QUANTUM STOCHASTIC DIFFERENTIAL EQUATIONS* **

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The physical origin and the main ideas of the theory of quantum stochastic differential equations are outlined. The related limit theorems are briefly discussed.

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In classical statistical physics the influence of an environment on the time evolution of a selected dynamical variable X_t is described in terms of the Langevine equation:

$$\dot{X}_t = F(X_t) + G(X_t)\,\chi_t\,, \tag{1}$$

where χ_t is a stochastic process describing noise. Typically $\{\chi_t, t \in [0,\infty)\}$ is assumed to be white noise *i.e.* "Gaussian stochastic process with δ-like autocorrelation function". Obviously such noise is a singular object and one needs a proper mathematical formulation of (1). It can be done using Itô formalism [1] which involves Itô integral

$$J = \int_{t_1}^{t_2} f_t dB_t \tag{2}$$

with respect to the normalized Brownian motion B_t ($\mathrm{E}(B_t) = 0$, $\mathrm{E}(B_t^2) = t$, $\mathrm{E}(\cdot)$ denotes expectation value). The random variable J in (2) is well defined in the case of adapted (nonanticipating) stochastic process f_t. Then

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the heuristic equation (1) with a white noise χ_t may be replaced by the Itô stochastic differential equation:

$$dX_t = \{F(X_t) + \tfrac{1}{2}G(X_t)\nabla_x G(X_t)\}dt + G(X_t)dB_t, \tag{3}$$

where in fact Eq. (3) is a short-hand notation for the associated integral equation. The term $^1/_2 G\nabla_x G$ in (3) is sometimes called Wong–Zakai correction [1] and may be derived from Eq. (1) with a regular noise using a limiting procedure. Starting with an equation of the type (3) one can find stochastic differentials for functions of X_t using the formal rule given by the famous Itô formula

$$dB_t^2 = dt. \tag{4}$$

Now we shall try to generalize the notion of stochastic differential equation into the quantum domain. First we consider the case of random time-dependent Hamiltonian $H(t)$ leading to a random Schrödinger equation

$$\frac{d\Psi(t)}{dt} = -iH(t)\Psi(t), \qquad H(t) = H^*(t), \tag{5}$$

or in an equivalent form

$$\frac{dU(t)}{dt} = -iH(t)U(t), \tag{6}$$

where $U(t)$ is a unitary operator. Again one can study a white noise perturbation

$$H(t) = H_0 + \chi_t V, \quad H_0 = H_0^*, \quad V = V^*. \tag{7}$$

The proper form of (6) in this case is given by the following operator valued "Itô–Schrödinger equation"

$$dU(t) = \{(-iH_0 - \tfrac{1}{2}V^2)dt - iVdB_t\}U(t). \tag{8}$$

The equivalent description is given by the reduced dynamics Λ_t, $t \geq 0$ on density matrices defined as

$$\rho_t = \Lambda_t\rho = \mathrm{E}\{U(t)\rho U^*(t)\}. \tag{9}$$

Using Eq. (8) and the Itô formula (4) one may easily check that Λ_t is a quantum dynamical semigroup [2] satisfying the Markovian master equation:

$$\frac{d\rho_t}{dt} = -i[H_0, \rho_t] - \tfrac{1}{2}[V,[V,\rho_t]]. \tag{10}$$

The class of master equations obtained from (9) is rather restricted. For example, if the entropy of ρ is given by

$$S(\rho) = -\mathrm{Tr}\rho \ln \rho, \tag{11}$$

then for Λ_t defined by equation (9) with any random $U(t)$ we have [2]:

$$S(\Lambda_t \rho) \geq S(\rho). \tag{12}$$

Hence the model with stochastic Hamiltonians of the type (7) cannot describe the open systems in contact with a heat bath at finite temperatures.

It is known that the most general (completely positive, trace preserving) master equation can be written as ([2] and references therein):

$$\frac{d\rho_t}{dt} = -i[H,\rho] + \frac{1}{2}\sum_{\alpha}\{[W_\alpha, \rho_t W_\alpha^*] + [W_\alpha \rho_t, W_\alpha^*]\} \tag{13}$$

with $H = H^*$, W_α operators acting on the Hilbert space $\mathcal{H}_0$. Quite formally one may derive master equation (13) (with a single $W_\alpha \equiv W$ for simplicity) using the following generalization of (8):

$$dU(t) = \{(-iH - \tfrac{1}{2}W^*W)dt + i(W^*dA_t + WdA_t^*)\}U(t), \tag{14}$$

if one assumes the following Itô rules for the "noises" A_t, A_t^*

$$\begin{aligned} dA_t^2 = dA_t^{*2} = dA_t^* dA_t = 0, \\ dA_t dA_t^* = dt. \end{aligned} \tag{15}$$

We see that this new type of noise is a noncomutative one (quantum noise) and therefore should be realized in terms of operators on certain Hilbert space. The presented below noncomutative generalization of the stochastic differential calculus was proposed by Hudson and Parthasarathy [3] and developed by many authors (see [4, 5] for reviews).

Let $\mathcal{F}(L^2(\boldsymbol{R}_+))$ be a Bose Fock space over a "single particle" Hilbert space $L^2(\boldsymbol{R}_+)$, ($\boldsymbol{R}_+ = [0,\infty)$). For any pair $f, g \in L^2(\boldsymbol{R}_+)$ we define annihilation and creation operators $a(f), a^*(g)$ satisfying CCR:

$$\begin{aligned} [a(f), a^*(g)] = \langle f, g\rangle, \\ [a(f), a(g)] = [a^*(f), a^*(g)] = 0. \end{aligned} \tag{16}$$

The family of operators $\{A_t, A_t^*, t \in \boldsymbol{R}_+\}$ defined as:

$$\begin{aligned} A_t \equiv a(\theta_t), \\ A_t^* \equiv a^*(\theta_t), \end{aligned} \tag{17}$$

($\theta_t(x) = 1$ if $0 \leq x \leq t$, and zero otherwise) is called *quantum Brownian motion* (or quantum Winer process). One can check using (16) that formally dA_t, dA_t^* satisfy quantum Itô formula (15). The quantum Brownian motion is a noncomutative generalization of the classical Wiener process in the sense that a family of random variables $\{B_t, t \geq 0\}$ is replaced by the family of operators $\{A_t, A_t^*, t \geq 0\}$ and the classical expectation $\mathbf{E}(\cdot)$ is replaced by the quantum one $\mathbf{E}_\Omega(\cdot)$ such that:

$$\mathbf{E}_\Omega(A_{t_1}^\sharp A_{t_2}^\sharp \cdots A_{t_n}^\sharp) = \langle \Omega, A_{t_1}^\sharp A_{t_2}^\sharp \cdots A_{t_n}^\sharp \Omega \rangle, \tag{18}$$

where $A_t^\sharp$ denotes A_t or A_t^* and Ω is the vacuum state in $\mathcal{F}(L^2(\boldsymbol{R}_+))$. Moreover one can easy check that the commuting family of operators $\widetilde{B}_t = A_t + A_t^*$ yields the quantum correlation functions at the state Ω which are equal to the correlation functions for the Brownian motion and hence $\widetilde{B}_t$ can be identified with B_t .

In order to define Itô integrals with respect to A_t, A_t^* one needs the notion of adapted quantum process. For $t \in [t, \infty)$ we have $L^2(\boldsymbol{R}_+) = L^2([0,t)) \oplus L^2([t,\infty))$ and hence

$$\mathcal{F}(L^2(\boldsymbol{R}_+)) = \mathcal{F}(L^2([0,t))) \otimes \mathcal{F}(L^2([t,\infty))).$$

The family $\{X_t\}$ of operators on $\mathcal{F}(L^2(\boldsymbol{R}_+))$ is a *quantum adapted (nonanticipative) process* if $X_t = \widetilde{X}_t \otimes \mathbf{1}_{[t,\infty)}$ where $\widetilde{X}_t$ acts on $\mathcal{F}(L^2([0,t)))$ and $\mathbf{1}_{[t,\infty)}$ is an identity operator on $\mathcal{F}(L^2([t,\infty)))$. For such quantum processes one may define the quantum Itô integral:

$$I^\sharp = \int_{t_1}^{t_2} X_t dA_t^\sharp, \tag{19}$$

which exists as an operator on $\mathcal{F}(L^2(\boldsymbol{R}_+))$ defined by its action on coherent states (exponential vectors):

$$\phi(f) = \Omega \oplus \frac{f}{\sqrt{1!}} \oplus \frac{f \otimes f}{\sqrt{2!}} \oplus \cdots. \tag{20}$$

Having the notion of Itô integral one can treat (14) as a short hand notation of the associated integral equation. The later may be solved in terms of Dyson series involving quantum stochastic integrals. The solution $U(t)$ is a family of unitary maps on $\mathcal{H}_0 \otimes \mathcal{F}(L^2(\boldsymbol{R}_+))$. Taking several independent (commuting) quantum Brownian noises $(A_t^\alpha, A_t^{\alpha *})$ one can generalize equation (14) and obtain the unitary evolution $U(t)$ such that:

$$\mathbf{E}_\Omega(U(t)\rho U^*(t)) = e^{tL}\rho, \tag{21}$$

with L given by the right-hand side of the equation (13).

The quantum stochastic differential equation (QSDE) describe open quantum systems driven by quantum white noise. It is well known that generally white noise cannot be realized in real physical systems and may be obtained by means of certain limit procedures. Therefore the natural question arises: How to derive QSDE from the realistic quantum dynamics of an open system coupled to a reservoir? Such program has been investigated in the pioneering work by Accardi, Frigerio and Lu [6] and is still realized (see for example Refs [7–9]).

We shall describe very briefly the main ideas of this approach in the case of the weak coupling limit [6]. Let $\mathcal{H}_{\mathrm{S}}$, $\mathcal{H}_{\mathrm{R}}$ denote the Hilbert spaces of the system S and the reservoir R respectively. The total Hamiltonian involving a coupling constant λ is given by:

$$H_\lambda = H_{\mathrm{S}} + H_{\mathrm{R}} + \lambda V \tag{22}$$

and the evolution in the interaction picture is governed by the family of unitary operators on $\mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{R}}$:

$$U^\lambda(t) = \exp\{i(H_{\mathrm{S}} + H_{\mathrm{R}})t\} \exp\{-iH_\lambda t\}. \tag{23}$$

The limit theorems state the convergence $U(t/\lambda^2) \to U(t)$ where $U(t)$ is a solution of QSDE of the type (14) (notice the rescaled time t/λ^2 — van Hove limit). Obviously $U^\lambda(t)$ and $U(t)$ act on different Hilbert spaces and the convergence should be understand in a weak sense. In the simplest case one proves the convergence of the matrix elements:

$$\langle u \otimes \psi_\lambda,\ U^\lambda(t/\lambda^2) v \otimes \psi'_\lambda\rangle \xrightarrow{\lambda \to 0} \langle u \otimes \phi,\ U(t) v \otimes \phi\prime\rangle \tag{24}$$

for any $u, v \in \mathcal{H}_{\mathrm{S}}$ and $\{\phi\} \subset \bigotimes_\alpha \mathcal{F}(L^2(\mathbf{R}_+))$. Moreover for any λ ψ_λ is obtained from fixed $\psi \in \mathcal{H}_{\mathrm{R}}$ by a certain time averaging procedure and there is a map $\psi \mapsto \phi$. Typically one chooses as $\{\phi\}$ coherent vectors while the form of ψ depends strongly on the interaction V. The known limit theorems have been proved for the free Bose and Fermi reservoirs with linear and bilinear interactions. In the linear (Bose) case ψ may be taken as coherent states while for the bilinear interaction one can choose certain squeezed states. Besides the weak coupling limit the low density limit has been studied [7–9]. In this case the varying coupling parameter λ^2 is replaced by the density of the reservoir's gas. The obtained QSDE is more complicated, it involves scattering matrix and the new type of quantum noise (quantum Poisson process) being a linear combination of A_t, A_t^* and the so-called number process:

$$N_t = \int_0^t a^*(x)a(x)dx.$$

Here $\int_0^\infty f(x)a^\sharp(x)dx \equiv a^\sharp(f)$. One should stress that these new limit theorems are essential extensions of the usual Markovian limit procedures for the reduced dynamics of given systems. The later involve a fixed equilibrium state of the reservoir while the former provide a simplified description of the total dynamics of the open system plus reservoir with large class of initial conditions.

The formalism of QSDE found already applications in quantum optics and measurement theory (see for example Refs [10, 11]) and for sure will be a powerful mathematical tool in the quantum theory of irreversible processes.

REFERENCES

[1] Z. Schuss, *Theory and Application of Stochastic Differential Equations*, Wiley, New York 1980.

[2] R. Alicki, K. Lendi, *Quantum Dynamical Semigroups and Applications*, LNP 286 Springer, Berlin 1987.

[3] R.L. Hudson, K.R. Parthasarathy, *Commun. Math. Phys.* **93**, 301 (1984).

[4] K.R. Parthasarathy, *Rev. Math. Phys.* **1**, 89 (1989).

[5] L. Accardi, *Rev. Math. Phys.* **2**, 127 (1990).

[6] L. Accardi, A. Frigerio, Y.G. Lu, *Commun. Math. Phys.* **131**, 537 (1990).

[7] L. Accardi, Y.G. Lu, R. Alicki, A. Frigerio, in *Quantum Probability VI*, World Scientific, Singapure 1991.

[8] L. Accardi, Y.G. Lu, *J. Phys. A* **24**, 3483 (1991).

[9] S. Rudnicki, R. Alicki, S. Sadowski, *J. Math. Phys.* (1992) in print.

[10] C.W. Gardiner, M.J. Collet, *Phys. Rev.* **A31**, 3761 (1985).

[11] A. Barchelli, *J. Phys. A* **20**, 6341 (1987).