ENTROPIC PROPERTIES OF QUANTUM DYNAMICAL SYSTEMS* **

W.A. MAJEWSKI

Institute of Theoretical Physics and Astrophysics, Gdańsk University
Wita Stwosza 57, 80-952 Gdańsk, Poland

(Received December 3, 1991)

A review of entropic properties of quantum dynamical systems is presented.

PACS numbers: 02.20.+b, 03.65.Fd,05.30.-d

The lecture notes presented here are an attempt to give an introduction to some concepts of quantum dynamical systems theory. As the subject of the theory is very vast I have had to restrict myself to a very limited subset. In these notes, I present only the latest developments in the study of entropic properties of infinite quantum dynamical systems. In particular, I shall concentrate upon relations between entropy increase, dynamical entropy, convergence to equilibrium and randomness.

In order to fix notations let me start with the following definition. Let (A, τ_t, ω) be a dynamical system where A is a C^* algebra, τ_t a one-parameter semigroup of completely positive unital maps over A, ω a τ_t - invariant state. Let me recall that A represents the set of obsrevables and τ_t a time evolution of physical system.

One of the most important concepts in the theory of classical dynamical systems is the so-called dynamical entropy of flows, Kolmogorov–Sinai (K-S) entropy [1-3]. This entropy gives a qualitative characterization of dynamical maps with respect to mixing. Moreover, it can be also considered as a quantitative measure how fast the system is mixed. Let me recall at this point that various mixing properties are important ingredient of attempts of physicists to understand the approach to equilibrium.

---

* Presented at the IV Symposium on Statistical Physics, Zakopane, Poland, September 19–29, 1991.

** This work has been partially supported by the grant 5-400-4-048-1.

Recently, a generalization of K-S entropy for quantum systems with a number of promising properties [2-8] was obtained. This definition of quantum dynamical entropy is very complicated and technical. Therefore, let me present here the basic points of simplified definition. Let A be a finite subalgebra of $\boldsymbol{A}$. Then

$$H_\omega(A) = \sup_{\sum \lambda_i \omega_i = \omega} \sum_i \lambda_i \, S(\omega|_A, \, \omega_i|_A),$$

where $S(\mu, \nu)$ is the relative entropy and the sup is taken over all decomposition of ω into states ω_i. Then, after the generalization of $H_\omega(A)$ to an arbitrary finite number of subalgebras, $H_\omega(A_1, \cdots, A_n)$, one can define

$$h_{\omega,\boldsymbol{A}}(\tau, A) = \lim_{n \to \infty} \frac{1}{n} H_\omega(A, \tau(A), \cdots, \tau^{n-1}(A)). \tag{1}$$

Then, the quantum dynamical entropy $h_{\omega,\boldsymbol{A}}(\tau)$ is defined as

$$h_{\omega,\boldsymbol{A}}(\tau) = \sup_{A \subset \boldsymbol{A} \quad A \,\text{finite}} h_{\omega,\boldsymbol{A}}(\tau, A). \tag{2}$$

Although this definition is rather complicated it was possible to prove promising properties of $h_{\omega,\boldsymbol{A}}$ and to compute this quantity for several models. In particular, quantum dynamical entropy was explicitly computed for:
(i) a quantum lattice model with the space translations as τ_t [2].
(ii) infinite fermi systems with a quasi-free dynamics [9].
(iii) an infinite fermi system with a "mixture" of translations as τ_t [10].
(iv) an infinite bose system where τ_t is the diffusion of an oscillator in the infinite heat bath [11].

In order to make these notes more clear let me present the last example in detail. Let $\boldsymbol{A}$ be the CCR algebra over one dimensional Hilbert space $\boldsymbol{C}$, *i.e.* $\boldsymbol{A}$ is generated by $\{W(z), z \in \boldsymbol{C}\}$ where

$$W^*(z) = W(-z), \quad W(z)W(z') = \exp\left(\frac{i}{2}\operatorname{Im} \bar{z}z'\right) \quad W(z+z')$$

Further, let Ψ_β denote the Gibbs state. The GNS representation associated with $(\boldsymbol{A}, \Psi_\beta)$ will be denoted by $(\Pi(\boldsymbol{A}), \mathcal{H}, \Omega_\beta)$. I shall consider a semigroup time evolution τ_t , which one can interpret as describing the diffusion of a quantum particle in a harmonic well [13, 14]

$$\tau_t : \quad W_\Pi(z) \longmapsto W_\Pi(\mathrm{e}^{-\lambda t} z) \quad \exp\Big(- \tfrac{1}{4} Q_\beta \, |z|^2 \big(- \exp(-2\lambda t)\big)\Big),$$

where λ , Q_β are positive constants, $t > 0$, $W_\Pi(\bullet) = \Pi \circ W(\bullet)$. τ_t has an extension to a completely positive map on $\mathcal{M} = \Pi(\boldsymbol{A})''$. I shall denote the extension by the same letter τ. Thus, the dynamical system $(\mathcal{M}, \tau_t, \Omega_\beta)$ has been defined. For this system $h_{\omega,A}(\tau) = 0$. Therefore this evolution can be considered as a regular one (in the sense of ergodic theory).

Now, let me pass to a brief description of the most important properties of quantum dynamical entropy $h_{\omega,A}(\tau)$. Again, as for classical dynamical systems, the basic property of $h_{\omega,A}(\tau)$ is its conjugacy invariance $h_{\omega,A}(\tau) = h_{\omega\circ\sigma,A}(\sigma^{-1} \circ \tau \circ \sigma)$ for an automorphism σ of A. Further, a very important property of $h_{\omega,\mathbf{A}}(\tau)$ is a quantum version of Kolmogorov–Sinai (K-S) theorem for the measure-theoretic entropy. As a corollary of the quantum version of K-S theorem, it was possible to prove the scaling property $h_{\omega,A}(\tau^k) = |k|\, h_{\omega,A}(\tau)$, k an arbitrary integer. I should add, that for an abelian C^* algebra the quantum dynamical entropy is equal to the original K-S entropy.

In examples *(iii)* and *(iv)* $h_{\omega,A}(\tau) = 0$. These results may surprise at first glance because τ in *(iii)* and *(iv)* are contractions. (The generalization of quantum dynamical entropy for completely positive maps was done in [12]). In other words, maps which improve the convergence to equilibrium diminish the quantum dynamical entropy. But this can be understood easily if one remembers that the contraction describing time evolution was obtained as a reduction of an automorphism (Hamiltonian evolution) of a larger system. Nevertheless, it should be pointed out that in example *(iv)* the quantum dynamical entropy of dilation of τ is equal to infinity! [15].

The entropic functional $H_\omega(A)$ satisfies the following inequality:

$$H_\omega(A_1, A_2) \le H_\omega(A_1) + H_\omega(A_2). \tag{3}$$

Moreover

$$H_{\omega_A \otimes \omega_B}(A) = S_{\omega_A}(A), \tag{4}$$

where $S_{\omega_A}(A)$ is the entropy of a normal state ω_A over algebra A.

In the classical case, (3) is also valid:

$$H_\mu^{\text{class}}(\xi_1 \cup \xi_2) \le H_\mu^{\text{class}}(\xi_1) + H_\mu^{\text{class}}(\xi_2), \tag{3a}$$

where μ is a measure on a phase space, ξ_i a partition and the equality holds for independent partitions only. Hence, H_μ^{class} can be considered as a measure of independence of partitions. For the quantum case, we want to keep this point of view, *i.e.* $H_\omega(A)$ is a measure of independence with the change of partition ξ for subalgebra A. The equality (4) implies that the difference between functionals S, H , *i.e.* $S - H$, measures the lack of product structure of a state. This remark was the starting point for serious study of the third law of thermodynamics [5]. Namely, it was possible to state the following quantum version of the third law: *the entropy density* *goes to zero when the temperature T approaches zero and equilibrium states cluster appropriately.* In other words, the third law is considered as a deeper property than a nondegeneracy of the ground state.

The theory of classical dynamical systems supplies us with K-systems which show how random behaviour can result from a deterministic time evolution [18,19]. Moreover, it seems to be a general agreement that classical K-systems exhibit those mixing and chaotic properties which are necessary for the foundation of statistical mechanics. Let me recall the definition [16]: a classical dynamical systems (M, τ_t, μ) is called a K-system if there exists a subalgebra $\mathcal{A}$ of the algebra $\mathcal{M}$ of all measurable sets satisfying

$$\begin{gathered} \mathcal{A} \subset \tau_t \mathcal{A} \quad \text{for any} \quad t \geq 0 \\ \bigwedge_{t=-\infty}^{\infty} \Phi_t \mathcal{A} = \widehat{O} \\ \bigvee_{t=-\infty}^{\infty} \Phi_t \mathcal{A} = \mathcal{M}, \end{gathered} \tag{5}$$

where $\widehat{O}$ is the algebra of sets of measure 0 or 1 , $\mathcal{A} \vee \mathcal{A}'$ denotes the algebra generated by $\mathcal{A}$ and $\mathcal{A}'$.

K-system can be defined by the following, different but equivalent, conditions:

(i) $\lim_{n\to\infty} h_\mu(\tau^n, \xi) = H_\mu(\xi)$

(ii) $h_\mu(\tau, \xi) > 0$ for each finite partition ξ

(iii) τ is K-mixing, *i.e.* for all finite $\{\mathcal{A}_i\}_0^n \subset \mathcal{M},\ n > 0$

$$\lim_{n\to\infty} \sup_{\mathcal{B}\in\bigvee_{i=1}^n \mathcal{A}_i} |\mu(\mathcal{A}_0 \cap \mathcal{B}) - \mu(\mathcal{A}_0)\,\mu(\mathcal{B})| = 0.$$

For the quantum case one can partially "translate" the above results. First of all, it is possible to introduce the quantum counterparts of K-systems [4,7]. Namely, an entropic K-system is a quantum dynamical system $(\boldsymbol{A}, \tau, \omega)$ such that for an arbitrary finite-dimensional subalgebra $\mathcal{A}$ one has

$$\lim_{n\to\infty} h_\omega(\mathcal{A}, \tau^n) = H_\omega(\mathcal{A}). \tag{6}$$

The idea standing behind this definition is that there is the full memory loss for every subalgebra.

We can also define an algebraic K-system as a quantum dynamical system $(\boldsymbol{A}, \tau, \omega)$ such that $\omega = \omega \circ \tau$,

$$\tau \boldsymbol{A}_0 \supset \boldsymbol{A}_0$$

$$\bigwedge_{n=0}^{-\infty} \tau^n A_0 = \lambda 1$$

$$\bigvee_{n=0}^{\infty} \tau^n A_0 = A \tag{7}$$

for any subalgebra A_0.

As we have seen, for classical dynamical systems these two notions are equivalent. On the other hand, for quantum systems the situation changes. Namely, there are some relations between algebraic K-systems and entropic K-systems, but still we do not know whether one class contains the other. In particular, one can prove [4] that

$$\lim_{n \to \infty} h_\omega(\mathcal{A}, \tau^n) = H_\omega(\mathcal{A}), \tag{8}$$

for each finite dimensional subalgebra $\mathcal{A} \subset A$ implies

$$h_\omega(\mathcal{A}, \tau) > 0. \tag{9}$$

It should be noted that (8) means that there is no gain of information after putting the dynamics into "the action". In other words such a system should be chaotic since it is impossible to gain any information from it. Thus, the above discussion about K-systems can be summarized by saying that the most important examples of deterministic chaotic classical systems can be partially translated into the quantum language. Partially, because there are open problems in this description,*e.g.* the question: *Does (9) imply (8)?* is still open. We close our discussion of entropic properties of quantum dynamical systems with the short description of relations between convergence to equilibrium and mixing-properties. One can prove [17] the following statements:

(i) The system is mixing, *i.e.*, $\lim_{t\to\infty} \omega(\tau_t(a)b) = \omega(a)\omega(b)$ if and only if all states restricted to suitable algebras converge to equilibrium.

(ii) The system is K-mixing if and only if all states restricted to suitable subalgebras converge strongly to equilibrium.

Let us remark that *(ii)* implies *(i)* but, in general, the reverse implication is false. So we can conclude our review of entropic properties of quantum dynamical system saying that the key tool of classical dynamical systems can be translated to the quantum theory.

The author thanks the Organizers of the IV Conference on Statistical Physics, Zakopane 1991, for the invitation.

REFERENCES

[1] A. Connes, *C.R. Acad. Sci. Paris* **301**, 1 (1985).
[2] A. Connes, H. Narnhofer, W. Thirring, *Commun. Math. Phys.* **112**, 691 (1987).
[3] A. Connes, E. Størmer, *Acta Math.* **134**, 288 (1987).
[4] H. Narnhofer, W. Thirring, *Commun. Math. Phys.* **125**, 565 (1989).
[5] H. Narnhofer, W. Thirring, *Lett. Math. Phys.* **15**, 261 (1988).
[6] H. Narnhofer, W. Thirring , *J. Stat. Phys.* **57**, 811 (1989).
[7] H. Narnhofer, W. Thirring, *Lett. Math. Phys.* **20**, 231 (1990).
[8] F. Benatti, H. Narnhofer, G.L. Sewell, *Lett. Math. Phys.* **21**, 157 (1991).
[9] H. Narnhofer, W. Thirring, *Lett. Math. Phys.* **14**, 89 (1987).
[10] W.A. Majewski, M. Kuna, *Lett. Math. Phys.* **20**, 337 (1990).
[11] W.A. Majewski, *On the entropic properties of the Ford-Kac-Mazur model*, preprint, Gdańsk 1990.
[12] F. Benatti, H. Narnhofer, *J. Stat. Phys.* **53**, 1273 (1988).
[13] G.W. Ford, M. Kac, P. Mazur, *J. Math. Phys.* **6**, 504 (1965).
[14] G.W. Emch, *Non-equilibrium Quantum Statistical Mechanics*, in *Current Problems in elementary particles and physics*, Springer Verlag, ed. P. Urban, 1976.
[15] F. Benatti, private communication.
[16] I.P. Cornfeld, S.V. Fomin, Ya G. Sinai, *Ergodic Theory*, Springer, Berlin, Heidelberg, New York, 1982.
[17] H.Narnhofer, A. Pflug, W.Thirring, *Scuola Norm. Superiore Pisa* p. 597 (1989).
[18] Ya B. Pesin, *Russian Math. Surveys* **32**, 55 (1977).
[19] V.I. Arnold, A.Avez, *Ergodic properties of Classical Mechanics*, Benjamin, New York, 1968.