THE GENERALIZED THERMODYNAMIC FORMALISM APPLIED TO HYPERBOLIC AND NONHYPERBOLIC MODELS*

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In this contribution, scaling properties of hyperbolic and nonhyperbolic model systems are discussed by using the generalized thermodynamic formalism. The central quantity for the investigation is the generalized entropy function. With the help of this approach, insight into the possible occurrence of phase transitions in the various entropy-like scaling functions can be gained. It is shown how this effect is determined by the existence of a critical line in the surface described by the generalized entropy function.

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1. Introduction

In this work, the generalized thermodynamic formalism, so far demonstrated to provide an elegant and complete means for characterizing chaotic dynamical systems, is applied to a simple hyperbolic and a nonhyperbolic system. The latter systems are of special interest because generic systems are believed to be of nonhyperbolic nature. In the pioneering work "thermodynamic formalism" of Ruelle [1] the topic is to deduce the macroscopic behaviour from the microscopic knowledge of a dynamical system (of axiom A-type). As the most prominent one, the phenomenon of phase transitions should be described. In going from the microscopic to the macroscopic

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point of view, an averaging process over a certain ensemble (canonical, microcanonical, grandcanonical ensembles) is involved, and the relevant probability measure, the Gibbs measure, is specified.

In subsequent years, this formalism has been applied to the characterization of experimental systems and artificial models focusing on two different aspects. From one point of view, the probability distribution in the phase space is described with the help of the family of Renyi dimensions [2–4], or their Legendre transform $f(\alpha)$. The other lends itself to the study of the distribution of exponential stretching rates in the tangent bundle along the flow of the system, which leads to a closely related formalism [5–9]. In spite of the similarity between the two approaches, only little use has been made of the possibility to unify the two approaches [9–16], and applications of the latter concept to real systems or models have been scarce so far. However, an understanding of this approach is advantageous for the discussion of important effects a dynamical system can undergo, such as, *e.g.*, phase transitions, which cannot only be detected for the probabilistic, but also for the temporal description of experimental systems [17, 18].

2. The generalized thermodynamic formalism

As a starting point, let us make the observation that, to incorporate both points of view, a generating partition suitable for symbolic dynamics should be chosen. Then the partition function [4] for a system can be written as

$$Z_G(q,\beta,n) = \sum_{j\in(1,\dots,M)^n} l_j^\beta p_j^q\,, \tag{1}$$

where the sum extends over all non-forbidden sequences of length n which can be obtained by using the M symbols needed for the symbolic description. Here, the size of the j-th region R_j of the partition is denoted by l_j, whereas the probability of falling into this region is denoted by p_j ($p_j = \int_{R_j} \rho(x)dx$, where $\rho(x)$ denotes the natural measure). Local scaling of ℓ and p in n (where n denotes the "level" of the partition) is expected. In this way, the length scale ℓ and the probability p give rise to scaling exponents ϵ and α through

$$\ell_j = e^{-n\epsilon_j}\,, \tag{2}$$

$$p_j = \ell_j^{\alpha_j}\,. \tag{3}$$

Let us underline that the above partition takes into account both the length scales and the probabilities in an independent way. This is necessary for a refined description of the scaling behaviour, since no general dependence between the two aspects is given. Owing to this fact, in contrast to partition functions involving only length or only probability scales, the treatment leading to the associated averages is termed the "generalized" or "bivariate" thermodynamic formalism. From the partition function the generalized free energy F_G can be derived [9, 10]:

$$F_G(q,\beta) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{j\in(1,\ldots,M)^n} e^{-n\epsilon_j(\alpha_j q+\beta)}, \qquad (4)$$

where log denotes a natural logarithm. Note that for $G(\beta, q) = -\frac{1}{\beta}F_G(q,\beta)$, the term "Gibbs potential" is used [11].

From the generalized free energy, the generalized entropy function can be derived as [10, 17]

$$S_G(\alpha, \epsilon) = F_G(q, \beta) + (\langle\alpha\rangle q + \beta)\langle\epsilon\rangle. \qquad (5)$$

The angular brackets indicate those values of α and ϵ which yield the most dominant contribution to Z_G (for the given values of q and β). The free energy F_G or the generalized entropy S_G describe in this way the scaling behaviour of the dynamical system in an equivalent way.

3. Hyperbolic and nonhyperbolic examples

As a first example for the application of the thermodynamic formalism, let us consider a hyperbolic model with a symbolic description by three symbols, with unrestricted grammar (for the effect of incomplete grammars on the generalized entropy see [14]). In Fig. 1 we show the associated generalized entropy function, where the contour lines indicate the values of the function.

Furthermore, the lines are pointed out along which different, more specific entropy functions are evaluated: $S_G(\epsilon) = S_G(\alpha,\epsilon)|_{q=0}$, $f(\alpha) = \frac{S_G(\alpha,\epsilon)}{\epsilon}|_{F_G(q,\beta)=0}$, and $g(\Lambda)$ [8], the Legendre transform associated with the Renyi-entropies $S_G(\alpha,\epsilon)|_{\beta=0}$. As a consequence of the hyperbolicity of the system, all of these entropy-like scaling functions (not to be mistaken with the scaling functions of the Feigenbaum type) are strictly convex.

This situation changes drastically, if a nonhyperbolic system is considered. To illustrate this, we shall investigate a second model (closely related to a model which has been first introduced by Kovács and Tél [15]). The support of this model is generated by the elements of a partition generated from a hyperbolic map, whereas the measure attributed to the support is

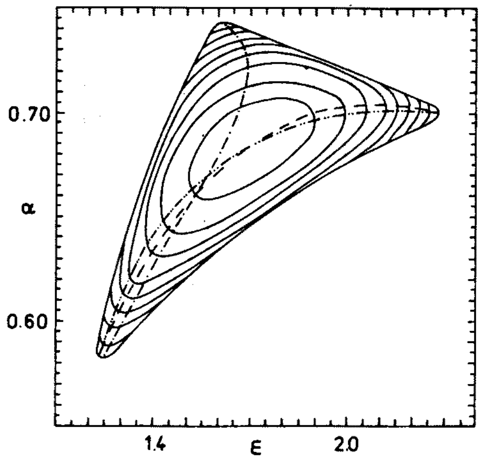

Fig. 1. Scaling behaviour of a hyperbolic three-scale Cantor set with restriction. The figure shows the $\alpha-\epsilon$—area on which the entropy $S_G(\alpha, \epsilon)$ is positive. The lines are shown along which the functions $S_G(\epsilon)$ (dashed-doubledotted), $g(\Lambda)$ (dashed) and $f(\alpha)$ (dashed-dotted) are evaluated. In the case of hyperbolic systems, these functions are analytic. To give more detailed information on the function S_G, contour lines are indicated, increasing in steps of 0.1.

given by the corresponding elements of a nonhyperbolic map. As a very simple example of a hyperbolic support map, the tent map

$$\begin{aligned} f\ : x &\to x/\ell_1\,, \text{ for } x \in [0, \ell_1/(\ell_1+\ell_2)]\,, \\ x &\to (1-x)/\ell_2\,, \text{ for } x \in [\ell_2/(\ell_1+\ell_2), 1]\,, \end{aligned} \tag{6}$$

is taken, where for the numerical treatment $\ell_1 = \frac{2}{9}$ and $\ell_2 = \frac{2}{7}$. The map of the measure, instead, is given by

$$g\ :\ x \to 4(1-x)x\,, \tag{7}$$

the fully developed logistic map. Note that, in this way, no restriction is imposed on the associated binary grammar and, since the measure map is nonhyperbolic, we deal with a nonhyperbolic system. It is easily seen that, in this way, a two-scale Cantor set with measure is obtained. The three-scale Cantor sets described above arise from a generalization of the latter model to three linear pieces. Moreover, also the behaviour of maps from the interval can be fitted into this description. While the behaviour of the support is given by the map itself, the measure map has to be determined from additional considerations involving specific properties of the map [15].

For this model, the generalized entropy function is displayed in Fig. 2, as obtained from a numerical approximation of level $n = 10$. As can be

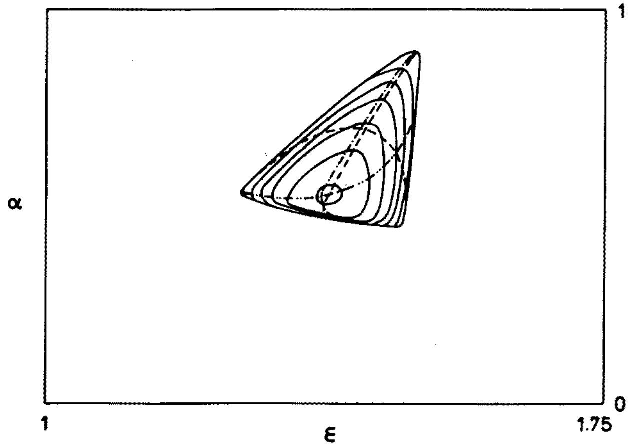

Fig. 2. Support of $S_G(\alpha, \epsilon)$ for the nonhyperbolic model, using an approximation of level $n = 10$. The lines are shown along which the functions $S_G(\epsilon)$, $g(\Lambda)$ and $f(\alpha)$ are evaluated (dashed-doubledotted, dashed, dashed-dotted, respectively). The contour lines obtained from (1) are indicated, increasing in steps of 0.1. The circular-shaped dashed line indicates the location of the critical line. From level $n = 10$, this property is not indicated sufficiently by the contour lines.

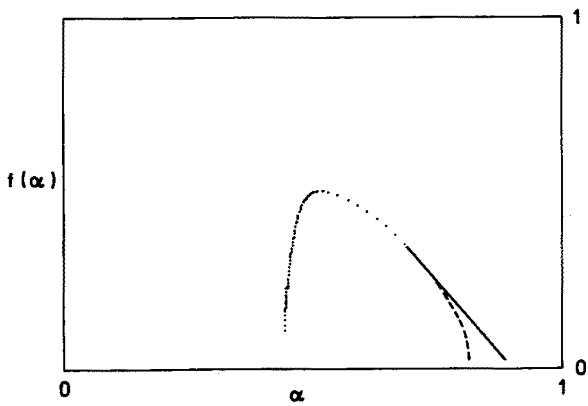

Fig. 3. $f(\alpha)$-spectrum for the nonhyperbolic model. The dashed branches indicate the hyperbolic contribution. The phase-transition point is situated at the intersection of the straight line with the curved lines.

demonstrated explicitly from an eigenvalue formalism involving a generalized Frobenius–Perron equation, a critical line emanates in the surface of the values provided by the generalized entropy function, which connects all points of nonanalytical behaviour (the circular-shaped dashed line in Fig. 2). Note that the results obtained in Ref. [15] also apply to the present model, where the phase transitions are shown to be generically of first order. Typically, the scaling functions are characterized by the form shown in Fig. 3 (for the $f(\alpha)$-spectrum), in many cases (but not necessarily) with a straight-line behaviour at one of the tails. From the approximation of a given level alone, it would be concluded that also the $S_G(\epsilon)$ spectrum undergoes a phase-transition-like behaviour. However, a more careful investigation of the asymptotic situation suggests a different insight, illustrated in Fig. 4. The critical line, in the asymptotic case $n \to \infty$, is situated exactly above the bottom border of the support of the generalized entropy function. Therefore, $S_G(\epsilon)$, being confined to this line in the asymptotic limit, shows no phase transition.

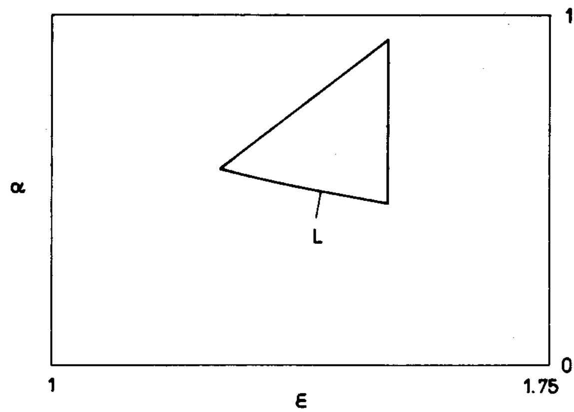

Fig. 4. Asymptotic form of the generalized entropy function $S_G(\alpha, \epsilon)$. The critical line has moved towars the bottom line. No contour lines are shown.

4. Conclusions

With the help of appropriate models, the theoretical tools have been outlined which permit one to predict and understand the different combinations of first-order phase transitions which appear for generic dynamical systems. Apart from this, using the generalized thermodynamic formalism, a more detailed and refined description of the system considered can be given. This fact, in addition, could be of valuable help for the development of realistic models for experimental systems.

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