beta) = \frac{\frac{1}{4}P^{(\zeta)} + \frac{1}{6} + \sqrt{\Delta(\zeta)}}{2\sqrt{\Delta(\zeta)}} \exp\left[\frac{\beta}{2}\left(\frac{1}{4}P^{(\zeta)} - \frac{1}{6} + \sqrt{\Delta(\zeta)}\right)\right]$$
$$- \frac{\frac{1}{4}P^{(\zeta)} + \frac{1}{6} - \sqrt{\Delta(\zeta)}}{2\sqrt{\Delta(\zeta)}} \exp\left[\frac{\beta}{2}\left(\frac{1}{4}P^{(\zeta)} - \frac{1}{6} - \sqrt{\Delta(\zeta)}\right)\right], \tag{49}$$

$$\Delta(\zeta) = \frac{1}{4}\left(\frac{1}{2}P^{(\zeta)} + \frac{1}{3}\right)^2 + \frac{1}{2}R^{(\zeta)},$$

$$R^{(\zeta)} = \frac{1}{3}\left(\frac{4}{\zeta-1} + \frac{3}{\zeta} - \frac{3}{\zeta+1} - \frac{4}{\zeta+2}\right) - 2\left(\frac{1}{\zeta^2} + \frac{1}{(\zeta+1)^2}\right). \tag{50}$$

During inversion the requirement on the constant c of Eq. (25) is $c > 1$. The analytical inversion of the Eq. (49)has not been found yet and probably does not exist.

We have done the numerical inversion of Eq. (49). In the soft limit the D^+ function reduces to the D^{NS} function (30), *i.e.*, the D^S function itself vanishes for $x \to 1$. As a consequence the $D^{S(\zeta)} \equiv \frac{1}{2}D^{+(\zeta)} - \frac{1}{2}D^{NS(\zeta)}$ vanishes quickly for $|\zeta| \to \infty$ and the integration can be done without difficulties. The result becomes almost independent of the upper integration limit A and the Gaussian numerical integration is stable. We took as the integration contour the line $z = c + it$ with c slightly higher than 1.

3. 2. 3. Comparison of solutions

The comparison of $\mathcal{O}(\beta^2)$ and $\mathcal{O}(\beta^3)$ solutions of Eq. (47) with the exact numerical solution is given in Fig. 4. The second order function, as the lowest possible order, is not very accurate, about a few per cent, even for medium and hard region, $x \leq 0.75$. The third order correction already reaches the level of accuracy below 0.1%, for $x \leq 0.75$. In the soft emission limit $x \to 1$ both solutions start to diverge. This is due to the lack of soft photons resummation, "exponentiation". This discrepancy is, however, completely irrelevant numerically since $D^S \to 0$ as $x \to 1$ and the relative weight of the region $x \sim 1$ is negligible. Moreover, as compared to D^{NS} the function D^S is a few orders of magnitude lower almost everywhere except for $x \sim 0$. The only region where D^S contribution could be of any importance

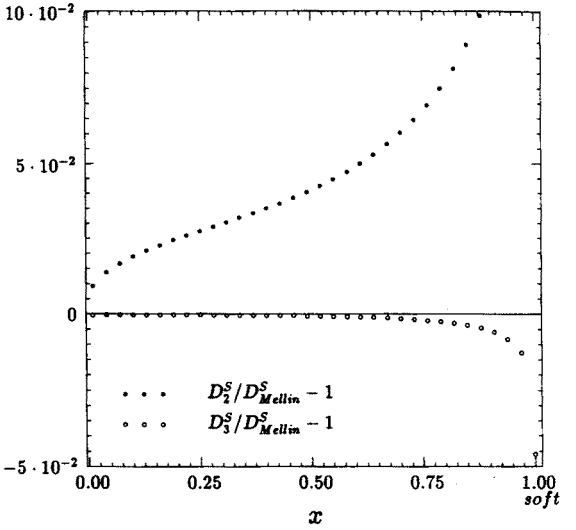

Fig. 4. The ratio of the approximate to exact solutions for the LL singlet electron structure function. The approximate solutions are the $\mathcal{O}(\beta^2)$ (filled circles) and $\mathcal{O}(\beta^3)$ (open circles) solutions respectively. The exact solution is obtained by means of the Mellin transform. Calculation was done for $\sqrt{s} = 92\text{GeV}$.

is $x \sim 0$, which corresponds to the emission of very high energy pairs. Only for very loose cut-offs and energies far from resonance such contribution may be significant.

4. Applications to total cross-section at LEP

The second part of the article is devoted to a certain application of the presented LL structure functions formalism, to the calculation of the total cross section $\sigma(s)$ for the process $e^+e^- \to \mu^+\mu^-$ at LEP. We shall look at the e^+e^- annihilation as the Drell–Yan process. Using the QCD-style parton model we may describe all the initial state QED corrections with the help of structure functions[8]. This approach was pioneered in Refs [2] and [4].

Let us concentrate first on the LL calculations. Later on we shall discuss the total cross-section in a complete way going beyond the LL approximation. In this extension we shall make use of the $\mathcal{O}(\alpha^2)$ results of Refs [5,6]. We should mention here that the generalization of the parton model like convolution formula (51) to the subleading orders is not proved to be correct. To be rigorous, one should follow the NLL factorization theorems, and

[8] Actually, at the LL level the direct resummation of leading, "ladder" graphs justifies the improved parton model as a proper "factorization theorem".

in particular use the corrected hard cross section instead of the Born one. On the other hand, however, such a solution would not reproduce the soft limit of the formulas in a correct way.

The LL calculations constitute the "skeleton" of the analysis. At the LL level we have estimated the size of $\mathcal{O}(\beta^3)$ corrections and compare various exponentiations. In the next step we shall assure the proper soft limit behaviour of our LL formula (based on the YFS result, see Section 3.1.3.). In the final step we shall add remaining nonleading terms, representing among others hard photons emission. Our strategy should be regarded as a phenomenological one, since it does not use the NLL factorization theorem but rather relies on the explicit $\mathcal{O}(\alpha^2)$ results of Refs [5,6].

4.1. Physical input

The requirements for theoretical accuracy of peak total cross-section formulas are about $\delta\sigma/\sigma \leq 0.3\%$ in the energy range $|\sqrt{s} - M_Z| \leq 2.5\text{GeV}$. There are three types of corrections: QED, electro-weak and QCD.

The QED photonic corrections at LEP have to be under control up to the second order. On the other hand there are almost no estimations of the accuracy of the $\mathcal{O}(\alpha^2)$ formulas available in the literature. For example in the review of Ref. [6] there is an extensive presentation of various analytical formulas for $\sigma(s)$ including emission of up to two additional photons from the initial state and the exponentiation of multiple soft photons. The additional fermion pair production and all other initial-final corrections are excluded. An accuracy quoted therein is $\delta\sigma/\sigma(M_Z^2) \sim 0.3\%$. There is, however, no satisfactory explanation where from this error comes.

The main goal of our work is to reduce this error and to justify the new result with the series of calculations. We restrict ourself to the initial state QED corrections. The initial-final state corrections as well as the pair production are left beyond present analysis and will be published separately, see Ref. [39].

As far as the initial state corrections are concerned the most important points to be discussed are:

- What is the size of $\mathcal{O}(\alpha^3)$ photonic corrections?
- Which exponentiation scheme is better?
- What is the magnitude of subleading terms?
- What is the dependence on the cut-off?

Using results obtained in the previous sections we are able to answer the above list of questions.

We shall also propose a new compact analytical formula for $\sigma(s)$, see Ref. [8]. It includes initial state LL photonic corrections up to the third order and the NLL ones up to the second order. This formula we shall call

"the pragmatic" third order formula. The overall accuracy of the formula is estimated conservatively to be $\delta\sigma/\sigma \leq 0.015\%$ for $|\sqrt{s} - M_Z| \leq 2.5\text{GeV}$.

4.2 Details of calculations

The QED parton model description of the $e^+e^- \to \mu^+\mu^-$ annihilation is, in analogy to the Drell–Yan process, of the form

$$\sigma(s) = \int dx_1 dx_2 \sum_{p=e,\bar{e},\gamma} D_e^p(x_1,s) D_{\bar{e}}^{\bar{p}}(x_2,s) \sigma_{p\bar{p}}(sx_1x_2) \tag{51}$$

$$= \int_{x_{cut}}^1 dx \Phi(x,s) \sigma_{\text{Born}}(sx). \tag{52}$$

In Eq. (52) we already omitted the term with $p = \gamma$, as contributing to the non resonant, two photon pair production mechanism. Up to the third order only the terms with $p = e$ contribute to Eqs (51)-(52). In Eq. (52) we shown explicitly the cut-off x_{cut}.

At the LL level the kernel $\Phi_{\text{LL}}(x,\beta)$ can easily be obtained from the structure function results of the previous sections. We get

$$\Phi_{\text{LL}}(x,\beta) = \Phi_{\text{LL}}^{\text{NS}}(x,\beta) + \Phi_{\text{LL}}^{S}(x,\beta), \tag{53}$$

where

$$\Phi_{\text{LL}}^{\text{NS}}(x,\beta) = D^{\text{NS}}(\cdot,\beta) \otimes D^{\text{NS}}(\cdot,\beta)(x) = D^{\text{NS}}(x,2\beta), \tag{54}$$

$$\Phi_{\text{LL}}^{S}(x,\beta) = D^{S}(\cdot,\beta) \otimes D^{S}(\cdot,\beta)(x) + 2D^{\text{NS}}(\cdot,\beta) \otimes D^{S}(\cdot,\beta)(x). \tag{55}$$

The non singlet kernel $\Phi_{\text{LL}}^{\text{NS}}$ is of the same form as the D^{NS} function. The whole modification is to substitute $\beta \to 2\beta$. It can be seen *e.g.* from the Eq. (28) for the moments of D^{NS}. All the exponentiated solutions to D^{NS} can be directly rewritten for $\Phi_{\text{LL}}^{\text{NS}}$. This way, we have at our disposal also the exact M.C. and numerical form of $\Phi_{\text{LL}}^{\text{NS}}$. It is either the M.C. algorithm of Sect. 3.1.4. or Mellin integration of Sect. 3.1.5. In actual comparisons we will use the parameterization of the exact results by the analytical formula (37), see also Fig. 1. This parameterization guaranties the accuracy of $\delta\Phi_{\text{LL}}^{\text{NS}}/\Phi_{\text{LL}}^{\text{NS}} \leq 3 \cdot 10^{-4}$ in the hard region $x < 1$ and $\delta\Phi_{\text{LL}}^{\text{NS}}/\Phi_{\text{LL}}^{\text{NS}} \leq 10^{-5}$ in the soft limit $x \to 1$.

On the contrary to $\Phi_{\text{LL}}^{\text{NS}}$ the Φ_{LL}^{S} does not obey a simple multiplication rule (compare Eq. (49)). Up to the β^3 terms it reads

$$\Phi_{\text{LL}}^{S}(x,\beta) = \left(\frac{\beta}{4}\right)^2 R(x) + \left(\frac{\beta}{4}\right)^3 \left(\frac{5}{3} P(\cdot) \otimes R(\cdot)(x) - \frac{2}{9} R(x)\right), \tag{56}$$

see Eqs (14) and (46) for R and $P \otimes R$ functions.

The exact $\mathcal{O}(\alpha^2)$ formula for $\Phi(x,s)$ has been calculated up to the second order in Refs [5,6].

In practical applications the integration in Eq. (52) is done numerically. There are, however, some approximate analytical results for the special forms of $\Phi(x,\beta)$, see Refs [5,6,29].

4.3. Higher order corrections

To obtain the size of $\mathcal{O}(\beta^3)$ photonic corrections we can use the exact, infinite order, Monte Carlo calculation of $\Phi_{\rm LL}^{\rm NS}$. Compared with the $\mathcal{O}(\beta^2)$ cross-section based on Eq. (16) we get the size of dominant $\mathcal{O}(\alpha^3)$ correction.

In Fig. 5 we present, among others, the ratio of LL iterative, not exponentiated, $\mathcal{O}(\beta^2)$ and $\mathcal{O}(\beta^3)$ cross-sections of Eqs (16) and (17) to the exact LL result, as parameterized by Eqs (37) and (38). A cut-off $x_{\rm cut} = 0.9$ on the energy of photons in the integration of Eq. (52) is used. One can clearly see from Fig. 5 that the $\mathcal{O}(\beta^3)$ photonic corrections are huge, about 1% near the peak. Even the $\mathcal{O}(\beta^4)$ terms are 0.2%. It is too much to be neglected. The situation for the looser cut-off, $x_{\rm cut} = 0.6$, is presented in Fig. 6. The size of $\mathcal{O}(\beta^3)$ and $\mathcal{O}(\beta^4)$ corrections is approximately the same as in the previous case.

Such a big corrections are mainly due to the soft photons emission. If it is really the case, the exponentiation procedure should substantially reduce their size. Indeed, as a second pair of curves in Figs 5 and 6 we plotted the LL exponentiated according to Jadach–Ward prescription $\mathcal{O}(\beta^1)$ and $\mathcal{O}(\beta^2)$ cross-sections with normalization and cut-offs as before. In the case of stronger cut $x_{\rm cut} = 0.9$, already the first order exponentiated formula is accurate to 0.2% whereas, the second order one almost coincides with the exact result. For the looser cut, $x_{\rm cut} = 0.6$, the exponentiated solutions close to the resonance preserve the same accuracy. Further from the peak, however, they start to diverge. It is mainly due to the hard photons included only to lower (first or second) orders. At the peak, the resonance itself serves as a strong cut-off.

The lesson from the above exercise is the following. In the case of strong cut-offs, or close to the resonance, the exponentiated formulas are more effective than not exponentiated ones. It could be especially useful in the case of Bhabha scattering where one uses the $\mathcal{O}(\alpha^1)$ matrix elements, see Ref. [40].

4.4 Comparison of exponentiation schemes

The results of the previous section pointed out that the actual size of higher order corrections is highly sensitive to exponentiation procedures. In

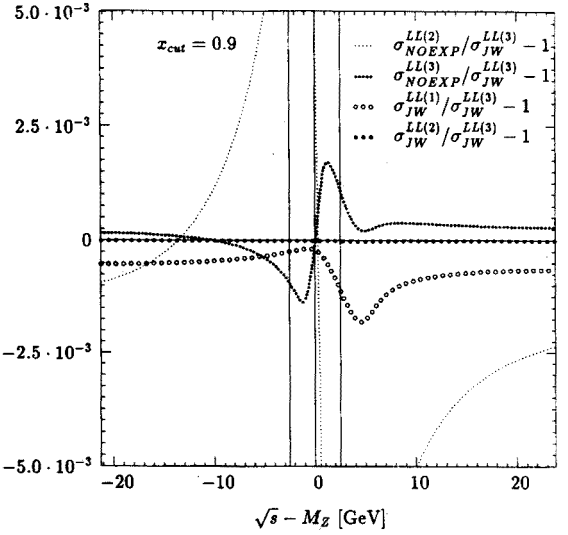

Fig. 5. The ratio of the various approximate LL cross-sections to the exact LL result as parameterized by the exponentiated third order formula Eq. (37) and (38) of the JW type. The approximate cross-sections are either not exponentiated $\mathcal{O}(\beta^2)$ (dots) and $\mathcal{O}(\beta^3)$ (small filled circles) ones or exponentiated JW $\mathcal{O}(\beta^1)$ (big open circles) and $\mathcal{O}(\beta^2)$ (big filled circles) ones. The strong cut-off $x_{\mathrm{cut}} = 0.9$ in the integration of Eq. (52) is used.

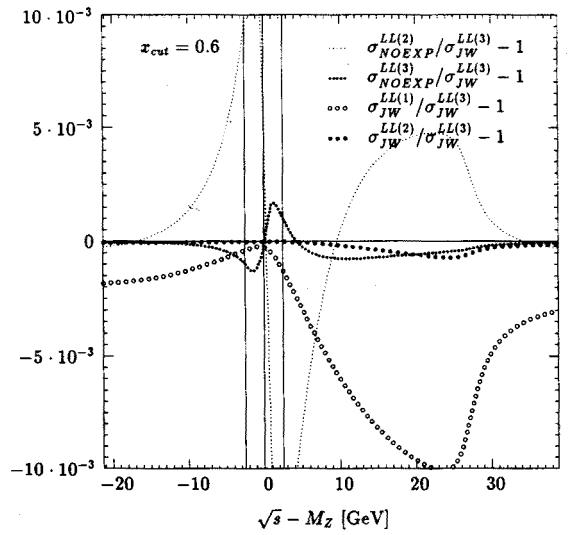

Fig. 6. The same as in Figure 5 for the looser cut $x_{\mathrm{cut}} = 0.6$.

the following we pursue this question and compare various LL-exponentiation

schemes in the total cross-section calculations. As before it will be done in the LL approximation with the help of the exact Monte Carlo form of $\Phi_{\mathrm{LL}}^{\mathrm{NS}}$. We hope that the results remain true also in the nonleading orders, where none direct analysis is possible, see, however, remarks in Sect. 4.5.

In Figs 7 and 8 we present the LL second order exponentiated total cross-sections resulting from two different prescriptions: Kuraev–Fadin of Eq. (32) and Jadach–Ward of Eq. (37). Both curves are normalized to the exact LL result as parameterized by formulas (37) and (38). Two different cut-offs $x_{\mathrm{cut}} = 0.9$ and $x_{\mathrm{cut}} = 0.6$ are employed.

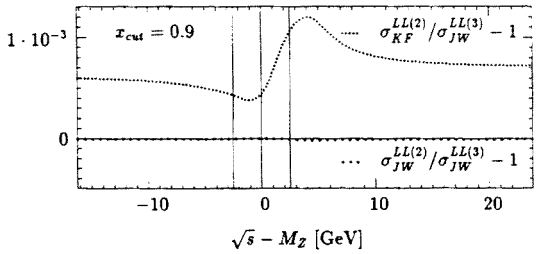

Fig. 7. The ratio of the two approximate LL second order cross-sections to the exact LL result as parameterized by the exponentiated third order formula Eq. (37) and (38) of the JW type. The approximate cross-sections are either the $\mathcal{O}(\beta^2)$ exponentiated KF (dots) or $\mathcal{O}(\beta^2)$ exponentiated JW (filled circles) ones. The strong cut-off $x_{\mathrm{cut}} = 0.9$ in the integration of (52) is used.

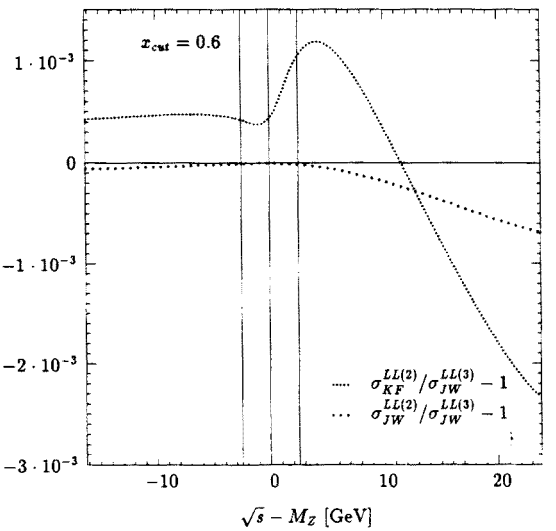

Fig. 8. The same as in Figure 7 for the looser cut-off $x_{\mathrm{cut}} = 0.6$.

The results of comparison remain in the perfect agreement with the

earlier analysis of Sect. 3.1.6 done for single structure functions. The KF prescription is definitely less accurate than the JW one. The quantitative results are the following:

- In the range $|\sqrt{s} - M_Z| \leq 2.5\text{GeV}$ the $\mathcal{O}(\beta^3)$ corrections to the KF formula are $\Delta\sigma^{(3)}_{\text{KF}}/\sigma_{\text{KF}} \leq 0.12\%$, no matter which cut-off is used.
- For the JW formula, in the same $\sqrt{s}$ range, the $\mathcal{O}(\beta^3)$ corrections are at least ten times smaller, *i.e.* $\Delta\sigma^{(3)}_{\text{JW}}/\sigma_{\text{JW}} \leq 0.01\%$.
- The third prescription of Eq. (39), not shown in the pictures, is somewhere in between ($\leq 0.05\%$).

The above points summarize the main results of our analysis of total cross-section at LEP. They represent the *actual* $\mathcal{O}(\alpha^3)$ *photonic corrections to the commonly used formulas of* Ref. [6]. With an eye on the 0.1% accuracy of the QED sector, we come to the conclusion that the most popular KF exponentiation exceeds at the second order the required accuracy. The third order corrections are *not negligible* in this scheme. In the JW prescription the situation is different. In this case, already at the second order, we reach the accuracy $\delta\sigma_{\text{JW}}/\sigma_{\text{JW}} \leq 0.01\%$ for $|\sqrt{s} - M_Z| < 2.5\text{GeV}$.

We stress here once more that there is nothing principally wrong with the KF formula. Both KF and JW ones are physically correct. The JW is, however, more effective in summing up higher orders.

The final remark concerns the dependence on the cut-offs. Comparing Fig. 7 for cut-off $x_{\text{cut}} = 0.9$ with Fig. 8 for cut-off $x_{\text{cut}} = 0.6$ we find again that exponentiated formulas are more accurate in the case of stronger cuts.

4.5. NLL *corrections and the "pragmatic"* $\mathcal{O}(\alpha^3)$ *formula for total cross-section*

Until this point our analysis was carried out within pure leading logarithmic framework. We have compared various exponentiation schemes and calculated the actual size of the QED $\mathcal{O}(\beta^3)$ corrections to corresponding formulas for the total cross-section. In order to make it realistic we have to go now beyond the LL approximation. The collection of formulas can be found in Refs [5] and [6]. In this section we propose a *new*, compact formula for initial state photonic corrections accurate to 0.015% in the resonance region.

We recall that according to Eq. (54) the $\Phi^{\text{NS}}_{\text{LL}}$ has the same functional form as the $D^{\text{NS}}_{\text{LL}}$. The whole modification amounts to substitution $\beta \to 2\beta$. The LL-exponentiation procedure can be therefore done for a single structure function as well as for a convolution of two structure functions. We will benefit from it in NLL level. As mentioned earlier, in our analysis of subleading corrections we will not follow the rigorous NLL factorization theorems and NLL evolution equations. Instead, following Refs [2,6], we

extend the parton model description of Eq. (52), modifying appropriately the kernel Φ and convoluting it with the Born cross-section. At this point we will make use of the exact calculations for the total cross-section of Ref. [6] and combine them with all the previously obtained results for the LL structure functions.

The exact treatment of the higher order soft nonleading corrections requires to use the YFS scheme of QED calculations (Sect. 3.1.3.), *i.e.* YFS exponentiation. In practice, however, one prefers to "improve" by hand the LL soft $\Phi_{\rm LL}^{\rm Gribov}$ result of Eqs (30) and (54) to ensure the correct soft behaviour. The NLL "improvement" is realized as follows. Comparing Eqs (30) and (34)-(36) we easily implement the necessary refinements (the ϵ is identified with $1-x$ for $x \sim 1$): the leading logarithmic $\beta = 2(\alpha/\pi)\ln(s/m_e^2)$ should be replaced by $\beta_S = 2(\alpha/\pi)[\ln(s/m_e^2) - 1]$ and the multiplicative form-factor $\exp\left((\alpha/\pi)(\pi^2/3 - 1/2)\right)$ should be added. The first replacement solely ensures the proper IR behaviour. It is a rigorous result of the YFS analysis. The second, multiplicative factor of Eq. (36) is IR regular and reflects conventions of YFS exponentiation. The remaining perturbative corrections to soft photon limit are hidden in the $\bar{\beta}_0$ term of Eq. (34). The actual criterion in establishing these corrections are the LL limit, *i.e.* $\Phi_{\rm LL}^{\rm Gribov}$ function and the first order iterative solution that must be properly reproduced. Finally, we get the following extension of Eq. (30)

$$\Phi_{\rm LL}^{\rm Gribov} \to \Phi^{\rm soft}(x) = \frac{\exp\left[\beta_S(\frac{3}{4} - C_{\rm Euler}) + \frac{\alpha}{\pi}(\frac{\pi^2}{3} - \frac{1}{2})\right]}{\Gamma(1+\beta_S)} \beta_S (1-x)^{\beta_S - 1}. \tag{57}$$

It has manifestly correct YFS and Gribov soft limits. In the following NLL *ad hoc* exponentiation it will be interpolated with the hard limit, as it was done with $D^{\rm Gribov}$ formula in the LL framework.

The soft nonleading corrections are numerically very important. The "-1" term, assuring the proper IR behaviour, contributes 2% correction to the total cross-section. The multiplicative term $\exp\left((\alpha/\pi)[(\pi^2/3) - (1/2)]\right)$ contributes another 0.7%.

In the next step we add remaining NLL corrections. It can be done according to any of the before discussed exponentiation scheme, as shown in Eqs (32), (37) or (39). The noninfrared terms are of both real and virtual type. In the KF prescription the modifications are of the form

$$\Phi_{\rm KF}^{\rm NS}(x) = \Phi^{\rm soft} \bar{\Delta}_{\rm virt}^{\rm KF} + \bar{\Delta}_{\rm real}^{\rm KF}, \tag{58}$$

$$\bar{\Delta}_{\rm virt}^{\rm KF} = 1 + \frac{\alpha}{\pi}\beta_S \left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2}\zeta(3) + \mathcal{O}(\alpha^2 L^0) \right),$$

$$\bar{\Delta}^{\mathrm{KF}}_{\mathrm{real}}$$
$$= -\beta_S \frac{(1+x)}{2} + \frac{1}{8}\beta_S^2\left[-4(1+x)\ln(1-x) - \frac{1+3x^2}{1-x}\ln x - 5 - x\right]$$
$$+ \frac{1}{8}\frac{\alpha}{\pi}\beta_S \frac{1}{1-x}\left[-(1+3x^2)\ln^2 x + 2(3+2x+x^2)\ln x\right.$$
$$+ 4(1+x^2)\Big(\mathrm{Li}_2(1-x) + \ln x \ln(1-x)\Big) + 2(1-x)(3-2x)$$
$$\left. - 4(1-x^2)\Big(\frac{\pi^2}{3} - \frac{1}{2}\Big) + \mathcal{O}(\alpha^2 L^0)\right]. \tag{59}$$

The nonlogarithmic $\mathcal{O}(\alpha^2 L^0)$ terms, as numerically negligible (see below) are already discarded in Eq. (59). The reader interested in the full formula we refer to Eq. (3.29) of Ref. [6]. The JW prescription looks as follows

$$\bar{\Phi}^{\mathrm{NS}}_{\mathrm{JW}}(x) = \bar{\Phi}^{\mathrm{soft}}(\bar{\Delta}^{\mathrm{JW}}_{\mathrm{virt}} + \bar{\Delta}^{\mathrm{JW}}_{\mathrm{real}}) = \bar{\Phi}^{\mathrm{soft}}\bar{\Delta}^{\mathrm{JW}}_{(2)}, \tag{60}$$

$$\bar{\Delta}^{\mathrm{JW}}_{(2)} = \frac{(1+x^2)}{2} + \frac{\alpha}{\pi}\beta_S\Big(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2}\zeta(3)\Big) + \frac{\beta_S}{4}\left[-\frac{(1+3x^2)}{2}\ln x\right.$$
$$\left. - (1-x)^2\right] + \frac{1}{8}\frac{\alpha}{\pi}\Big[4(1+x^2)\Big(\mathrm{Li}_2(1-x) + \ln x \ln(1-x)\Big)$$
$$- (1+3x^2)\ln^2 x + 2(3+2x+x^2)\ln x + 2(1-x)(3-2x)\Big]. \tag{61}$$

In Figs 9 and 10 we compare the size of the second order NLL corrections to total cross-section. The ratios of cross-sections resulting from Eqs (58) and (60) with the noninfrared NLL corrections in $\bar{\Delta}$ functions, switched on and off are presented (the nonleading nonlogarithmic terms are discarded in both cases). Note that we do not touch the NLL terms in $\bar{\Phi}^{\mathrm{soft}}$. It is so, since the $\bar{\Phi}^{\mathrm{soft}}$ is not an *ad hoc* guesswork but rather a result of calculations. Two cuts $x_{\mathrm{cut}} = 0.9$ and $x_{\mathrm{cut}} = 0.6$ are shown.

The striking feature of Figs 9 and 10 is the smallness of NLL $\mathcal{O}(\alpha^2 L)$ corrections. At the resonance region they do not exceed the 0.03% (JW) and 0.05% (KF) level, respectively. It is already below the required accuracy and, at least, allows us to safely neglect the nonleading contributions. It is important *e.g.* in the case of M.C. programs, where their inclusion might complicate and slow down the algorithms. Naive counting rules would suggest that the NLL terms are of the order $\alpha \sim 0.7\%$. The actual, ten times smaller, size is due to the exponentiation that has already resummed some NLL contributions, all the first order NLL corrections in particular.

Another interesting information comes from the comparison of Figs 9 and 10 with Figs 7 and 8. One can see, that in the most popular KF scheme

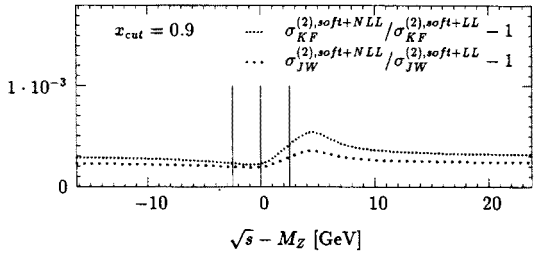

Fig. 9. The ratio of the exact second order total cross-sections with the NLL terms in $\bar{\Delta}$ functions of Eqs (58) and (60) switched on and off. The cross-sections are the $\mathcal{O}(\alpha^2)$ exponentiated ones of the KF type, based on Eq. (58) (dots) and JW type, based on Eq. (60) (filled circles). The nonleading (nonlogarithmic) terms are discarded. The strong cut-off $x_{\rm cut} = 0.9$ in the integration of (52) is used.

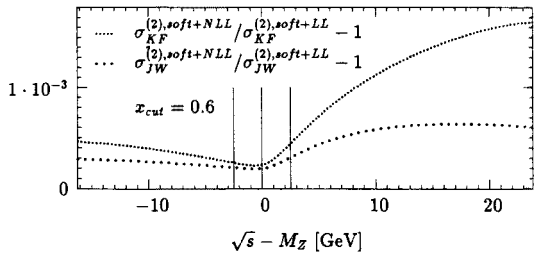

Fig. 10. The same as in Figure 9 for the looser cut-off $x_{\rm cut} = 0.6$.

the second order NLL corrections are significantly smaller than the $\mathcal{O}(\beta^3)$ terms. This has been overlooked in the literature, see for instance Refs [5,6]. The main emphasis was put on the full second order calculations of $\mathcal{O}(\alpha^2 L^1)$ and $\mathcal{O}(\alpha^2 L^0)$ corrections. The formulas were built according to the powers of coupling constant α rather than to actual accuracy.

We propose a *new* arrangement of the total cross-section formula, see also Ref. [8]. There are two scales involved, α and $L \equiv \ln(s/m_e^2)$. Instead of calculating order by order in α we try to pick up all the terms of similar size, determined in fact by both scales. We show it schematically in the Table I.

TABLE I

1			
αL	α		
α^2L^2	α^2L	α^2	
α^3L^3	α^3L^2	α^3L	α^3
...	...	...	...

Rows symbolize subsequent perturbative orders in α. It is the traditional arrangement which is used in the renormalization technique. It is quite useful in practical calculations in the absence of the big logarithms, *i.e.* for L$\sim$ 1. Every result in $\mathcal{O}(\alpha^n)$ can be divided into $\mathcal{O}(\alpha^n L^n)$, $\mathcal{O}(\alpha^n L^{n-1})$, ..., $\mathcal{O}(\alpha^n L^0)$ terms and the corresponding approximations are called LL, NLL and so on. For electron and $s \sim M_Z^2$, terms of equal magnitude are roughly on the diagonal. This means that to reach a given accuracy level we may keep only $\mathcal{O}(\alpha^n L^n)$, *i.e.* LL contribution of the highest order, $\mathcal{O}(\alpha^{n-1} L^{n-2})$, *i.e.* NLL one of lower by one order and so on. We call this selection a "pragmatic" $\mathcal{O}(\alpha^n)$ expansion.

The *third order pragmatic expansion* which we shall use includes, in descending magnitude order, $\mathcal{O}(\alpha^0 L^0)$, *i.e.* Born, $\mathcal{O}(\alpha^1 L^1)$, $\mathcal{O}(\alpha^2 L^2)$, $\mathcal{O}(\alpha^1 L^0)$, $\mathcal{O}(\alpha^3 L^3)$ and $\mathcal{O}(\alpha^2 L^1)$ terms, as depicted in Table I. The dominant omitted terms are $\mathcal{O}(\alpha^4 L^4)$, $\mathcal{O}(\alpha^3 L^2)$ and $\mathcal{O}(\alpha^2 L^0)$.

With the help of our third order LL results we present here the third order "pragmatic" formula for the total cross-section [8], see Table I,

$$\Phi_{(3)}^{\text{"pragm"}}(x) = \Phi^{\text{soft}}(x)\bar{\Delta}_{(3)}^{\text{"pragm"}}, \tag{62}$$

$$\begin{aligned}
\Delta_{(3)}^{\text{"pragm"}} &= \frac{1}{2}(1+x^2) + \frac{\alpha}{\pi}\beta_S\left(\frac{3}{32} - \frac{\pi^2}{8} + \frac{3}{2}\zeta(3)\right) \\
&+ \frac{\beta_S}{4}\left(-\frac{1}{2}(1+3x^2)\ln x - (1-x)^2\right) + \frac{1}{8}\frac{\alpha}{\pi}\Big(-(1+3x^2)\ln^2 x \\
&+ 4(1+x^2)\big(\mathrm{Li}_2(1-x) + \ln x \ln(1-x)\big) + 2(1-x)(3-2x) \\
&+ 2(3+2x+x^2)\ln x\Big) + \frac{1}{8}\beta_S^2\left(\frac{1}{2}(3x^2-4x+1)\ln x\right. \\
&+ \left.\frac{1}{12}(1+7x^2)\ln^2 x + (1-x^2)\mathrm{Li}_2(1-x) + (1-x)^2\right),
\end{aligned} \tag{63}$$

where Φ^{soft} is defined in Eq. (57).

The formula includes the initial state photonic corrections. Its accuracy we estimate in the following way. The size of the $\mathcal{O}(\alpha^4 L^4)$ term we estimate conservatively to be no more than 0.5 times the third order correction. It gives the numbers 0.001% for $|\sqrt{s} - M_Z| \leq 2.5$GeV and 0.01% for $|\sqrt{s} - M_Z| \leq 7.5$GeV, in the wide range of cuts $1 > x_{\text{cut}} > 0.1$. In analogy the third order NLL contribution we estimate by 0.5 of the second order one. It reads 0.015% for $|\sqrt{s} - M_Z| \leq 2.5$GeV and 0.025% for $|\sqrt{s} - M_Z| \leq 7.5$GeV. Adding absolute values we get the following conservative accuracy of pragmatic formula (62): $\delta\sigma/\sigma \leq 0.015\%$ for $|\sqrt{s} - M_Z| \leq 2.5$GeV and $\leq 0.035\%$ for $|\sqrt{s} - M_Z| \leq 7.5$GeV. To our knowledge, it is the up to date

the best available formula for the total-cross section at LEP[9]. Compared to any formulas presented in the review [6] the above one is more accurate by factor ten and shorter at least by factor three!

5. Conclusions and final remarks

In the presented paper we have discussed the method of structure functions in QED. The work was oriented on the practical calculation of total cross-section at LEP (Z line shape). We described a series of *new* solutions of the QED evolution equations for the electron LL structure function:

- The exact Monte Carlo solution, based on the semi-Poissonian distribution of soft photons (for non singlet function).
- The exact numerical inversion of the Mellin Transforms (for both non singlet and singlet functions).
- The complete, analytical $\mathcal{O}(\beta^3)$ corrections to the non singlet and singlet functions.
- The general form of the iterative series of singlet function.

Having at hand the above results we compared accuracy of various $\mathcal{O}(\beta^2)$ formulas for non singlet electron structure function available in literature. We were looking for the best exponentiation prescription. By comparison with the exact results we showed that the most accurate is the prescription due to Jadach and Ward.

In the second part we applied the structure functions formalism to the analysis of the QED corrections to total cross-section at LEP. With the help of exact and $\mathcal{O}(\beta^3)$ forms of electron structure functions we were able to answer the problem of the size of higher order corrections to commonly used formulas. We have repeated the comparison of various exponentiated formulas (at the level of cross-section) and found again that the ansatz of JW is the most accurate one. The actual size of $\mathcal{O}(\alpha^3)$ corrections we calculated to be: $\Delta\sigma^{(3)}/\sigma \leq 0.12\%$ for the KF formula and $\leq 0.01\%$ for the JW formula, in the range $|\sqrt{s} - M_Z| \leq 2.5\text{GeV}$. We have also checked the influence of exponentiation on the subleading corrections. Our comparison showed that already the next-to-leading terms contribute $\sigma_{\text{NLL}}/\sigma_{\text{LL}} \leq 0.05\%$ for the KF prescription and $\sigma_{\text{NLL}}/\sigma_{\text{LL}} \leq 0.03\%$ for the JW one. It enables us to discard the nonlogarithmic (NNL) $\mathcal{O}(\alpha^2 L^0)$ terms completely.

As a practical consequence of this analysis we proposed a *new* compact formula for initial state QED photonic corrections to the total cross-section at LEP. We estimate its accuracy to be $\delta\sigma/\sigma \leq 0.015\%$ in the range

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[9] The $\mathcal{O}(\beta^3)$ pragmatic formula (62-63) has been recently extended to simultaneous inclusion of lepton pairs, featuring the same high precision, see Ref. [39].

$|\sqrt{s} - M_Z| \leq 2.5\text{GeV}$ and 0.035% for $|\sqrt{s} - M_Z| \leq 7.5\text{GeV}$. We called this formula the "pragmatic" $\mathcal{O}(\alpha^3)$ which contains all the terms contributing at the same accuracy level, *i.e.* $\mathcal{O}(\alpha^0 L^0)$, (Born), $\mathcal{O}(\alpha^1 L^1)$, $\mathcal{O}(\alpha^2 L^2)$, $\mathcal{O}(\alpha^1 L^0)$, $\mathcal{O}(\alpha^3 L^3)$ and $\mathcal{O}(\alpha^2 L^1)$ terms, as depicted in Table I.

Finally, we have discussed the cut-off dependence of the cross-section. We came to the conclusion that in the case of the exponentiated formulas, as expected, the stronger cut-off eliminate significantly better higher order corrections.

In the presented work we confined ourselves to the initial state corrections to total cross-section. We left over final state and initial-final QED contributions. We also have not discussed in detail the additional fermion pair production. In the following we would like to comment shortly on some of these points.

The first order final state and box+interference corrections to total cross-section can be found in Refs [41,42]. The final state radiation can be described in terms of the structure functions and parton model in a similar way as the initial state ones, compare *e.g.* Refs [11,13]. This approach was used *e.g.* in Ref. [19]. The qualitative estimation of the final state contribution is the following. If no cuts are used, *i.e.* the integration is done over the entire phase space, the KLN theorem [34] guarantees that all mass singularities ($\ln(s/\mu^2)$ in our case) have to disappear in total cross-section. It means that the entire final state correction is $\sim \alpha$. Introduction of any cut-off x_{cut} will lead, however, to the terms $\sim \ln x_{\text{cut}}$ spoiling the above argument.

The last contribution to be discussed is the additional lepton pair production. At the LL level, confined to the initial state, this process is already built in the presented structure functions (parton model) formalism. The practical prescription for their inclusion is the following:

1. Replace $\alpha = 1/137$ with the appropriate running coupling constant of Eq. (10).
2. Add the singlet structure function contribution.
3. Calculate the structure function $D^{\gamma}_{e(\bar{e})}$ and include it into the sum of Eq. (51), combined with the appropriate hard cross-section $\sigma_{\gamma\gamma \to e\bar{e}}$, see Ref. [22].

The bremsstrahlung-like process (1), *i.e.* the contribution to the D^{NS} function from the running coupling constant is the dominant one.

The difficulties with the pair production appear however, if one wants to go beyond the LL approximation. The exact cross-section is known only in the lowest $\mathcal{O}(\alpha^2)$ order, see Refs [5,43]. From the $\mathcal{O}(\alpha^2)$ result one can learn that the nonleading contributions (correct soft behaviour) reduce the LL result even to 50%. On the other hand the higher order terms (multiple soft photons) can reduce the $\mathcal{O}(\alpha^2)$ result to 50% as well, see Ref. [39].

Some solution to this difficulty is an ansatz proposed by Kuraev and Fadin in Ref. [2]. It consists of the $\mathcal{O}(\alpha^2)$ cross-section appropriately matched with the soft photons contribution. The more systematic analysis of pair contribution to total cross-section, based on a certain extension of YFS framework has been recently done in Ref. [39].

The other interesting extension of the presented work would be to find the exact solution of the NLL evolution equation. We have in mind the Monte Carlo algorithm for non singlet function. The basis of the algorithm, the quasi-Poissonian distribution of soft photons, may remain unchanged in NLL calculations. We hope therefore, that the NLL corrections could be smoothly included by the standard Monte Carlo techniques. Such an exact NLL result would be very important from the practical point of view.

The author would like to thank dr. S. Jadach for the supervision and constant help in this research, prof. J. Kalinowski and dr. M. Jeżabek for critical reading of the manuscript, dr. Z. Wąs for instructive discussions and prof. J.J. Aubert for his hospitality in CPPM-Marseille, where parts of this work were done.

Finally, I want to thank my wife for pains she took while this work was in progress.

Note added in proof

After this work was completed the paper by M. Jeżabek [44] has been published. The author of Ref. [44], making use of the ansatz (37), derives general perturbative recurrence formula for $\Delta^{\mathrm{JW}}(x,\beta)$, being the exact solution to Eq. (12). Also the terms singular at $x = 0$ are resummed to the Bessel function and it is shown that factorizing of this function further simplifies the series $\Delta^{\mathrm{JW}}(x,\beta)$.

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