A NOTE ON CANONICAL GRAVITY

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The canonical Hamiltonian form of the Einstein–Cartan theory with gauge group SO(3,1) is developed. The connection of the field variables with those of Arnowitt, Deser, and Misner and with Ashtekar's canonical variables is worked out.

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1. Introduction and motivation

The advent of Ashtekar's variables in 1987 ([2], [3]) has been a significant step in the development of the theory of gravitation. The most important work based on these variables is perhaps the development of a loop representation by Rovelli and Smolin who found a quantum theory of gravity using these variables ([14]); unfortunately, the theory's contact to reality is unclear since the observables of the theory are unknown yet.

Another recent step in understanding gravity was Witten's proof that quantum gravity in 2+1 dimensions is a solvable theory ([17]). He showed the equivalence of the Einstein–Cartan Lagrangian with a Chern–Simon term and solved the theory with the aid of knot theory. Since gravity in 2+1 dimensions does not propagate, the only solutions of the theory are flat spaces (therefore the diffeomorphism constraint has a simple form) and the only events are quantum tunneling between these spaces. In four dimensions things are more involved.

For both of these works it was essential that the authors did not work with the metric as a fundamental variable but with the soldering form (the "square root" of the metric) and a gauge connection, treating gravity as a gauge theory with an additional soldering structure. This idea was first proposed by Cartan ([5]), and because of its significance it is of interest to develop a canonical theory of gravity based on these variables.

Different views have been taken in literature concerning the gauge group of gravity. By the equivalence principle it should contain at least the Lorentz group SO(3,1), but larger group like SO(3,2), ISO(3,1) (the Poincaré group), or GL(4) will also do the job, allowing additional variables like torsion and nonmetricity ([4], [9], [10]). The SO(3,1) gauge connection and the soldering form can be considered to be different parts of a Cartan connection of the (Anti-) de Sitter group ([8]). In this note an SO(3,1) gauge theory with an independent soldering form will be used.

I have been motivated by the above mentioned developments to work out the canonical Hamiltonian framework for the Einstein–Cartan theory. As an additional advantage this work should be comparable to the Ashtekar formulation of gravity, since he used similar variables. Therefore, a further reason for this work was to try to understand Ashtekar's work from the point of view of canonical Einstein–Cartan theory. Indeed, I found that Ashtekar's variables can be viewed as components of the three-connection of the Einstein–Cartan theory and their canonical momenta.

In Section 2, a summary of the ADM formalism of gravity is presented to give the reader the possibility of comparing it with the Hamiltonian theory developed in the later sections. In 1962 Arnowitt, Deser, and Misner ([1]) worked out a Hamiltonian theory of gravity, using the metric as canonical variable. Splitting space-time into space and time, they found the canonically conjugated variable to be essentially the second fundamental form which describes the imbedding of space into space-time. This is well known and can also be found in textbooks, *e.g.* [12], so the summary is short. In Section 3, a Hamiltonian formalism is developed for the Einstein–Cartan theory starting from the Lagrange density.

$$\mathcal{L} = R_{\mu\nu}{}^{\alpha\beta}\theta^\mu_\alpha\theta^\nu_\beta\,, \tag{1}$$

θ^α_μ is the soldering form, soldering the tangent space of the space-time manifold to a four dimensional space M^4; and the curvature is expressed in terms of an independent SO(3,1) connection. The analogues of the ADM equations of motion are regained. In Section 4, the relation between the canonical Einstein–Cartan theory and Ashtekar's formulation is investigated. The latter will be recovered in the special case where the four dimensional space soldered to the tangential space is taken to be the space of complex 2×2 matrices(or End($\mathbb{C}^2$, $\bar{\mathbb{C}}^2$)) with basis 1, $\vec{\tau}$; $\vec{\tau}$ being the Pauli matrices. The spatial part of the soldering form θ turns out to be essentially Ashtekar's soldering form $\tilde{\sigma}_i^{AB}$, while Ashtekar's connection A_i^{AB} is a combination of some parts of the SO(3,1) connection.

Some remarks on nomenclature: Greek indices run from 0 to 3. Letters from the middle of the Greek alphabet (λ, μ, ν, ρ) are space-time indices (of tensors in TM), letters from the beginning of the alphabet

$(\alpha, \beta, \gamma, \delta)$ are indices in the four-dimensional space M^4, soldered to TM. To tell a space-time index 0 from an internal space index $\hat{0}$ the latter is dressed with a hat. Lower case Latin letters run from 1 to 3, they describe the space part of their Greek pendants, so i, j, k, l are indices on a spatial hyperplane of the space-time manifold, and a, b, c, d are indices in a three dimensional Euclidean space E^3 soldered to a spatial hyperplane of M^4 by the spatial part of the soldering form, ϑ.

Primed and unprimed capital Latin letters take the values 0 and 1, they are spinor indices of two dimensional complex spinors.

Left upper parenthesized indices (3) or (4) indicate whether the quantity they index is associated to a three- or four-dimensional manifold, *e.g.* ${}^{(3)}R_{ijkl}$.

Differentiation is indicated by a comma $\left(A_{,\mu} = \frac{\partial}{\partial\mu}A\right)$, covariant derivation with the Levi–Civita connection in four dimensions is indicated by a semicolon, $A_{;\mu}$, in three-dimensions with a dash, $A_{|\mu}$.

The Minkowski metric is taken to be $(-+++)$, the determinant of a matrix is indicated by parenthesis: *e.g.* $(h) = \det(h_{mn})$.

2. Review of the ADM formalism in empty space

In a canonical theory it is necessary to distinguish between space and time, that is, to choose a coordinate system in space-time to split it into space and time. In mathematical language this means to provide a foliation of the space-time manifold M into space-like hypersurfaces Σ_t which are all diffeomorphic to a reference three-manifold Σ. Since all possible foliations of space-time are equally admitted (Wheeler called this *many-fingered time*) the theory remains covariant. The metric $g_{\mu\nu}$ in M induces a metric h_{mn} (the first fundamental form) on Σ (and on each Σ_t). The imbedding of Σ in M is described by the second fundamental form $K_i{}^j$, defined by

$$-{}^{(4)}\nabla_i e_0 = K_i{}^j e_j \,,$$

where e_0 is a time-like unit vector normal to Σ. Lowering one index with h_{mn}, it can be seen easily that K_{ij} is symmetric. The situation of imbedding is described by the Gauss–Weingarten equation

$$e_{j;i} = -K_{ij}e_0 + {}^{(3)}\Gamma^k{}_{ij}e_k$$

and the Gauss–Codazzi equations

$$\begin{aligned} {}^{(4)}R^l{}_{ijk} &= {}^{(3)}R^l{}_{ijk} - (K_{ij}K_k{}^l - K_{ik}K_j{}^l) \\ {}^{(4)}R^0{}_{ijk} &= K_{ij|k} - K_{ik|j} \,. \end{aligned}$$

Further, the component ${}^{(4)}R^i_{0i0}$ can be expressed in terms of three dimensional quantities ([12]):

$${}^{(4)}R^{i}{}_{0i0} = (\operatorname{Tr} K)^2 - \operatorname{Tr}(K^2) - (n^\mu(\operatorname{Tr} K) + n^\mu_{,\nu} n^\nu)_{;\mu}\,.$$

Here the unit vector orthogonal to Σ is denoted by n. With these equations the Riemann curvature scalar ${}^{(4)}R$ of M and the Einstein–Hilbert Lagrangian can be expressed by three dimensional quantities:

$${}^{(4)}R\sqrt{-(g)} = \sqrt{-(g)}\left({}^{(3)}R - \operatorname{Tr}(K^2) + (\operatorname{Tr} K)^2 - 2\left(n^\mu(\operatorname{Tr} K) + n^\mu_{;\nu} n^\nu\right)_{;\mu}\right).$$

Integration over space-time leads to the action

$$S = \int\limits_M d^4x\sqrt{-(g)}\,{}^{(4)}R = \int\limits_M d^4x\sqrt{-(g)}\left({}^{(3)}R - \operatorname{Tr}(K^2) + (\operatorname{Tr} K)^2\right). \quad (2)$$

The total derivative has been omitted from the integral. In general, this is not unproblematic since it may contain boundary terms necessary for the dynamics of a quantum theory based on this approach. This is seen for example in Hawking's path-integral approach to gravity, where the Gibbons–Hawking boundary term is needed to render the path-integral meaningful. (The Gibbons–Hawking term is a part of the omitted total derivative.)

Now let a time function $\tau : M \to \mathbb{R}$ and a foliation of M into hyperplanes $\Sigma_t = \tau^{-1}(t)$ be given. Further let t^μ be a time-like vector-field satisfying $t^\mu \tau_{;\mu} = -1$, then the *lapse* and *shift* functions N and N^i, respectively, are defined by

$$(t_\mu) = N e_0 + N^i e_i\,.$$

With this vector-field as a new time-like basis-vector, *i.e.*, using $((t_\mu),\ e_i)$ as a basis of the tangential space of space-time, the metric becomes:

$$(g_{\mu\nu}) = \begin{pmatrix} N_i N^i - N^2 & N_m \\ N_n & h_{mn} \end{pmatrix} \quad (3)$$

with inverse

$$(g^{\mu\nu}) = \begin{pmatrix} -\frac{1}{N^2} & \frac{N^m}{N^2} \\ \frac{N^n}{N^2} & h^{mn} - \frac{N^m N^n}{N^2} \end{pmatrix}. \quad (4)$$

The spatial indices $(i,\ j,\ n, \ldots)$ are raised and lowered with h_{mn} and its inverse h^{mn}. The square root of the determinant of the metric is given by

$$\sqrt{-(g)} = N\sqrt{(h)}\,.$$

In this 3+1 decomposition the second fundamental form is expressed by

$$K_{ij} = \frac{1}{2N}\left(N_{i|j} + N_{j|i} - h_{ij,0}\right). \tag{5}$$

Knowing that the equations of motion for the Lagrange density $\mathcal{L} = \sqrt{-(g)}\,{}^{(4)}R$ are the Einstein equations

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 0\,, \tag{6}$$

one can see that the time–time part and the space–time part of this equation, $G_{00} = 0$ and $G_{0i} = 0$ respectively are equivalent to

$${}^{(3)}R + (\operatorname{Tr} K)^2 - \operatorname{Tr}(K^2) = 0\,, \tag{7}$$

$$\left(K^{ik} - \eta^{ik}(\operatorname{Tr} K)\right)_{|i} = 0\,, \tag{8}$$

which will be the constraints of the theory; the equations of motion in the three space–time dimensions correspond to $G_{ij} = 0$.

To formulate the canonical theory of gravity the Lagrange density $\mathcal{L}$ is integrated over the spatial manifold Σ to give the Lagrangian of the theory:

$$L = \int_{\Sigma} d^3x N \sqrt{(h)} \left({}^{(3)}R + \operatorname{Tr}(K^2) - (\operatorname{Tr} K)^2\right). \tag{9}$$

The term ${}^{(3)}R$ still contains second order spatial derivations of the metric, $h_{ij|k|l}$. Like the time derivations before, these are removed by a partial integration, leading again to a surface term which is omitted. To be able to omit it, the fields in the theory have to fall off fast enough at infinity to render this term finite. This yields some essential conditions for the configuration space of the theory. Since, as we will see, the Hamiltonian of the theory will be a combination of constraints and therefore will vanish on the physical hypersurface of the theory, the boundary term is the only nonvanishing energy of the theory, so the question whether one is allowed to omit it is again a crucial one. See Ashtekar ([2]) for more details.

The canonical momenta conjugated to the variables N, N_i and h_{ij} are defined by:

$$\pi := \frac{\delta L}{\delta N_{,0}} = 0\,, \tag{10}$$

$$\pi^i := \frac{\delta L}{\delta N_{i,0}} = 0\,, \tag{11}$$

$$\pi^{ij} := \frac{\delta L}{\delta h_{ij,0}} = -\sqrt{(h)}\left(K^{ij} - h^{ij} \operatorname{Tr} K\right).$$

To perform a Legendre transformation, the last equation is solved for $h_{ij,0}$, leading to

$$h_{ij,0} = N_{i|j} + N_{j|i} - \frac{N}{\sqrt{(h)}}(h_{ij}\,\mathrm{Tr}\,\pi - 2\pi_{ij}),$$

and

$$K_{ij} = \frac{1}{\sqrt{(h)}}\left(\tfrac{1}{2}h^{ij}\,\mathrm{Tr}\,\pi - \pi^{ij}\right). \tag{12}$$

Using Dirac's formulation of constrained Hamiltonian systems ([6], [16]), the constrained Hamiltonian is given by:

$$\begin{aligned} H &= \int d^3x\left(\pi N_{,0} + \pi^i N_{i,0} + \pi^{ij}h_{ij,0}\right) - L \\ &= \int d^3x\left(\pi N_{,0} + \pi^i N_{i,0} + N\mathcal{H} + N_i\mathcal{C}^i\right). \end{aligned} \tag{13}$$

The following abbreviations have been used in the previous line:

$$\mathcal{H} = \frac{1}{\sqrt{(h)}}\left(\mathrm{Tr}(\pi^2) - \tfrac{1}{2}(\mathrm{Tr}\,\pi)^2\right) + \sqrt{(h)}\,{}^{(3)}R, \tag{14}$$

$$\mathcal{C}^i = -2\pi^{ij}{}_{|j}. \tag{15}$$

Since the primary constraints have to be time independent their Poisson brackets with the Hamiltonian yields secondary constraints:

$$\{\pi, H\} = \frac{\delta H}{\delta N} = \mathcal{H} \approx 0, \tag{16}$$

$$\{\pi^i, H\} = \frac{\delta H}{\delta N_i} = \mathcal{C}^i \approx 0. \tag{17}$$

As usual the symbol $\approx$ means *equal modulo constraints.*

The primary constraints (10), (11) have no dynamical content in themselves. They merely express the fact that the Lagrangian is independent of $N_{,0}$ and $N_{i,0}$, so they can be removed from the theory without loss. On the contrary, the secondary constraints are of utmost importance, they are called the Hamiltonian and diffeomorphism constraints; they constitute the dynamical evolution of the theory and show its invariance under the diffeomorphism group, respectively.

Without going into further details of these matters, I give the canonical equations of the theory as worked out by Arnowitt, Deser, and Misner, neglecting the source term they considered:

$$\frac{\partial h_{ij}}{\partial t} = \frac{\delta H}{\delta \pi^{ij}} = \frac{2N}{\sqrt{(h)}}(\pi_{ij} - \tfrac{1}{2}h_{ij}\,\mathrm{Tr}\,\pi) + N_{i|j} + N_{j|i}, \tag{18}$$

$$\frac{\partial \pi^{ij}}{\partial t} = \frac{\delta H}{\delta h_{ij}} = -N\sqrt{(h)}\left({}^{(3)}R^{ij} - \tfrac{1}{2}h^{ij}R\right) + \frac{N}{2\sqrt{(h)}}h^{ij}\left(\operatorname{Tr}\pi^2 - \tfrac{1}{2}(\operatorname{Tr}\pi)^2\right)$$
$$- \frac{2N}{\sqrt{(h)}}\left(\pi^{ik}\pi_k{}^j - \tfrac{1}{2}\pi^{ij}\operatorname{Tr}\pi\right)$$
$$+ \sqrt{(h)}\left(N^{|ij} - h^{ij}N^{|k}{}_{|k}\right) + \left(\pi^{ij}N^k\right)_{|k}$$
$$- N^i{}_{|k}\pi^{kj} - N^j{}_{|k}\pi^{ki}\,. \tag{19}$$

These equations will be called the ADM equations in the rest of this paper.

Although these equations formally seem to be more involved than the Einstein equations, $G_{\mu\nu} = 0$, they must be used to work out an initial value problem of gravity: given a hypersurface Σ with metric h_{ij} and the lapse and shift functions, solving these equations will yield the space–time M with a metric $g_{\mu\nu}$ fulfilling (3) and the Einstein equations. (At least when developed for, in some sense, *small* times, otherwise singularities may and will develop.)

3. Canonical theory of the Einstein–Cartan action

Einstein–Cartan gravity is a theory described by the Lagrange density (1):

$$\mathcal{L} = R_{\mu\nu}{}^{\alpha\beta}\theta^\mu_\alpha\theta^\nu_\beta\,.$$

Here $R^{\mu\nu}{}_{\alpha\beta}$ is the curvature form of a SO(3,1) connection $\omega_\mu{}^{\alpha\beta}$, represented in a four-dimensional space. The simplest choice is the Minkowski–space M^4, which carries a canonical representation of SO(3,1). θ^μ_α is the soldering form, an invertible mapping from TM to M^4; θ^α_μ is its inverse. It is well known that the variation of this Lagrangian with respect to the soldering form and the connection yield the Einstein equations and the condition of vanishing torsion. The flat metric imposed on M^4, $\eta_{\alpha\beta}$ pulled back to TM by θ^α_μ, gives the space–time metric of M:

$$g_{\mu\nu} = \theta^\alpha_\mu\eta_{\alpha\beta}\theta^\beta_\nu\,, \tag{20}$$

therefore $(g) = -(\theta)^2$. Since η is fixed the composite field, $g_{\mu\nu}$ is completely determined by the soldering form.

Now I shall carry out the $3+1$ decomposition of this Lagrangian, that is separating the field components with time-like indices from the others. Therefore, I introduce some notation: I define:

$$\omega_0{}^{ab} =: L^{ab}\,,$$
$$\omega_0{}^{\hat{0}a} =: L^{\hat{0}a}\,.$$

Since $\omega_\mu{}^{\alpha\beta}$ is antisymmetric, it suffices to consider the components with the smaller index first, so let always $\omega_\mu{}^{a\hat{0}} = -\omega_\mu{}^{\hat{0}a}$. Then there are four different kinds of components of $\omega_\mu{}^{\alpha\beta}$, namely $L^{\hat{0}a}$, L^{ab}, $\omega_i{}^{\hat{0}a}$, $\omega_i{}^{ab}$, the latter being a SO(3) connection on a space-like hypersurface Σ of M.

The decomposition of the soldering form is guided by the form of the metric (3), which should be regained under (20). This condition yields:

$$(\theta^\alpha_\mu) = \begin{pmatrix} N & N^k \vartheta^a_k \\ 0 & \vartheta^a_m \end{pmatrix} \tag{21}$$

with inverse

$$(\theta^\mu_\alpha) = \begin{pmatrix} \frac{1}{N} & -\frac{N^m}{N} \\ 0 & \vartheta^m_a \end{pmatrix} . \tag{22}$$

Here ϑ^a_m is a three-dimensional soldering form, soldering the three-dimensional hypersurface $\Sigma \subset M$ to a three-dimensional Euclidean subspace E^3 of M^4. ϑ^m_a again denotes its inverse. N and N^k are the previously defined lapse and shift functions.

It has to be mentioned that there is a one-parameter freedom of choice in the decomposition of θ. The most general solution of the decomposition problem which gives the metric (3) is not (21) but

$$(\theta^\mu_\alpha) = \begin{pmatrix} \xi & \left(-\xi \pm \sqrt{\frac{\xi^2 N^2 - 1}{N_k N^k}}\right) N^m \\ \pm\sqrt{\xi^2 - \frac{1}{N^2}}\, e_a & \left(\pm\sqrt{\xi^2 - \frac{1}{N^2}}\, N^k \vartheta^a_k e_a + \xi N\right) \vartheta^m_a \end{pmatrix} ,$$

e_a being an orthonormal basis in $E^3 \subset M^4$ and ξ the parameter. This reduces to (22) for $\xi = \frac{1}{N}$. Fixing the parameter is a partial gauge fixing of the boost part of the Lorentz group in the inner space M^4. This has the consequence that the $L^{\hat{0}a}$ parts of the connection do not disturb the dynamics because they can be eliminated from the equations of motion by an additional primary first class constraint, which would be second class without gauge fixing, as we shall see. This constraint will be called the gauge constraint. In the general case, the four quantities with time-like space-time index, N, N_i, $L^{\hat{0}a}$ and L^{ab} would be treated on equal footing, leading to ten secondary constraints [1]. But the lapse and shift functions are essential parts of the metric (21), while $L^{\hat{0}a}$ and L^{ab} describe boosts and rotations in the inner space. Fixing the boost part by choice of ξ is

[1] This was pointed out to me by M. Tielke who conjectured *a priori* that one should obtain ten secondary constraints reflecting the structure of gravity as a gauge theory of the Poincaré group.

at least convenient, it might even be necessary, since the choice of a three-dimensional subspace restricts the boost freedom (but the *many fingered time* idea should be taken into account) (*cf.* [7], where the gauge constraint is also used). In any case the gauge constraint does not generate gauge transformations, as should be expected for primary first class constraints.

Before proceeding, the three-dimensional Christoffel symbols expressed in terms of the soldering form are given. By construction the three-dimensional metric on Σ is

$$h_{kl} = \vartheta^a_k \eta_{ab} \vartheta^b_l \,.$$

Here $\eta_{ab} = \mathrm{diag}(1, 1, 1)$. So

$$\Gamma^i{}_{jk} = \tfrac{1}{2}\Big[h^{il}\big((\partial_k \vartheta^c_l)\eta_{cd}\vartheta^d_j - (\partial_l \vartheta^c_j)\eta_{cd}\vartheta^d_k\big) + \vartheta^i_a(\partial_k \vartheta^a_j)\Big] + (j \leftrightarrow k) \,. \quad (23)$$

After these preparations the Lagrange density is decomposed:

$$\begin{aligned}\mathcal{L} = {} & 2\big(\partial_0 \omega_n{}^{\hat{0}a} - \partial_n L^{\hat{0}a} + L^{ac}\eta_{cd}\omega_n{}^{\hat{0}d} + L^{\hat{0}c}\eta_{cd}\omega_n{}^{da}\big)\big(\theta^0_{\hat{0}}\theta^n_a - \theta^n_{\hat{0}}\theta^0_a\big) \\ & + \big(\partial_0 \omega_n{}^{ab} - \partial_n L^{ab} + 2L^{ac}\eta_{cd}\omega_n{}^{db} + 2L^{\hat{0}a}\omega_n{}^{\hat{0}b}\big)\big(\theta^0_a\theta^n_b - \theta^0_b\theta^n_a\big) \\ & + \big(2\partial_m \omega_n{}^{\hat{0}b} - \omega_m{}^{db}\eta_{cd}\omega_n{}^{\hat{0}c} + \omega_n{}^{db}\eta_{cd}\omega_m{}^{\hat{0}c}\big)\big(\theta^m_{\hat{0}}\theta^n_b - \theta^n_{\hat{0}}\theta^m_b\big) \\ & + \big(\partial_m \omega_n{}^{ab} + \omega_m{}^{ac}\eta_{cd}\omega_n{}^{db} + \omega_m{}^{\hat{0}a}\omega_n{}^{\hat{0}b}\big)\big(\theta^m_a\theta^n_b - \theta^n_a\theta^m_b\big) \end{aligned} \quad (24)$$

or, using the decomposition (21) of the soldering form,

$$\begin{aligned}\mathcal{L} = {} & \frac{2}{N}\big(\partial_0 \omega_n{}^{\hat{0}a} - \partial_n L^{\hat{0}a} + L^{ac}\eta_{cd}\omega_n{}^{\hat{0}d} + L^{\hat{0}c}\eta_{cd}\omega_n{}^{da}\big)\vartheta^n_a \\ & + \frac{2N^n}{N}\big(\partial_m \omega_n{}^{\hat{0}b} + \omega_m{}^{bc}\eta_{cd}\omega_n{}^{\hat{0}d} - \partial_n \omega_m{}^{\hat{0}b} - \omega_n{}^{bc}\eta_{cd}\omega_n{}^{\hat{0}d}\big)\vartheta^m_b \\ & + \big(\partial_m \omega_n{}^{ab} + \omega_m{}^{ac}\eta_{cd}\omega_n{}^{db} + \omega_m{}^{\hat{0}a}\omega_n{}^{\hat{0}b}\big)\big(\vartheta^m_a\vartheta^n_b - \vartheta^n_a\vartheta^m_b\big) \,. \end{aligned} \quad (25)$$

With the abbreviations

$$\begin{aligned} C_{mn}{}^b &:= D_m \omega_n{}^{\hat{0}b} = \partial_m \omega_n{}^{\hat{0}b} + \omega_m{}^{bc}\eta_{cd}\omega_n{}^{\hat{0}d} \,, \\ H_{mn}{}^{ab} &:= \partial_m \omega_n{}^{ab} + \omega_m{}^{ac}\eta_{cd}\omega_n{}^{db} + \omega_m{}^{\hat{0}a}\omega_n{}^{\hat{0}b} \end{aligned}$$

the Lagrange density becomes

$$\begin{aligned}\mathcal{L} = {} & \frac{2}{N}\big(\partial_0 \omega_n{}^{\hat{0}a} + L^{ac}\eta_{cd}\omega_n{}^{\hat{0}d} - \partial_n L^{\hat{0}a} - \omega_n{}^{ad}\eta_{dc}L^{\hat{0}c}\big)\vartheta^n_a \\ & + 2\frac{N^n}{N}\big(C_{mn}{}^b - C_{nm}{}^b\big)\vartheta^m_b \\ & + H_{mn}{}^{ab}\big(\vartheta^m_a\vartheta^n_b - \vartheta^n_a\vartheta^m_b\big) \,. \end{aligned} \quad (26)$$

Now the Lagrangian L of the theory is

$$L = \int_{\Sigma} d^3x N(\vartheta)\mathcal{L}\,. \tag{27}$$

To proceed in the canonical framework the conjugated momenta are defined. These all lead to primary constraints:

$$\begin{aligned}
p_0 &= \frac{\delta L}{\delta N_{,0}} \approx 0\,,\\
p_i &= \frac{\delta L}{\delta N^i{}_{,0}} \approx 0\,,\\
p^i_a &= \frac{\delta L}{\delta \vartheta^a_{i,0}} \approx 0\,,\\
p_{\hat{0}a} &= \frac{\delta L}{\delta L^{\hat{0}a}{}_{,0}} \approx 0\,,\\
p_{ab} &= \frac{\delta L}{\delta L^{ab}{}_{,0}} \approx 0\,,\\
p^i{}_{\hat{0}a} &= \frac{\delta L}{\delta \omega_i{}^{\hat{0}a}{}_{,0}} \approx 2(\vartheta)\vartheta^i_a\,,\\
p^i{}_{ab} &= \frac{\delta L}{\delta \omega_i{}^{ab}{}_{,0}} \approx 0\,.
\end{aligned} \tag{28}$$

These constraints are distinguished as first and second class constraints, with Poisson brackets:

$$\{p^i_a,\, p^j{}_{\hat{0}b}\} = 2(\vartheta)\left(\vartheta^i_a\vartheta^j_b - \vartheta^i_b\vartheta^j_a\right).$$

All other Poisson brackets vanish, therefore all other constraints are first class. If the gauge fixing would not have been imposed on θ, there would have been an additional constraint, namely $p^a_0 = \frac{\delta L}{\delta\theta^0_{a,0}}$, and there would be two more non vanishing Poisson brackets, namely $\{p^i_c, p^i_{ab}\} \neq 0$ and $\{p^a_0, p^i{}_{ab}\} \neq 0$. Imposing the gauge fixing makes the first of these vanish, $p_i{}^{ab}$ becomes first class, and the second disappears altogether, because p^a_0 does not exist. Since the primary first class constraints lead to secondary constraints, the gauge condition imposes a new secondary constraint on the theory, it will be called the gauge constraint in the following.

The canonical Hamiltonian is now given by

$$H = \int d^3x \Big\{ (\vartheta)\Big[2\vartheta^n_a\left(\partial_n L^{\hat{0}a} - L^{ac}\eta_{cd}\omega_n{}^{\hat{0}d} + \omega_n{}^{ac}\eta_{cd}L^{\hat{0}d}\right)$$

$$+ 2N^n\vartheta^m_b\big(C_{nm}{}^b - C_{mn}{}^b\big) + NH_{mn}{}^{ab}\big(\vartheta^n_a\vartheta^m_b - \vartheta^m_a\vartheta^n_b\big)\Big]$$
$$+ c^0 p_0 + c^i p_i + c^a_i p^i_a + c^{\hat{0}a} p_{\hat{0}a}$$
$$+ c^{ab} p_{ab} + c_i{}^{\hat{0}a}\big(p^i{}_{\hat{0}a} - 2(\vartheta)\vartheta^i_a\big) + c_i{}^{ab} p^i{}_{ab}\Big\}\,. \qquad (29)$$

The first part of the Hamiltonian will be called H°. The physical submanifold of configuration space should be preserved, so the Poisson brackets of the primary constraints should vanish, at least weakly. In the case of second class constraints this condition will be used to find the Lagrange multipliers c^a_i and $c_i{}^{\hat{0}a}$, while the first class primary constraints lead to secondary constraints:

$$\{H, p_0\} = (\vartheta)H_{mn}{}^{ab}(\vartheta^m_a\vartheta^n_b - \vartheta^n_a\vartheta^m_b) =: \mathcal{H}\,,$$

$$\{H, p_i\} = -2(\vartheta)\vartheta^j_a(C_{ij}{}^a - C_{ji}{}^a) =: \mathcal{C}_i\,,$$

$$\{H, p_{\hat{0}a}\} = 2\partial_n[(\vartheta)\vartheta^n_a] - 2(\vartheta)\vartheta^n_b\omega_n{}^{bd}\eta_{ad}\,,$$

$$\{H, p_{ab}\} = (\vartheta)\big(\vartheta^n_a\omega_n{}^{\hat{0}d}\eta_{bd} - \vartheta^n_b\omega_n{}^{\hat{0}d}\eta_{da}\big)\,,$$

$$\{H, p^i_{ab}\} = (\vartheta)\big(\vartheta^i_a\eta_{bc}L^{\hat{0}c} - \vartheta^i_b\eta_{ac}L^{\hat{0}c}\big)$$

$$+ 2(\vartheta)N^m\vartheta^n_c\frac{\delta}{\delta\omega_i{}^{ab}}(C_{nm}{}^c - C_{mn}{}^c)$$

$$+ N(\vartheta^n_c\vartheta^m_d - \vartheta^n_d\vartheta^m_c)\frac{\delta}{\delta\omega_i{}^{ab}}H_{mn}{}^{cd}\,.$$

The two first constraints are by construction the Hamiltonian and the diffeomorphism constraint, and the last one is the gauge constraint. Since this constraint vanishes weakly, it can be solved for $L^{\hat{0}a}$ on the constrained hypersurface of configuration space:

$$L^{\hat{0}a} \approx (\partial_n N)\vartheta^n_b\eta^{ba} - \tfrac{2}{3}N(\partial_m\vartheta^c_k)(\vartheta^m_c\vartheta^k_b - \vartheta^k_c\vartheta^m_b)\eta^{ba} - N^k\omega_k^{\hat{0}a}\,. \qquad (30)$$

This makes it possible to discard $L^{\hat{0}a}$ from the equations of motion, as mentioned before.

The Poisson brackets of the second class primary constraints can be solved (weakly) for the Lagrange multipliers:

$$c_i{}^{\hat{0}a} \approx \frac{1}{2(\vartheta)} \frac{\delta H^\circ}{\delta \vartheta^b_k} (\vartheta^b_i \vartheta^a_k - \tfrac{1}{2} \vartheta^b_k \vartheta^a_i) ,$$

$$c^a_i \approx \frac{1}{2(\vartheta)} \frac{\delta H}{\delta \omega_k{}^{\hat{0}b}} (\vartheta^b_i \vartheta^a_k - \tfrac{1}{2} \vartheta^b_k \vartheta^a_i) .$$

Now the principal equations of motion are given, that is, the equations of motion that do not reduce to constraints or Lagrange multipliers but contain the physical part of the theory. These equations are

$$\begin{aligned}\frac{d}{dt}\omega_k{}^{\hat{0}a} &= c^a_k \\ &= \partial_k L^{\hat{0}a} - L^{ac}\eta_{cd}\omega_k{}^{\hat{0}d} + \omega_k{}^{ac}\eta_{cd}L^{\hat{0}d} + N^n(C_{nk}{}^a - C_{kn}{}^a) \\ &+ N\big(\tfrac{1}{2}(-H_{mk}{}^{ac}\vartheta^m_c + H_{kn}{}^{ac}\vartheta^n_c + H_{mk}{}^{ca}\vartheta^m_c - H_{kn}{}^{ca}\vartheta^n_c) \\ &- \tfrac{1}{2}H_{mn}{}^{cd}(\vartheta^m_c\vartheta^n_d - \vartheta^m_d\vartheta^n_c)\vartheta^a_k\big) \end{aligned} \tag{31}$$

$$\begin{aligned}\frac{d}{dt}\vartheta^a_k &= c_k{}^{\hat{0}a} \\ &= -\vartheta^b_k L^{ac}\eta_{cb} - N\omega_k{}^{\hat{0}a} + N^n\omega_n{}^{ac}\eta_{cb}\vartheta^b_k + (\partial_k N_l)h^{ln}\vartheta^a_n \\ &- N_l(\partial_k\vartheta^c_n)\vartheta^n_d\vartheta^l_c\eta^{da} . \end{aligned} \tag{32}$$

As mentioned before, $L^{\hat{0}a}$ can be discarded from these equations by means of equation (30).

To compare this result with the ADM equations presented in the previous section, the time derivative of the metric h_{ij} and its ADM canonically conjugated variable π^{ij} are worked out. These should become the ADM equations (18) and (19). It is clear that

$$\frac{d}{dt}h_{ij} = \Big(\frac{d}{dt}\vartheta^a_i\Big)\eta_{ab}\vartheta^b_j + \vartheta^a_i\eta_{ab}\Big(\frac{d}{dt}\vartheta^b_j\Big) .$$

As one might presume, it turns out that

$$K_{ij} = \tfrac{1}{2}(\omega_i{}^{\hat{0}a}\eta_{ab}\vartheta^b_j + \omega_j{}^{\hat{0}a}\eta_{ab}\vartheta^b_i) ,$$

or equivalently

$$\pi^{ij} = (\vartheta)\big(h^{ij}\omega_k{}^{\hat{0}a}\vartheta^k_a - \tfrac{1}{2}(h^{ik}\omega_k{}^{\hat{0}a}\vartheta^j_a + h^{jk}\omega_k{}^{\hat{0}a}\vartheta^i_a)\big) ,$$

lead to the correct ADM equations. Inserting (31) and (32) into the equations for $\frac{d}{dt}h_{ij}$ and $\frac{d}{dt}\pi^{ij}$ leads, after a straightforward but tedious calculations, to the ADM equations (18) and (19). $L^{\hat{0}a}$ has been eliminated by the gauge constraint. The terms containing L^{ab} cancel by the symmetry of h_{ij} and π^{ij}, which follows from the symmetry of the second fundamental form. The covariant derivatives in the ADM equations are, of course, Levi–Civita derivatives with the connection given in (23).

In this way the ADM theory is completely regained.

4. Comparison with Ashtekar's variables

In this Section, the special case where the four dimensional space M^4 is the *spinor space* End($\mathbb{C}^2$, $\bar{\mathbb{C}}^2$) is investigated. It turns out that this choice leads in a natural way to a description of gravity which is, in some sense, analogous to Ashketar's. The internal space End($\mathbb{C}^2$, $\bar{\mathbb{C}}^2$) will be realized by multiplying all internal space tensor indices (α, β, ...) with Pauli matrices $\tau_\alpha{}^A{}_{A'}$, $\tau_{\hat{0}} = Id$, *e.g.* the soldering form ϑ^a_μ is replaced by

$$\sigma_\mu{}^A{}_{A'} := \vartheta^a_\mu \tau_a{}^A{}_{A'} .$$

So there is the following commutative diagram of soldering forms:

$$\begin{array}{ccc} TM & \xrightarrow{\theta^\alpha_\mu} & M^4 \\ \sigma_\mu{}^A{}_{A'} \searrow & & \swarrow \tau_\alpha{}^A{}_{A'} \\ & \mathrm{End}(\mathbb{C}^2,\, \bar{\mathbb{C}}^2) & \end{array}$$

End($\mathbb{C}^2$, $\bar{\mathbb{C}}^2$) carries a representation of the Lorentz group: a Lorentz transformation $L^{\alpha\beta}$ in M^4 is translated into

$$L^{AB}{}_{A'B'} := L^{\alpha\beta}\tau_\alpha{}^A{}_{A'}\tau_\beta{}^B{}_{B'} = L^{AB}\varepsilon_{A'B'} + L_{A'B'}\varepsilon^{AB} .$$

For the last decomposition see [13]. By antisymmetry of $L^{\alpha\beta}$ the last quantity is seen to lie in SU(2)⊗SU(2)≅SO(3)⊗SO(3)≅SO(3,1).

Here $\varepsilon^{(AB)} = \varepsilon^{(A'B')} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. These matrices are used to raise and lower indices in spinor space, according to

$$\varepsilon^{AC}\varepsilon_{BC} = \delta^A_B, \quad \varepsilon^{AC}\varepsilon_{CB} = -\delta^A_B ,$$

$$X_{AB'} = X^B{}_{B'}\varepsilon_{BA} = -X^B{}_{B'}\varepsilon_{AB} .$$

Analogous relations hold for primed indices. One should realize that

$$\eta_{\alpha\beta}\tau^{\alpha}{}_{A}{}^{A'}\tau^{\beta}{}_{B}{}^{B'} = \eta_{AB}{}^{A'B'} = \varepsilon_{AB}\varepsilon^{A'B'} .$$

It is noteworthy that for a spatial index a, $\tau_a^{AB'} = \tau_a{}^{A}{}_{A'}\varepsilon^{B'A'}$ is a symmetric matrix. This will be tacitly used, *e.g.* in Eq. (36).[2]

It is well known that every *world tensor* can be written as a spinorial quantity (*cf. e.g.* [13], vol. 1, or [11]), for example

$$R_{\mu\nu}{}^{\alpha\beta}\tau_{\alpha}{}^{A}{}_{A'}\tau_{\beta}{}^{B}{}_{B'} = R_{\mu\nu}{}^{AB}{}_{A'B'} .$$

The antisymmetry of $R_{\mu\nu}{}^{\alpha\beta}$ allows the following decomposition of this tensor:

$$R_{\mu\nu}{}^{AB}{}_{A'B'} = R_{\mu\nu}{}^{AB}\varepsilon_{A'B'} + \bar{R}_{\mu\nu A'B'}\varepsilon^{AB} . \tag{33}$$

Since $R_{\mu\nu}{}^{\alpha\beta}$ is real, $R_{\mu\nu}$ and $\bar{R}_{\mu\nu}$ on the r.h.s. of Eq. (33) are complex conjugated quantities.

In the same spirit the connection form is written as

$$\omega_{\mu}{}^{\alpha\beta}\tau_{\alpha}{}^{A}{}_{A'}\tau_{\beta}{}^{B}{}_{B'} = \omega_{\mu}{}^{AB}\varepsilon_{A'B'} + \bar{\omega}_{\mu A'B'}\varepsilon^{AB} . \tag{34}$$

Since

$$R_{\mu\nu}{}^{\alpha\beta} = \partial_{\mu}\omega_{\nu}{}^{\alpha\beta} + \omega_{\mu}{}^{\alpha\gamma}\eta_{\gamma\delta}\omega_{\nu}{}^{\delta\beta} - (\mu \leftrightarrow \nu)$$

equations (33) and (34) can be compared to give

$$R_{\mu\nu}{}^{AB} = \partial_{\mu}\omega_{\nu}{}^{AB} + \omega_{\mu}{}^{AC}\varepsilon_{CD}\omega_{\nu}{}^{DB} - (\mu \leftrightarrow \nu) ,$$

$$\bar{R}_{\mu\nu A'B'} = \partial_{\mu}\bar{\omega}_{\nu A'B'} + \omega_{\mu A'C'}\varepsilon^{C'D'}\bar{\omega}_{\nu D'B'} - (\mu \leftrightarrow \nu)$$

because the mixed terms of the form $\omega_{\nu}{}^{AB}\bar{\omega}_{\nu A'B'}$ cancel. In this way $R_{\mu\nu}{}^{AB}$ and $\omega_{\mu}{}^{AB}$ can be interpreted as curvature and connection in a GL(2)-bundle. The analogue holds for the barred quantities.

In order to make contact to Ashtekar's formulation of canonical gravity, it is necessary to realize that the decomposition (33) of $R_{\mu\nu}{}^{ab}$ splits the curvature into its self-dual and anti-self-dual part. In his description of canonical gravity Ashtekar works with a self-dual Lagrangian, which leads, due to the Bianchi identity, to the same equations of motion. Therefore nothing is lost when the theory is restricted to its self-dual part. In this

[2] The only point in this paper, where the use of this symmetry cannot be avoided, is the transcription of the Hamiltonian constraint into its terminal form (40).

way Ashtekar's formulation may be considered to be more economic because he works in a smaller bundle as we shall see.

Performing the 3 + 1-decomposition, the equations (33) and (34) still hold for the three-dimensional quantities $R_{mn}{}^{ab}$, $\omega_m{}^{ab}$. In four-dimensions, for every $n \in \{1,2,3\}$, $\omega_n{}^{\alpha\beta}$ has six components (the number of generators of the Lorentz group). They build the SO(3)-connection $\omega_n{}^{ab}$ and the second fundamental form $\omega_n{}^{\hat{0}a}$. (For simplicity and to provide a name for these quantities I shall call $\omega_m{}^{\hat{0}a}$ the second fundamental form because of its close relation to K_{nm}.)

The connection ω_n is written as

$$\omega_n{}^{ab}\tau_a{}^A{}_{A'}\tau_b{}^B{}_{B'} = \omega_n{}^{AB}\varepsilon_{A'B'} + \bar{\omega}_{nA'B'}\varepsilon^{AB} \,. \tag{35}$$

Counting the degrees of freedom for every n gives four from the four combinations $(AB) \in \{(00), (01), (10), (11)\}$ minus one, since the Pauli matrices are traceless. This makes three for $\omega_n{}^{AB}$ and in the same way three for $\bar{\omega}_{nA'B'}$, which are the three degrees of freedom of $\omega_n{}^{ab}$ plus the three conditions that these quantities are real.

Now $R_{mn}{}^{AB}$ and $\omega_n{}^{AB}$ are curvature and connection in an SU(2)-bundle.

The decomposition of $\omega_n{}^{\hat{0}a}$ may be performed in two ways: as before it can be written as

$$\omega_n{}^{\hat{0}a}\tau_{\hat{0}}{}^D{}_{D'}\tau_a{}^A{}_{A'} = \pi_n{}^{DA}\varepsilon_{D'A'} + \bar{\pi}_{nD'A'}\varepsilon^{DA}$$

($\pi_n{}^{AB}$ will be used to distinguish this quantity from $\omega_n{}^{AB}$ which is related to $\omega_n{}^{ab}$). This equation can be simplified since

$$\tau_{\hat{0}}{}^D{}_{D'} = \frac{1}{\sqrt{2}}\delta^D_{D'} \,.$$

Therefore

$$\omega_n{}^{\hat{0}a}\tau_a{}^A{}_{A'} = \frac{1}{\sqrt{2}}(\pi_n{}^A{}_{A'} + \bar{\pi}_{nA'}{}^A) =: \kappa_n{}^A{}_{A'} \,.$$

(Note that $\kappa_n{}^T = \bar{\kappa}_n$.) In this way the quantity $\omega_n{}^{\hat{0}a}$ with one internal space index is related to $\kappa_n{}^A{}_{A'}$ with one pair of indices (AA'). Since the trace of κ vanishes it has three degrees of freedom, as necessary.

Before translating the constraint equations into this spinorial form, I shall detour to inspect the bundle structure of this theory after the introduction of time. The time-like vectorfield t^μ is related to

$$t^\mu\sigma_\mu{}^A{}_{A'} = \tau_{\hat{0}}{}^A{}_{A'} = \frac{1}{\sqrt{2}}\delta^A_{A'} =: G^A{}_{A'} \,.$$

This time-like vectorfield (fixed from now on) therefore leads to the distinguished fixed scalar product $G^A{}_{A'} : \mathbb{C}^2 \to \bar{\mathbb{C}}^2$, see [15]. This scalar product can be used to convert primed indices into unprimed ones and vice versa:

$$X \ldots^{A'} G^A{}_{A'} = X \ldots^{A} ,$$
$$Y \ldots_{A} G^A{}_{A'} = Y \ldots_{A'} ,$$

where the ellipses stand for any set of indices. (One has to realize that expressions like $\sigma_n{}^{AB}$ or $\tau_a{}^{A'}{}_{B'}$ only become meaningful after the introduction of time.) The Pauli matrices solder the internal space E^3 to SU(2), while the SO(3,1)-connection is reduced to an SO(3)-connection which is embedded by Eq. (35) in an SU(2)⊗SU(2)′-bundle, associated to the End($\mathbb{C}^2$, $\bar{\mathbb{C}}^2$)-bundle, where the prime refers to the primed indices. This imbedding will be more closely investigated. Let X^{ab} be a generator of SO(3), *e.g.*

$$(X^{ab}) = \begin{pmatrix} 0 & A & -B \\ -A & 0 & C \\ B & -C & 0 \end{pmatrix} ,$$

then writing

$$(\tau_a{}^{AA'}) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -i & 0 \\ 1 & i & 0 \\ 0 & 0 & -1 \end{pmatrix} ,$$

where the columns are labeled by $a = (1,2,3)$ and the rows by $(AA') = ((00'), (01'), (10'), (11'))$, the quantity corresponding to X^{ab} in SU(2) ⊗SU(2)' is

$$(\tau_a{}^{AA'} X^{ab} \tau_b{}^{BB'}) = \begin{pmatrix} 0 & B+iC & B-iC & 0 \\ -B-iC & 0 & 2iA & B+iC \\ B+iC & -2iA & 0 & B-iC \\ 0 & -B-iC & -B+iC & 0 \end{pmatrix} ,$$

where the rows are labeled by $(AA') = ((00'), (01'), (10'), (11'))$ and the columns by $(BB') = ((00'), (01'), (10'), (11'))$. To decompose this quantity in the form $\tau_a \otimes \tau'_b$, it is rewritten as

$$\begin{pmatrix} 0 & B+iC & -B-iC & 0 \\ B-iC & 0 & 2iA & B+iC \\ -B-iC & -2iA & 0 & -B-iC \\ 0 & B-iC & -B+iC & 0 \end{pmatrix} .$$

Now the rows are labeled by $(AB) = ((00), (01), (10), (11))$ and the columns by $(A'B') = ((0'0'), (0'1'), (1'0'), (1'1'))$. This last form is seen to decompose according to

$$A(\tau_1\otimes\tau_2'-\tau_2\otimes\tau_1')+B(\tau_3\otimes\tau_1'-\tau_1\otimes\tau_3')+C(\tau_2\otimes\tau_3'-\tau_3\otimes\tau_2') \in \mathrm{SU}(2)\otimes\mathrm{SU}(2)' .$$

In the spirit of this decomposition $R_{mn}{}^{AB}{}_{A'B'}$ and $\omega_n{}^{AB}{}_{A'B'}$ can be regarded as curvature and connection in the SU(2)⊗SU(2)'-bundle, while $R_{mn}{}^{AB}$ and $\omega_n{}^{AB}$ are curvature and connection in the SU(2) part of this bundle, and correspondingly the barred quantities are curvature and connection in the SU(2)' part. In this language Ashtekar's theory is the one restricted to the first factor of the product bundle. Since the scalar product provided by the introduction of time, $G^A{}_{A'}$, is an isomorphism between both factors it seems reasonable to consider only the first factor. Ashtekar's work showed that this is indeed sufficient. Nevertheless, I shall continue to work in the larger bundle and we shall see how the equations or the connection decompose.

Because of the reality of $M^4 \cong \mathbb{R}^4$ this space is soldered to the set of real multiples of the Pauli-matrices, $\tau_{\hat{0}}, \ldots, \tau_3$. Instead of insisting on the reality of $M^4 \cong TM$, the theory may be expanded to describe complex gravity, or, following Ashtekar, to work in the complex theory and add some reality conditions. The internal three dimensional spatial space E^3 is now soldered to the set of complex 2×2 matrices which are multiples of the three Pauli matrices τ_1, τ_2, τ_3, that is, the fundamental representation of SU(2), or, working in complex gravity, the traceless complex 2×2 matrices.

I shall now proceed to transcribe the constraints of the Hamiltonian formulation worked out in the previous section into this SU(2)⊗SU(2)' formulation of the theory. The gauge constraint is excluded here: as it was discussed before in Section 3, it is not on the same footing and does not represent a gauge freedom of the theory.

The Hamiltonian constraint was written as

$$\begin{aligned}\mathcal{H} &= (\vartheta)\Big(R_{mn}{}^{ab} + \omega_m{}^{\hat{0}a}\omega_n{}^{\hat{0}b} - \omega_m{}^{\hat{0}b}\omega_n{}^{\hat{0}a}\Big)\vartheta^m_a\vartheta^n_b \\ &=: (\vartheta)\big(R_{mn}{}^{ab} + \Pi_{mn}{}^{ab}\big)\vartheta^m_a\vartheta^n_b .\end{aligned}$$

Since, besides $R_{mn}{}^{ab}$, $\Pi_{mn}{}^{ab}$ is also antisymmetric in a, b this can also be written as:

$$\begin{aligned}&=: (\sigma)\Big(\big(R_{mn}{}^{AB} + \Pi_{mn}{}^{AB}\big)\varepsilon_{A'B'} \\ &+ \big(\bar{R}_{mnA'B'} + \bar{\Pi}_{mnA'B'}\big)\varepsilon^{AB}\Big)\sigma^m{}_A{}^{A'}\sigma^n{}_B{}^{B'} \\ &= (\sigma)\big(R_{mn}{}^{AB} + \Pi_{mn}{}^{AB}\big)\sigma^m{}_A{}^{C'}\sigma^n{}_{C'B} \\ &+ (\sigma)\big(\bar{R}_{mnA'B'} + \bar{\Pi}_{mnA'B'}\big)\sigma^{mA'C}\sigma^n{}_C{}^{B'} .\end{aligned} \tag{36}$$

The above mentioned decomposition into SU(2) and SU(2)′ parts is manifestly seen. Here

$$\Pi_{mn}{}^{ab} = \omega_m{}^{\hat{0}a}\omega_n{}^{\hat{0}b} - \omega_n{}^{\hat{0}a}\omega_m{}^{\hat{0}b}\,,$$

so

$$\begin{aligned}\Pi_{mn}{}^{AB}{}_{A'B'} &= \tfrac{1}{2}(\pi_{mA'}{}^{A} + \bar{\pi}_m{}^{A}{}_{A'})(\pi_{nB'}{}^{B} + \bar{\pi}_n{}^{B}{}_{B'}) - (m \leftrightarrow n)\\ &= \kappa_m{}^{A}{}_{A'}\kappa_n{}^{B}{}_{B'} - \kappa_n{}^{A}{}_{A'}\kappa_m{}^{B}{}_{B'}\\ &= \varepsilon_{A'B'}\tfrac{1}{2}\Pi_{mn}{}^{AB}{}_{C'}{}^{C'} + \varepsilon^{AB}\tfrac{1}{2}\Pi_{mn}{}^{C}{}_{CA'B'}\\ &= \varepsilon_{A'B'}\tfrac{1}{2}(\kappa_m{}^{A}{}_{C'}\kappa_n{}^{BC'} - \kappa_n{}^{A}{}_{C'}\kappa_m{}^{BC'})\\ &+ \varepsilon^{AB}\tfrac{1}{2}(\kappa_m{}^{C}{}_{A'}\kappa_{nCB'} - \kappa_n{}^{C}{}_{A'}\kappa_{mCB'})\,,\end{aligned}$$

see [13]. Therefore

$$\begin{aligned}\Pi_{mn}{}^{AB} &= \tfrac{1}{2}\left(\kappa_m{}^{A}{}_{C'}\kappa_n{}^{BC'} - \kappa_n{}^{A}{}_{C'}\kappa_m{}^{BC'}\right),\\ \bar{\Pi}_{mnA'B'} &= \tfrac{1}{2}\left(\kappa_m{}^{C}{}_{A'}\kappa_{nCB'} - \kappa_n{}^{C}{}_{A'}\kappa_{mCB'}\right).\end{aligned}$$

This should be substituted in Eq. (36).

The vanishing of the covariant derivative of the soldering form,

$$D_n\tilde{\vartheta}^n_a = \partial_n((\vartheta)\vartheta^n_a) + (\vartheta)\omega_n{}^{bc}\eta_{ca}\vartheta^n_b \approx 0\,,$$

(Ashtekar's convention of writing $(\vartheta)X = (\sigma)X = \tilde{X}$ for the densitized — that is, multiplied with the determinant — quantities is used) is by multiplication with the (constant) Pauli matrices translated into

$$\begin{aligned}0 &\approx \partial_n\tilde{\sigma}^n{}_A{}^{A'} + (\sigma)\left(\omega_n{}^{BD}\varepsilon_{B'D'} + \bar{\omega}_{nB'D'}\varepsilon^{BD}\right)\varepsilon_{AD}\varepsilon^{A'D'}\sigma^n{}_B{}^{B'}\\ &= \partial_n\tilde{\sigma}^n{}_A{}^{A'} + \omega_n{}^{B}{}_A\tilde{\sigma}^n{}_B{}^{A'} + \bar{\omega}_{nB'}{}^{A'}\tilde{\sigma}^n{}_A{}^{B'} =: D_n\tilde{\sigma}^n{}_A{}^{A'}\,, \qquad (37)\end{aligned}$$

where the covariant derivative in the product bundle is defined in the previous equation. The connection is seen to split into an SU(2) and an SU(2)′ part.

The symmetry of the second fundamental form is expressed in the Hamiltonian formulation by the constraint

$$\omega_n{}^{\hat{0}d}(\eta_{db}\vartheta^n_a - \eta_{da}\vartheta^n_b) \approx 0\,.$$

This is equivalent to

$$\omega_n{}^{\hat{0}d}\eta_{db} - \omega_m{}^{\hat{0}d}\eta_{da}\vartheta^m_b\vartheta^a_n \approx 0\,,$$

and it translates to

$$\kappa_n{}^D{}_{D'}\varepsilon_{DB}\varepsilon^{D'B'} - \kappa_m{}^D{}_{D'}\varepsilon_{DA}\varepsilon^{D'A'}\sigma^m{}_B{}^{B'}\sigma^{nA}{}_{A'}$$
$$= \kappa_{nB}{}^{B'} - \kappa_{nA}{}^{A'}\sigma^m{}_B{}^{B'}\sigma_n{}^A{}_{A'} \approx 0\,,$$

or, multiplied by $\sigma^n{}_{B'}{}^C$:

$$\kappa_{nB}{}^{B'}\sigma^n{}_{B'}{}^C - \kappa_{nB'}{}^C\sigma^n{}_B{}^{B'} \approx 0\,. \tag{38}$$

The diffeomorphism constraint,

$$\mathcal{C}_i = -2(\vartheta)\vartheta^m_b\left(D_m\omega_i{}^{\hat{0}b} - D_i\omega_m{}^{\hat{0}b}\right)$$

it is easily seen to become

$$\mathcal{C}_i = -2(\sigma)\sigma^m{}_B{}^{B'}\left(D_m\kappa_i{}^B{}_{B'} - D_i\kappa_m{}^B{}_{B'}\right), \tag{39}$$

where the covariant derivative is the one encountered before in Eq. (37).

In the final part of this paper these constraints are written in a form analogous to Ashtekar's formulation ([2]). He writes the constraints as

$$H = \mathcal{F}_{mn}{}^A{}_B\tilde{\sigma}^{mB}{}_C\tilde{\sigma}^{nC}{}_A\,,$$
$$\mathcal{C}_i = \mathcal{F}_{in}{}^A{}_B\tilde{\sigma}^{nB}{}_A\,,$$
$$\mathcal{D}_n\tilde{\sigma}^{nA}{}_B \approx 0\,.$$

Here $\mathcal{D}$ and $\mathcal{F}$ are a connection form and its curvature, respectively. The first equation describes the Hamiltonian constraint, the second the diffeomorphism constraint. The third one is related to the symmetry of the second fundamental form. The covariant constancy of the (densitized) soldering form is achieved by Ashtekar's choice of a connection D. The connection $\mathcal{D}$ and its curvature $\mathcal{F}$ are defined by

$$\mathcal{D}_n = D_n + [A_n, .]\,, \quad A_n = \omega_n + \frac{i}{\sqrt{2}}\pi_n\,.$$

In an analogous way I define a covariant derivative on the SU(2)⊗SU(2)′ -bundle by

$$\mathcal{D}_nX^A{}_{B'} = \partial_nX^A{}_{B'} + \omega_n{}^A{}_CX^C{}_{B'} + \bar{\omega}_n{}^{C'}{}_{B'}X^A{}_{C'}$$
$$+ \frac{i}{\sqrt{2}}\left(\kappa_n{}^A{}_{C'}\delta^{C'}_C X^C{}_{B'} - \kappa_n{}^C{}_{B'}\delta^{C'}_C X^A{}_{C'}\right).$$

(Recall $G^{A'}{}_A = \frac{1}{\sqrt{2}}\delta^{A'}_A$.) The connection form is

$$\Gamma_n{}^{AB}{}_{A'B'} = \omega_n{}^{AB}\varepsilon_{A'B'} + \bar{\omega}_{nA'B'}\varepsilon^{AB} + \frac{i}{\sqrt{2}}\left(\kappa_n^{AB}\varepsilon_{A'B'} - \kappa_{nB'A'}\varepsilon^{AB}\right).$$

One should realize here that the splitting of the connection form into an SU(2) and an SU(2)′ part is always possible.

The associated curvature is

$$\begin{aligned}\mathcal{F}_{mn}{}^{AB}{}_{A'B'} &= \partial_m\Gamma_n{}^{AB}{}_{A'B'} + \Gamma_m{}^{AC}{}_{A'C'}\varepsilon_{CD}\varepsilon^{C'D'}\Gamma_n{}^{DB}{}_{D'B'} - (m \leftrightarrow n)\\ &= R_{mn}{}^{AB}{}_{A'B'} + \Big[\frac{i}{\sqrt{2}}\big(\partial_m(\kappa_n{}^{AB}\varepsilon_{A'B'} - \kappa_{nB'A'}\varepsilon^{AB}\big)\\ &\quad +(\omega_m{}^{AC}\varepsilon_{A'C'} + \bar{\omega}_{mA'C'}\varepsilon^{AC})\varepsilon_{CD}\varepsilon^{C'D'}(\kappa_n{}^{DB}\varepsilon_{D'B'} - \kappa_{nB'D'}\varepsilon^{DB}))\\ &\quad -\tfrac{1}{2}(\kappa_m{}^{AC}\varepsilon_{A'C'} - \kappa_{mC'A'}\varepsilon^{AC})\varepsilon_{CD}\varepsilon^{C'D'}(\kappa_n{}^{DB}\varepsilon_{D'B'} - \kappa_{nB'D'}\varepsilon^{DB})\\ &\quad -(m \leftrightarrow n)\Big]\\ &= (R_{mn}{}^{AB} + \tfrac{1}{2}\kappa_{[m}{}^{AD'}\kappa_{n]D'}{}^{B})\varepsilon_{A'B'} + (\bar{R}_{mnA'B'} + \tfrac{1}{2}\kappa_{[m}{}^{D}{}_{A'}\kappa_{n]B'D})\varepsilon^{AB}\\ &\quad + \frac{i}{\sqrt{2}}D_{[m}(\kappa_{n]}{}^{AB}\varepsilon_{A'B'} - \kappa_{n]B'A'}\varepsilon^{AB}).\end{aligned}$$

To relate this to the constraints, realize that

$$\mathcal{D}_n\tilde{\sigma}^{nA}{}_{A'} = D_n\tilde{\sigma}^{nA}{}_{A'} + \frac{i}{\sqrt{2}}(\kappa_n{}^{A}{}_{C}\sigma^{nC}{}_{A'} - \kappa_n{}^{C}{}_{A'}\sigma^{nA}{}_{C}).$$

So $\mathcal{D}_n\tilde{\sigma}^n{}_A{}^{A'} \approx 0$ and $D_n\tilde{\sigma}^n{}_A{}^{A'} \approx 0$ together are equivalent to (37) and (38).

To obtain the other constraints in a formulation analogous to Ashtekar's, realize that

$$\begin{aligned}&\tfrac{1}{2}\mathcal{F}_{mn}{}^{AB}{}_{A'B'}(\tilde{\sigma}^m{}_{BC'}\delta^{C'}_A\varepsilon^{A'B'} + \tilde{\sigma}^{mA'C}\delta^{B'}_C\varepsilon_{AB})\\ &= R_{mn}{}^{AB}\tilde{\sigma}^m{}_{BA} + \tfrac{1}{2}\kappa_n{}^{AD}\kappa_{mD}{}^{B}\tilde{\sigma}^m{}_{BA} - \tfrac{1}{2}\kappa_m{}^{AD}\kappa_{nD}{}^{B}\tilde{\sigma}^m{}_{BA} + \text{c.c.}\\ &+ \frac{i}{\sqrt{2}}\Big[D_{[m}(\kappa_{n]}{}^{AB})\tilde{\sigma}^m{}_{BA} - D_{[m}(\kappa_{n]B'A'})\tilde{\sigma}^{mA'B'}\Big].\end{aligned}$$

The first line of this vanishes because of the vanishing of trace $\tilde{\sigma}^m$ and constraints (37) and (38). Taking this into account this expression is equivalent to the diffeomorphism constraint:

$$\approx -i\sqrt{2}(\tilde{\sigma}^m{}_A{}^D D_{[m}\kappa_{n]}{}^A{}_D).$$

Finally, the Hamiltonian constraint is given by

$$\begin{aligned}
&\tfrac{1}{2}\mathcal{F}_{mn}{}^{AB}{}_{A'B'}\big(\tilde{\sigma}^m{}_B{}^{C'}\tilde{\sigma}^n{}_{C'A}\varepsilon^{A'B'} + \tilde{\sigma}^{mB'}{}_C\tilde{\sigma}^{nCA'}\varepsilon_{AB}\big) \\
&= (\sigma)H + \frac{i}{2\sqrt{2}}\big(D_{[m}\kappa_{n]}{}^{AB}\varepsilon_{A'B'} - D_{[m}\kappa_{n]B'A'}\varepsilon^{AB}\big) \\
&\times \big(\tilde{\sigma}^m{}_B{}^{C'}\tilde{\sigma}^n{}_{C'A}\varepsilon^{A'B'} + \tilde{\sigma}^{mB'}{}_C\tilde{\sigma}^{nCA'}\varepsilon_{AB}\big)\,.
\end{aligned}$$

The last term vanishes because of the diffeomorphism constraint and the symmetry of the second fundamental form, since $\tilde{\sigma}^{nA'}{}_A$ is covariantly constant. So

$$\tfrac{1}{2}\mathcal{F}_{mn}{}^{AB}{}_{A'B'}\big(\tilde{\sigma}^m{}_B{}^{C'}\tilde{\sigma}^n{}_{C'A}\varepsilon^{A'B'} + \tilde{\sigma}^{mB'}{}_C\tilde{\sigma}^{nCA'}\varepsilon_{AB}\big) \approx (\sigma)H\,. \quad (40)$$

In this way all four constraints are cast in a form analogous to Ashtekar's. Here I list them once again:

$$\begin{aligned}
D_n\tilde{\sigma}^{nA}{}_{A'} &\approx 0\,, \\
\mathcal{D}_n\tilde{\sigma}^{nA}{}_{A'} &\approx 0\,, \\
\tfrac{1}{2}\mathcal{F}_{mn}{}^{AB}{}_{A'B'}\big(\tilde{\sigma}^m{}_{BC'}\delta^{C'}_A\varepsilon^{A'B'} + \tilde{\sigma}^{mA'C}\delta^{B'}_C\varepsilon_{AB}\big) &\approx 0\,, \\
\tfrac{1}{2}\mathcal{F}_{mn}{}^{AB}{}_{A'B'}\big(\tilde{\sigma}^m{}_B{}^{C'}\tilde{\sigma}^n{}_{C'A}\varepsilon^{A'B'} + \tilde{\sigma}^{mB'}{}_C\tilde{\sigma}^{nCA'}\varepsilon_{AB}\big) &\approx 0\,.
\end{aligned}$$

In all these constraints the "self-dual" and "antiself-dual" parts may be separated, that is, in the SU(2)⊗SU(2)'-bundle all forms used in this theory can be projected onto the first or second factor without loss of generality. Since the Bianchi identity was used by Ashtekar to enable him to work in the self-dual part only, it seems to be a reason for this behaviour, but further work is needed to understand completely the rôle played by Bianchi identity in this theory.

5. Summary and conclusions

Viewing gravity as a gauge theory, it is natural to consider the dynamical variables to be the soldering form and the connection form. This viewpoint recently gave progress in the field, especially on the road toward quantization. It was therefore of interest to work out a canonical formulation *à la* Arnowitt, Deser, and Misner, using these variables, instead of the metric, as they did. It was clear that the theory obtained should bear some relation to Ashtekar's work, since he did an analogous analysis in a spinor bundle.

The Hamiltonian theory of gravitation was worked out in a gauge in which the dynamical part of the connection $L^{\hat{0}a}$ does not appear in the equations of motion. On the constraint surface this theory is equivalent to the one obtained by Arnowitt, Deser, and Misner. This was shown by regaining the vacuum ADM-equations.

The equivalence with Ashtekar's formulation was shown by using the special case of spinor space as inner space, soldered to TM. The representation of the Lorentz group in this space $\mathrm{End}(\mathbb{C}^2, \bar{\mathbb{C}}^2)$ decomposed into a representation of $SU(2)\otimes SU(2)'$, therefore the associated principal bundle is an $SU(2)\otimes SU(2)'$-bundle, a product bundle where the first factor contains variables with unprimed indices (the "self-dual" part) and the second factor primed ones (the "antiself-dual" part). This theory, projected onto the first part, reduces to Ashtekar's formulation of gravity. In the last part of the paper the constraints of the theory were written down in that special case, obtaining a form analogous to Ashtekar's.

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