Vol. 23(1992) *ACTA PHYSICA POLONICA B* No 5

BALIAN–BLOCH REPRESENTATION, SEMICLASSICAL EXPANSIONS AND BOREL SUMMABILITY IN ONE-DIMENSIONAL QUANTUM MECHANICS*

S. Giller

Institute of Physics, University of Łódź
Pomorska 149/152, 90–236 Łódź, Poland

(Received November 26, 1991)

To memory of Itka

A rigorous theory of semiclassical limit of the one-dimensional Schrödinger equation based on the Balian–Bloch representation is developed. It is shown that for a large class of potentials a global fundamental solution to the Schrödinger equation can be constructed which can be Lapalace transformed with respect to $\hbar^{-1}$ (or to some other relevant variable). This global solution has a definite asymptotic series expansion for $\hbar^{-1} \to +\infty$. The series is shown to be Borel summable to the global solution itself. Primitive coefficients — some other quantities basic for the quantum one-dimensional theory — are shown to be Borel summable, too. An efficient technique is developed to show both the analytic properties and the Borel summability of energy levels for a large class of potentials. The technique combines the analytic properties of the Stokes graphs and the primitive coefficient identities and is used together with the Bender–Wu method to determine the large order behaviour of the semiclassical series coefficients. The method is extended to a class of perturbing potentials which admit semiclassical treatment. The cubic- quartic single- and double-well potentials are studied in details. Our approach is generalized to $\hbar^{-1}$-dependent potentials.

PACS numbers: 02.30.+g, 03.65.Ge, 03.65.Sq

1. Introduction

The Balian–Bloch representation [1] has been introduced by its inventors into the quantum mechanics as an expected powerful tool for advanced

* Supported by CPBP 01.09

and refined applications of the semiclassical (JWKB) approximation. The heart of the method is to consider the Laplace transforms of physical quantities rather than the quantities themselves. As a transformation variable the inverse of the Planck constant $\hbar^{-1}$ is then used whilst the conjugate variable is the action S. The main advantages of such an approach are the following:

(i) a direct relation to the Feynman path integral method. Namely, the inverse Laplace transformation can be considered as the last integration in the Feynman path integral with the action chosen as the one of the integration variables. This allow to interpret the method in terms of classical paths both real and complex;

(ii) a possibility to exploit the firmly established theory of the Laplace transformation; and

(iii) a possibility to represent calculated quantities not only by the dominant, conventional JWKB contributions but also by the additional contributions having subdominant character in comparison with the former ones.

The last property of the Balian–Bloch representation is very important both from the technical and from the practical points of view and means that this representation can be used as a tool for obtaining a "complete" semiclassical description of relevant physical quantities. The way of using the representation to this end has been demonstrated by its inventors in their original paper [1] as well as by Balian *et al.* [2].

Closely related to the Balian–Bloch approach to the semiclassical limit of quantum mechanics is the problem of resummation of typical semiclassical series. It is well known, due to the pioneering paper of Bender and Wu [5], that these series are in general divergent but can, in many cases, be resumed and the Borel resummation method appears frequently to be powerful in obtaining desired sums. One of the ways of establishing the applicability of the Borel resummation technique is the rate-of-growth investigation of the late asymptotic expansion coefficients. In particular, this method was applied effectively to investigate the resummation problem of divergent perturbation series. It was found that many such series showed factorial low of growth of their coefficients *i.e.* the property which is necessary for the Borel resumming of the series [4–15, 20–21, 31]. It was also found, however, that there were perturbation series expansions, coefficients of which deviated from the factorial rule growing much faster [35–39]. These other cases call for more sophisticated summation tools [37–39].

All these results, despite the very spectacular character of some of them, cannot, however, pretend to be considered as a complete theory of the semiclassical/perturbative expansions even in the simplest case of the one-dimensional quantum mechanics. Specifically, the problems of analytical and asymptotic properties of relevant physical quantities (wave functions, Green functions, energy levels *etc.* in the complex planes of the asymptotic series expansion parameters (so important for the asymptotic series resummation problem) still call for some systematic and successful methods of their investigations.

In this paper we are going to formulate the corresponding theory just in the case of the one-dimensional quantum mechanics.

Our limitation to the one-dimensional quantum mechanics is not essential but is justified rather by the relative simplicity of the case. In fact, the rigorous and powerful methods used to investigate the one-dimensional case can be also applied effectively to the relevant quantum-mechanical problems having any finite degrees of freedom [43]. Although such an application appears to be a highly nontrivial procedure containing the one-dimensional case, as its exceptional simplification, the main ideas and notions developed in the corresponding one-dimensional theory contribute widely to the general case. Therefore, the one-dimensional theory developed in this paper can be considered as an introduction to this much more complicated general theory [43].

In our investigations we have used the wave function formalism. The main results we have obtained are the following:

1. for a large class of potentials there are sets of the wave functions corresponding to the problem and admitting:
   a. the Balian–Bloch representations
   b. semiclassical expansions
   c. Borel summability of the corresponding semiclassical series expansions to the wave functions themselves;
2. the quantized energy level semiclassical expansions appearing in the considered problems are Borel summable to the energies themselves;
3. there is a large class of perturbing potentials admitting semiclassical treatment and producing that way Borel summable perturbation series;

In our considerations we follow closely the theory of the Laplace–Borel transformations and related Borel summability procedure (see [40] for an outline of the theory). This theory to be applicable demands quantities having definite properties in the complex λ-plane. We were nice surprised to find out that all the desired properties are carried by sets of the fundamental solutions to the one-dimensional Schrödinger equation we have introduced and considered in our earlier papers [27–28].

Let us stress at this moment that the full advantage of the Balian–Bloch representation can be exploited only if the relevant quantum theory is formulated as a complex theory *i.e.* as a quantum theory in complex space of the space variables and other parameters (λ, E (energy), *etc.*). Such a point of view has been adopted in this paper where the stationary one-dimensional

Schrödinger equation is treated as genuinely defined in a complex space of the variable x, the parameter λ and the energy E. Such an approach allows to built a complete semiclassical theory in the case of the one-dimensional quantum mechanics. An unique objection is related to the inherently singular nature of the global fundamental solutions themselves and is expressed as the existence of turning points *i.e.* singular points of the global fundamental solutions (see, however, [29] where the Bargman representation is used to remove this apparent disadvantage of the fundamental solutions). On the other hand, it is just this singular structure of the theory which allows to construct relevant extended JWKB approximations for considered quantities [41].

Another crucial point of the paper is the full use of the analytic properties of the Stokes graphs. The usefulness of the graphs is demonstrated in all sections of the paper. We have borrowed the idea of using the Stokes graphs in this way from Voros [25].

The primitive canonical coefficients introduced and used successfully in our earlier papers are the third "working" element of our method. Each one-dimensional quantum mechanical problem can be solved in terms of them. Their well established analytical properties as a function of the complex variables x, λ and E allow to develop a relatively simple technique of determining both the analytic and the asymptotic properties of energy levels considering as functions of λ.

The last basic element of the techniques used in the paper are the abundance of identities satisfied by the primitive canonical coefficients (see Appendix A). They are these identities which allow to avoid a detailed analysis of the semiclassical wave functions so characteristic for the approach in Bender and Wu's [5, 30] and other papers [21–23, 32].

The properties of our approach enumerated above used altogether make our formulation of the semiclassical theory as compact and efficient as elegant.

The organization of the paper is the following. In the next section the global solution to the Schrödinger equation is constructed and its Laplace–Borel transformation is defined. The modified Borel summation method (similar in its idea to that of Cruchfield [24]) is applied to the global solution as well. It is shown also there that the global solution semiclassical expansion is Borel summable in definite sectors of the Stokes graph. In the same section the primitive canonical coefficients are introduced and their Laplace transformations are discussed.

In Section 3 a general discussion of the quantization of energy levels within the framework of our formalism is performed and the strategy of establishing the energy level asymptotic properties in the λ-plane is formulated.

In Section 4 the method worked out in Section 3 is applied to discuss the existence of Laplace transformations and Borel functions for the anharmonic oscillator energy levels regarded as functions of the complex variable $\lambda = \beta^{-1}$ with β being the anharmonicity parameter. The considered cubic-quartic potential is of the form: $U(x,\lambda) = \lambda V(x\lambda^{-1/2})$ where $V(x) = (x^2 + 1/4)^2 - bx^3$. It allows to study simultaneously its both physical channels *i.e.* its single- and double-well versions. In the single-well case the results of Bender and Wu [5, 30] (see also [26]) are reproduced. In the double-well one it is shown that when the wells have unequal depths ($b \neq 0$) then the energy levels are Borel summable but for b approaching zero (*i.e.* for the symmetric double-well configuration) the Borel transformation becomes singular. The modified Borel sum can be successfully used in such singular case. Nevertheless, the Borel summation technique is shown to provide also the solution to this Borel unsummable case.

In Section 5 a generalization of the Balian–Bloch representation to a class of λ-dependent potentials is developed. It is shown that the main results obtained for the cubic-quartic oscillator remain essentially unchanged. In particular it is shown that for the single-well potentials their energy levels $E(\lambda)$ are holomorphic in the cut λ-plane for $|\arg\lambda| > \pi$ and λ large enough and therefore, Borel summable. It is also argued that these results can be extended to a family of perturbing potentials admitting semiclassical treatment. Finally, we conclude with Section 6.

2. Laplace transforms of global fundamental solution and of primitive canonical coefficients

2.1. A global solution to the Schrödinger equation and its Laplace transform

Consider the Schrödinger equation:

$$\psi''(x,\lambda,E) - \lambda^2 q(x,E)\psi(x,\lambda,E) = 0\,, \tag{2.1}$$

where $q(x,E) = 2m\big(V(x) - E\big)/\hbar^2$. For convenience, we have introduced into (2.1) a formal variable λ. It can play a role of varying $\hbar^{-1}$ or of any other suitable variable which can enter Schrödinger equation (see, for example, [27]). Both λ and E take on complex values. Both for definiteness and simplicity we shall assume $V(x)$ to be a polynomial of any degree $n \geq 1$ with all its zeros being simple. A global Stokes line pattern corresponding to a given $q(x,E)$ depends, of course, on E and n. However, we select out from the pattern only a part consisting at least of three neighbouring sectors 1, 2 and p (see [27–28] for necessary definitions). This is shown in Fig. 1.

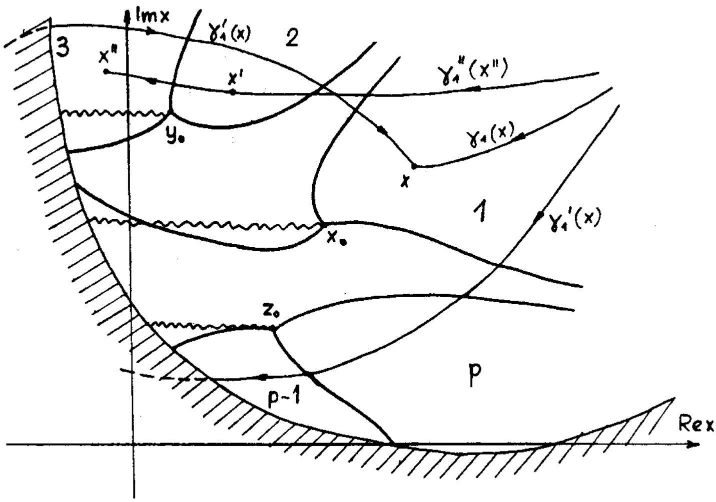

Fig. 1. The Stokes graph for a general polynomial potential

Then, we can associate with the sector 1 the following fundamental solution $\psi_1^\sigma(x,\lambda)$ to (2.1):

$$\psi_1^\sigma(x,\lambda,E) = q^{-\frac{1}{4}}(x,E)\exp\left(\sigma\lambda\int_{x_0}^{x} q^{\frac{1}{2}}(y,E)dy\right)\chi_1^\sigma(x,\lambda,E)$$

$$\mathrm{Re}\left(\sigma\int_{x_0}^{x} q^{\frac{1}{2}}(y,E)dy\right) < 0, \quad x \in \text{sector } 1, \quad \sigma = \pm 1$$

$$q(x_0,E) = 0 \tag{2.2}$$

and with the "amplitude factor" $\chi_1^\sigma(x,\lambda,E)$ given by the following functional series:

$$\chi_1^\sigma(x,\lambda,E) = 1 + \sum_{n\geq 1}\left(\frac{\sigma}{2\lambda}\right)^n \int_{\gamma_1^\sigma(x)} dy_1 \ldots \int_{\gamma_1^\sigma(y_{n-1})} dy_n \omega(y_1)\cdots\omega(y_n)$$
$$\times\left(1-e^{-2\lambda\xi(x,y_1)}\right)\left(1-e^{-2\lambda\xi(y_1,y_2)}\right)\ldots\left(1-e^{-2\lambda\xi(y_{n-1},y_n)}\right), \tag{2.3}$$

where:

$$\omega(y) = \frac{1}{4}\left(\frac{q''(y)}{q^{3/2}(y)} - \frac{5}{4}\frac{q'^2(y)}{q^{5/2}(y)}\right)$$

and

$$\xi(x_0,x) = -\sigma \int_{x_0}^{x} q^{\frac{1}{2}}(y,E)dy \tag{2.4}$$

and where the dependence of ω, q, ξ *etc.* on E is also understood. (From now on and up to the end of this section we shall assume the energy E to be fixed at some complex value and the obvious dependence on it of different quantities discussed below will not be marked explicitly.) We shall also choose $\sigma = -1$ in (2.2) and (2.3), by assumption.

Similar fundamental solutions can be associated with the sectors 2 and p (as well as with the remaining $p-2$ sectors of the Stokes graph corresponding to the case) but with corresponding signatures σ_2 and σ_p alternated *i.e.* $\sigma_2 = \sigma_p = 1$. When λ is real and positive, all p fundamental solutions are defined by (2.2) – (2.4) independently of each other constituting in this way a set of local solutions to the Schrödinger equation. However, a similar set of fundamental solutions can be constructed for any complex $\lambda \neq 0$ accompanied, of course, with suitably changed patterns of the corresponding Stokes graphs (deformed properly in comparison with its initial form given by Fig. 1).

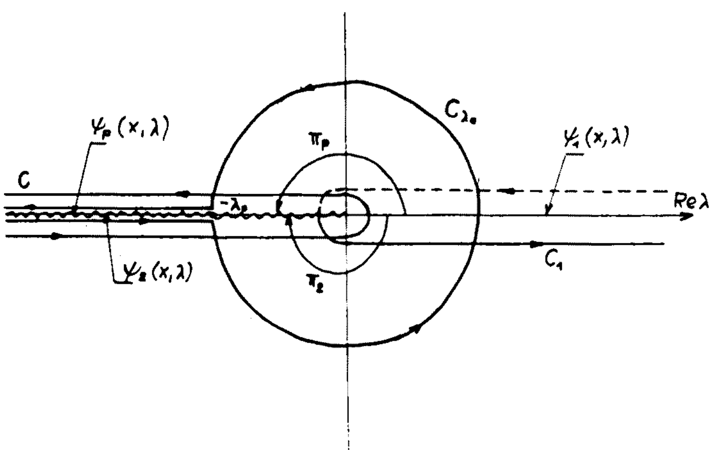

Fig. 2. The cut λ-plane corresponding to the global solution $\psi(x,\lambda)$

On the other hand, any such set of fundamental solutions can be obtained directly from the one constructed for real $\lambda > 0$ and corresponding to the Stokes graph of Fig. 1 by analytic continuation in λ of both the solutions $\psi_k^{\sigma_k}(x,\lambda)$, $k = 1,\ldots,p$, and the graph of Fig. 1 itself along suitably chosen path in the complex λ-plane. Moreover, beginning with any of the solutions $\psi_k^{\sigma_k}(x,\lambda)$ $k = 1,\ldots,p$, with $\lambda > 0$ and continuing it with λ along suitably chosen path one can obtain all the remaining solutions. In particular, if, by convention, the solution $\psi_1^-(x,\lambda)$ corresponds to the positive real axis of the λ-plane then the solutions $\psi_2^+(x,\lambda)$ and $\psi_p^+(x,\lambda)$ are defined correspondingly below and above the negative real axis of the λ-plane cut along this axis (see Fig. 2) and can be obtained by analytic continuations of $\psi_1^-(x,\lambda)$ along the paths π_2 and π_p, correspondingly. It is now obvious that continuing $\psi_1^-(x,\lambda)$ with λ along π_2 (*i.e.* clockwise) and encircling the point $\lambda = 0$ by the angle $-p \cdot \pi$ we have to come back again to the sector 1 *i.e.* again to the solution $\psi_1^-(x,\lambda)$. However, this coincidence is not exact and the final solution differs from $\psi_1^-(x,\lambda)$ by a factor $(-i)^n \exp(-\lambda \oint\limits_K q^{1/2}(x,E)dx)$ (*i.e.* $\psi_1^{\rm cont}(x,\lambda) \equiv (-i)^n \exp(-\lambda \oint\limits_K q^{1/2}dx)\psi_1^-(x,\lambda))$ where the closed contour K encircles (anticlockwise) all n zeros of q in the cut x-plane (*i.e.* all these zeros stay inside the contour K). The correctness of the last statement follows from the fact that in the case of $\psi_1^{\rm cont}(x,\lambda)$ the integration path $\gamma_1'(x)$ in the corresponding formula (2.3) runs in the x-plane as it is shown in Fig. 1 being the final form of the deformation of $\gamma_1(x)$ in the analytic continuation procedure in λ. However, since x is fixed the JWKB factor $q^{-1/4}\exp(-\lambda\xi(x_0,x))$ in (2.2) does not change when it is continued with λ. Contrary to that it changes by $i^n \exp(\lambda \oint_K q^{1/2}dx)$ when $\psi_1^-(x,\lambda)$ is continued to x along $\gamma_1'(x)$. But $\psi_1^-(x,\lambda)$ continued with x is always the same at x independently of the path which it is continued along. Therefore, these two analytic continuations (*i.e.* with x and with λ) do not commute but are related as it was stated above.

A conclusion which follows from the above discussion is, therefore, that for fixed x the point $\lambda = 0$ is a branch point for $\psi_1^-(x,\lambda,E)$ considered as a function of the two complex variables x an λ. This branch point is of the logarithmic type.

However, a much more general conclusion is that (2.2) constitutes, in fact, a global solution to the Schrödinger equation defined on the complex Riemann surface parameterized by x and λ. We shall denote this solution by $\psi(x,\lambda)$. It has the following properties:

(i) it is a holomorphic function of x for any fixed $\lambda \neq 0$;

(ii) it is, for any fixed x being not a zero for $q(x)$, a holomorphic function of λ on the infinitely many sheeted Riemann surface with the point $\lambda = 0$ removed;

(iii) it vanishes (approaching its JWKB approximation $q^{-1/4}\exp(-\lambda\xi(x_0,x))$ when $x \to \infty$ in any direction of the x-plane for which $\mathrm{Re}(-\lambda\xi(x_0,x)) \to -\infty$;

(iv) it approaches its JWKB approximation $q^{-1/4}\exp(-\lambda\xi(x_0,x))$ when $\lambda \to \infty$ and $\lambda \in \Lambda(x)$, where $\Lambda(x)$ is such a set of λ's that for every

$\lambda \in \Lambda(x)$ there exists a canonical set $K_{D(\lambda)}$ which contains x *i.e.* $x \in \prod_{\lambda \in D(\lambda)} K_{D(\lambda)}$ (see [27, 28] for a necessary information).

In particular, if x stays in sector 1 of Fig. 1 then it belongs to each of the canonical domains K_{D_1}, K_{D_2} and K_{D_p} corresponding to the solutions ψ_1^-, ψ_2^+ and ψ_p^+, respectively. Therefore, x belongs then also to each canonical domain $K_{D_i}(\lambda)$ which arises from the domain K_{D_i}, $i = 1, 2, p$ as its continuous deformation when moves to any point of the Riemann sheet depicted in Fig. 2. The canonical domain $K_{D_i(\lambda)}$, $i = 1, 2, p$, corresponds, of course, to the solution $\psi(x, \lambda)$ defined at the point λ. It follows further from the property *(iv)* above, that $\psi(x, \lambda)$ approaches its JWKB approximation when $\lambda \to \infty$ on the Riemann sheet shown in Fig. 2, since the amplitude factor $\chi(x, \lambda)$ approaches then unity. Therefore, we can define for Re $s < 0$ (with x still kept in sector 1) the following Laplace transform of the amplitude factor $\chi(x, \lambda)$:

$$\tilde{\chi}(x, s) = \frac{1}{2\pi i} \int_C e^{2\lambda s} \chi(x, \lambda) d\lambda \,, \tag{2.5}$$

where the integration contour C is shown in Fig. 2. (The factor 2 in the exponential in (2.5) is introduced for convenience.) Let us note, however, that the discontinuity of $\chi(x, \lambda)$ on the cut in Fig. 2 (*i.e.* the difference $\chi(x, \lambda e^{i\pi}) - \chi(x, \lambda e^{-i\pi}) \equiv \chi_p(x, \lambda) - \chi_2(x, \lambda)$, $\lambda > 0$) can be easily determined by noticing that the fundamental solution $\psi_1^-(x, \lambda)(\equiv \psi(x, \lambda)$ for $\lambda > 0)$ can be represented by the following linear combination of $\psi_2^+(x, \lambda)$ and $\psi_p^+(x, \lambda)$:

$$\psi_1^-(x, \lambda) = i \frac{\psi_2^+(x, \lambda) - \psi_p^+(x, \lambda)}{\chi_{2 \to p}(\lambda)} \,, \tag{2.6}$$

where $\chi_{2 \to p}(\lambda) \equiv \chi_2^+(\infty_p, \lambda)$ is the value of $\chi_2^+(x, \lambda)$ taken at $x = \infty_p$ when $\chi_2^+(x, \lambda)$ is continued to ∞_p (the infinity point in the sector D_p) along the canonical path $\gamma_{2 \to p}$ (see Fig. 3). From (2.6) and (2.2) it follows then that:

$$\chi_p^+(x, \lambda) - \chi_2^+(x, \lambda) = i \chi_{2 \to p}(\lambda) \chi_1^-(x, \lambda) e^{-2\lambda \xi(x_0, x)}, \tag{2.7}$$

so that (2.5) can be rewritten as:

$$\begin{aligned} \tilde{\chi}(x, s) = &-\frac{1}{2\pi} \int_{\lambda_0}^{\infty} e^{2\lambda(s - \xi(x_0, x))} \chi_{2 \to p}(\lambda) \chi_1^-(x, \lambda) d\lambda \\ &+ \frac{1}{2\pi i} \int_{C_{\lambda_0}} e^{-2\lambda s} \chi(x, \lambda) d\lambda \,, \end{aligned} \tag{2.8}$$

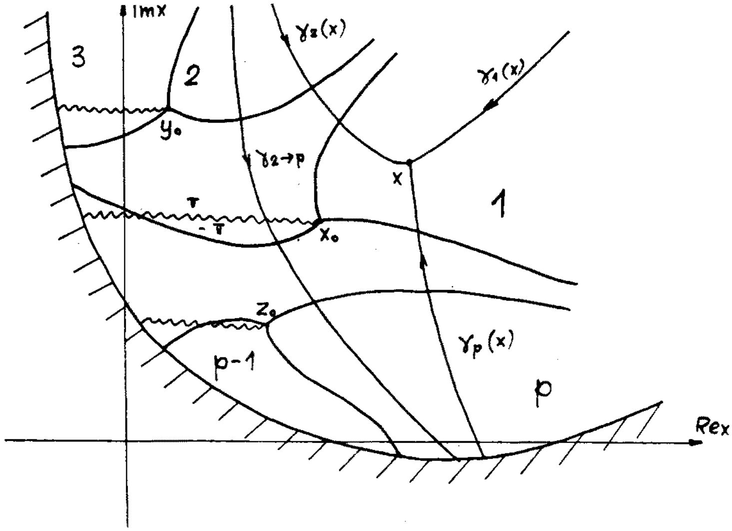

Fig. 3. The integration paths corresponding to the formula (2.6)

where C_{λ_0} is an open contour encircling (anticlockwise) the point $\lambda = 0$ and having both its ends attached to the point $\lambda = -\lambda_0$ on different sides of the cut (see Fig. 2). By the form (2.8) $\tilde{\chi}(x,s)$ is defined in the half-plane $\operatorname{Re} s < \operatorname{Re}\xi(x_0,x)$ is positive $\tilde{\chi}(x,s)$ appears to be, in fact, the Borel transform of $\chi(x,\lambda)$. It is, however, a good opportunity to show this fact by direct calculation of the large order behaviour of coefficients $\chi_n(x)$, $n \geq 0$, of the asymptotic series expansion for $\chi(x,\lambda)$. Namely, putting:

$$\chi(x,\lambda) \sim 1 + \sum_{n\geq 0} \chi_n(x)\lambda^{-n-1} \tag{2.9}$$

and using the Bender-Wu formula [5] we get:

$$\begin{aligned}\chi_n(x) &= \frac{1}{2\pi}\int_{\lambda_0}^{\infty} e^{-2\lambda\xi(x_0,x)}\chi(x,\lambda)\chi_{2\to p}(\lambda)(-\lambda)^n d\lambda \\ &+ \frac{1}{2\pi i}\int_{C_{\lambda_0}} (\chi(x,\lambda)-1)\lambda^n d\lambda \\ &\sim \frac{1}{2\pi} n!\big(-2\xi(x_0,x)\big)^{-n}\left(1+O\left(\frac{1}{n}\right)\right).\end{aligned} \tag{2.10}$$

Thus, indeed, according to the general properties of the Laplace–Borel transform (see [40], Appendix 1) the Borel series:

$$\sum_{n\geq 0}(-2s)^n \chi_n(x)\frac{1}{n!} \tag{2.11}$$

is convergent in the circle $|s| < |\xi(x_0, x)|$ with $\tilde{\chi}(x, s)$ as its sum. The latter function has at the point $s_0(x) = \xi(x_0, x)$ a singularity closest to the origin. The relevant Borel transformation can be also defined for $\psi(x, \lambda)$ itself to give:

$$\begin{aligned}\tilde{\psi}(x, s) &= q^{-\frac{1}{4}}(x)\tilde{\chi}(x, s + \tfrac{1}{2}\xi(x_0, x)) \\ &\equiv \frac{1}{2\pi i} \int_C e^{-2\lambda s} \psi(x, \lambda) d\lambda\,. \end{aligned} \tag{2.12}$$

Both the functions $\tilde{\chi}(x, s)$ and $\tilde{\psi}(x, s)$ are holomorphic for $\operatorname{Re} s < \operatorname{Re} \xi(x_0, x)$ and $\operatorname{Re} s < \frac{1}{2} \operatorname{Re} \xi(x_0, x)$ respectively (when x is still kept in sector 1).

The transformations (2.5) and (2.12) can be inverted to give:

$$\chi(x, \lambda) = 2 \int_{\tilde{C}} e^{2\lambda s} \tilde{\chi}(x, s) ds \tag{2.13}$$

and

$$\psi(x, \lambda) = 2 \int_{\tilde{C}'} e^{2\lambda s} \tilde{\psi}(x, s) ds\,, \tag{2.14}$$

where the contour $\tilde{C}$ starts at the infinity $\operatorname{Re}(\lambda s) = -\infty$ and ends at $s = 0$ and $\tilde{C}' = \tilde{C} - {}^1\!/_2 \xi(x_0, x)$. Since the contours $\tilde{C}$ and $\tilde{C}'$ can be freely deformed in the half plane $\operatorname{Re} s \leq 0$ the formulae (2.13) and (2.14) define $\chi(x, \lambda)$ and $\psi(x, \lambda)$ in the whole sheet shown in Fig. 2 excluding the points of the negative half of the real axis.

We can also express $\chi(x, \lambda)$ and $\psi(x, \lambda)$ by the inverse Laplace transformations integrating in s-plane along contours with their both ends anchored at the infinity. (We shall call this kind of transformations — modified Laplace–Borel transformations.) Namely defining:

$$\Pi_\varepsilon(x, s) = \frac{1}{2\pi i} e^{-\varepsilon s} \int_{\tilde{C}} e^{\varepsilon s'} \chi(x, s)(s' - s)^{-1} ds' \tag{2.15}$$

with (any) fixed $\varepsilon > 0$ and with $\tilde{\chi}(x, s)$ as the discontinuity of $\Pi_\varepsilon(x, s)$ across the cut along C we get:

$$\psi(x, \lambda) = 2 \int_{K_{\tilde{C}'}} e^{2\lambda s} \tilde{\Pi}_\varepsilon(x, s) ds\,, \tag{2.16}$$

where $\widetilde{\Pi}_\varepsilon(x,s) = q^{-1/4}\Pi_\varepsilon(x, s+\xi(x_0,x))$ and the contour $K_{\widetilde{C}'}$ encircles the cut along $\widetilde{C}'$ clockwise. The function $\widetilde{\Pi}_\varepsilon(x,s)$ can be also defined directly by:

$$\widetilde{\Pi}_\varepsilon(x,s) = \frac{1}{(2\pi)^2}\int_{C_\varepsilon} e^{-2\lambda s} Ei\Big((\varepsilon-\lambda)(s-\xi(x_0,x))\Big)d\lambda\,, \qquad (2.17)$$

where C_ε is any of the contours C in Fig. 2 which crosses the real axis between $\lambda = 0$ and $\lambda = \varepsilon$ and where $Ei(z)$ is the integral exponential function [42].

Summarizing the above discussion we have shown that for the solution $\psi(x,\lambda)$ with x kept in the sector 1 it is possible to define its Laplace transformation (2.16) with $\widetilde{\Pi}_\varepsilon(x,s)$ being holomorphic in the s-plane cut along $\widetilde{C}$ and having $\widetilde{\psi}(x,s)$ as its discontinuity across the cut.

The following two comments are in order here.

1. The formulae (2.13) and (2.14) are certainly valid for $\operatorname{Re}\lambda > 0$ when the contours $\widetilde{C}$ and $\widetilde{C}'$ stay in the half planes $\operatorname{Re} s < 0$ and $\operatorname{Re} s < -\xi(x_0,x)$ respectively. They can be continued, however, to other domains of the Riemann λ-surface corresponding to $\psi(x,\lambda)$ if accompanied with suitable changes of the variable x. Thus, for example, when continuing x to the sector 2 and deforming the contour C in Fig. 2 into C_1 the formulae (2.5) and (2.12) will then define $\widetilde{\chi}(x,s)$ and $\widetilde{\psi}(x,s)$ in the half planes $\operatorname{Re} s > 0$ and $\operatorname{Re} s > -\xi(x_0,x)$, respectively. On the other hand the inverse formulae (2.13) and (2.14) define then $\chi(x,\lambda)$ and $\psi(x,\lambda)$ in the half plane $\operatorname{Re}\lambda < 0$ with the contour $\widetilde{C}$ in the formulae deformed (anticlockwise) from its position in the left half plane to its new position in the right half of the s-plane. The function $\psi(x,\lambda)$ fulfils then for $\lambda > 0$ the condition: $\psi(x,-\lambda) \equiv \psi_2^+(x,\lambda)$. Possible singularities of $\widetilde{\chi}(x,s)$ and $\widetilde{\psi}(x,s)$ existing in the corresponding half planes $\operatorname{Re} s > 0$ and $\operatorname{Re} s > -\xi(x_0,x)$ when x stays in the sector 1 move to the half planes $\operatorname{Re} s < 0$ and $\operatorname{Re} s < -\xi(x_0,x)$ respectively when x moves to the sector 2.
2. Our proof that the series (2.9) is Borel summable needs only local arguments *i.e.* that the neighbouring solutions ψ_2 and ψ_p are analytic continuations of ψ_1 in λ. No particular properties of the potential are necessary apart of those which provide some "regular" picture of the Stokes graphs *i.e.* those which, roughly speaking, can ensure that the critical forms of the Stokes graphs (see below) are exceptional. Therefore, our limitation to the polynomial potentials seems to be too modest in this respect and, in fact, as we shall argue in Section 6, the results of this section can be extended to a family of λ-dependent potentials

which includes both non-polynomial entire potentials and the singular ones.

2.2. Primitive canonical coefficients and their Laplace transforms

For a given Stokes graph corresponding to fixed λ and E let K_{D_i} and K_{D_j} be any two of its communicating canonical domains (see [27,28] for a definition of the latter notion) corresponding to fundamental solutions $\psi_i^{\sigma_i}$ and $\psi_j^{\sigma_j}$ defined by (2.2)–(2.4). Then, the coefficient function $\chi_{i\to j}^{\sigma_i}(\lambda, E) \equiv \chi_i^{\sigma_i}(\infty_j, \lambda, E)$ $(= \chi_{j\to i}^{\sigma_j}(\lambda, E))$ will be called a primitive canonical coefficient. The canonical coefficient defined in our earlier paper [27, 28] are simple functions (quotients) of the primitive canonical coefficients. As it follows from the formulae (2.3)–(2.4) $\chi_{i\to j}^{\sigma_i}(\lambda, E)$ is, for fixed E, a ramified holomorphic function of λ on the same Riemann surface on which the global solution $\psi(x, \lambda)$ is *i.e.* the point $\lambda = 0$ is its unique singularity (branch point).

To follow the behaviour of $\chi_{i\to j}^{\sigma_i}(\lambda, E)$ with λ it is necessary to follow the relevant deformations in the x-plane of the integration path $\gamma_{i\to j}$ corresponding to the coefficient considered. The latter deformations are enforced, of course, by the deformations of the corresponding Stokes graph caused by the change of λ and are ruled by the following two conditions: (a) the ends of $\gamma_{i\to j}$ have to follow the (moving) infinities $\infty_i(\lambda)$ and $\infty_j(\lambda)$ (lying in the respective sectors $D_i(\lambda)$ and $D_j(\lambda)$); and (b) no one turning points can cross the path $\gamma_{i\to j}$ when λ is changed.

If energy E is fixed the deformations of the Stokes graph are simply the rotations of each triad of the Stokes lines attached to each turning point around these points by some angles. These rotations depend only on the argument of λ. If $\arg\lambda$ changes by $\pm\pi$ the Stokes graph comes back after the rotation again to its initial position (*i.e.* the Stokes lines rotate by $\mp 2\pi/3$ around the corresponding turning points). However, each sector of the graph moving continuously (clockwise for $+\pi$, anticlockwise for $-\pi$) occupies after the rotations the position of its nearest neighbours (see, for example, a set of relevant figures drawn by Voros in his paper [25]). If the initial pattern of the Stokes graph is "typical" *i.e.* if any of its Stokes lines starting from some turning point goes to infinity (that is, does not end at some other turning point) then any two sectors of the graph can be joined by a canonical path. In such a case the behaviour of $\chi_{i\to j}^{\sigma_i}(\lambda, E)$ when $\lambda \to \infty$ (with $\arg\lambda$ fixed) is also typical *i.e.* $\chi_{i\to j}^{\sigma_i}(\lambda, E) = 1 + O(\lambda^{-1})$. We shall call such a typical behaviour a normal asymptotic behaviour. However, during the continuation of $\chi_{i\to j}^{\sigma_i}(\lambda, E)$ with λ this normal asymptotic behaviour can be lost since the integration path $\gamma_{i\to j}$ can cease to be the canonical one. In fact, this has always to happen for any path $\gamma_{i\to j}$ if the change of $\arg\lambda$ is large enough and it takes place when the deformed path has to cross two Stokes lines emerging from the same turning point. (Remember, that both ends of $\gamma_{i\to j}$ have to follow the movement of the sectors D_i, D_j in the infinities of which they are anchored.) Such a case can happen only when some initially different Stokes lines crossed by $\gamma_{i\to j}$ (they have to emerge from different turning points) coincide for some $\arg\lambda$ in the course of the analytical continuation in the λ-plane. If this is the case the corresponding values of $\arg\lambda$ as well as the graph itself are called critical.

At the critical values of $\arg\lambda$ the asymptotic behaviour of $\chi_{i\to j}$ ($\equiv \chi^{\sigma_i}_{i\to j}(\lambda, E)$) changes and becomes oscillating. If the critical value of $\arg\lambda$ is overcrossed $\chi_{i\to j}$ exponentially diverges for $\lambda \to \infty$. Besides, $\chi_{i\to j}$ stays to be perfectly holomorphic for $\lambda \neq 0, \infty$. Therefore, the conclusion is that for each $\chi_{i\to j}$ there exists in the λ-plane (or, rather, on the corresponding Riemann surface) a sector $\lambda > 0$ bounded by two critical values of $\arg\lambda$ (obtained by rotating the Stokes graph in the two possible directions) at the infinity of which $\chi_{i\to j}$ has the normal asymptotic behaviour. We shall call such a sector a normal sector of $\chi_{i\to j}$. It then follows from Appendix A that $\chi_{i\to j}$ can be Laplace (or Borel) transformed from its normal sector if the size of the latter is greater than π. If this critical angles are Φ and Ψ with $\Phi - \Psi > \pi$ then for the Laplace transform $\tilde{\chi}_{i\to j}(\equiv \tilde{\chi}^{\sigma_i}_{i\to j}(s, E))$ of $\chi_{i\to j}$ we

$$\tilde{\chi}_{i\to j}(s, E) = \frac{1}{2\pi i} \int\limits_{C_{ij}} e^{-2\lambda s} \chi_{i\to j}(\lambda, E) d\lambda\,, \tag{2.18}$$

where C_{ij} is shown in Fig. 4. The function $\tilde{\chi}_{i\to j}(s, E)$ is holomorphic in the sector $\tilde{D}_{ij} = \{s : \ |s| > 0, 3\pi/2 - \Phi < \arg s < \pi/2 - \Psi\}$. The inverse transformation is:

$$\chi_{i\to j}(\lambda, E) = 2 \int\limits_{\tilde{C}_{ij}} e^{2\lambda s} \tilde{\chi}_{i\to j}(s, E) ds\,, \tag{2.19}$$

with $\tilde{C}_{ij}$ shown in Fig. 5. It can be shown [41] that then $\tilde{\chi}_{i\to j}$ is also the Borel transform of $\chi_{i\to j}$.

Conversely, it the size of normal sector of $\chi_{i\to j}$ is smaller than π (*i.e.* $\Phi - \Psi < \pi$) then the Laplace or Borel transformations are impossible and this is a clear sign that $\chi_{i\to j}$ in the considered normal sector cannot be obtained from its asymptotic series expansion by the Borel resummation method.

The case when the size of the normal sector is equal to π needs a careful treatment (see [40] (Appendix 1) for the relevant discussion).

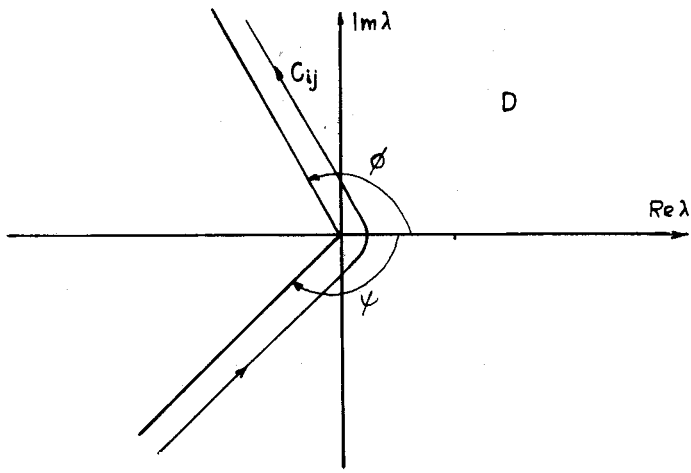

Fig. 4. The integration contour corresponding to the formula (2.18)

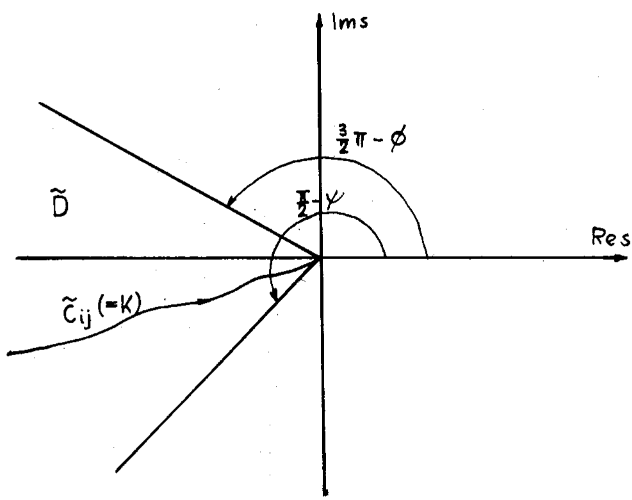

Fig. 5. The integration contour corresponding to the inverse Laplace transformation (2.19)

3. Quantization of energy levels and their analytic properties

A quantized energy $E(\lambda)$ is, certainly, this quantity which Laplace transformation properties in the λ-plane are most important. The existence of the latter transformation is, as usual (see Appendix A), determined by the existence of a sector in the λ-plane (with its angular size no less than π) inside which $E(\lambda)$ is holomorphic and shows suitable asymptotic behaviour when $\lambda \to \infty$ in the sector. In this section we are going to describe a method allowing: (a) to establish in a relatively simple way the existence of such sectors; (b) to determine their sizes; and (c) to determine the analytical and asymptotic properties of $E(\lambda)$ inside the sectors. The method is next used in the subsequent sections.

3.1. Analytic properties of the primitive canonical coefficients as a function of energy

As a function of energy E the primitive canonical coefficients $\chi_{i\to j}(\lambda, E)$ are holomorphic almost in the entire E-plane. Possible exceptional values of energy are:

(a) those for which some zeros of $q(x, E)$ coincide pinching simultaneously an integration path $\gamma_{i\to j}$; and

(b) the infinity of the E-plane approached from some directions if such approaching causes an infinite deformation of the path $\gamma_{i\to j}$ pinched by some zero (or zeros) running to the infinity.

Therefore, for a polynomial potential the total number of such points is finite. Under our assumption both the position and the character of these singularities are independent of λ. For example, a unique singular point of $\chi_{1\to 3}(\lambda, E)$ for the harmonic potential is $E = 0$ being a branch point of the type E^{-E} [25].

It is also worth to note that varying E we do not affect the asymptotic form of the Stokes lines. Only their patterns in the vicinity of turning points are disturbed.

3.2. Quantization of energy and its analytic properties as a function of λ

The quantization of energy is, in our formalism, a constraint put on a global solution $\psi(x, \lambda, E)$. Such a constraint is simply a matching of $\psi(x, \lambda\sigma_j, E)$ ($\lambda > 0$, $\sigma_j = \pm 1$) given in some sector D_j with $\psi(x, \lambda\sigma_k, E)$ defined in some other one D_k with $\sigma_j = \sigma_k \exp(i\pi r_{jk})$, $r_{jk} = \pm 2, \pm 3, \cdots, \pm(p-2)$. It has therefore, the following form:

$$\psi(x, \lambda\sigma_j, E) = C_{jk}(\lambda)\psi(x, \lambda\sigma_k, E). \tag{3.1}$$

For each of the solutions $\psi(x, \lambda\sigma_j, E)$ and $\psi(x, \lambda\sigma_k, E)$ the condition (3.1) means the following:

$$\psi(\infty_k, \lambda\sigma_j, E) = 0 \quad \text{and} \quad \psi(\infty_j, \lambda\sigma_k, E) = 0. \tag{3.2}$$

Since for the solution $\psi(x, \lambda\sigma_j, E)$ its JWKB factor

$$q^{-\frac{1}{4}}(x,E)\exp(\sigma_j\lambda_j\int_{x_0}^{x} q^{\frac{1}{2}}(y,E)dy)$$

vanishes only in the sector D_j (growing exponentially in the remaining ones) then (3.2) can be fulfilled iff the "amplitude" factor $\chi_j(x, \lambda\sigma_j, E)$ of $\psi(x, \lambda\sigma_j, E)$ given by (2.3)) vanishes for $x \to \infty_k$ *i.e.*

$$\chi_j(\infty_k, \lambda\sigma_j, E) = 0 \quad \text{or, equivalently,} \quad \chi_k(\infty_j, \lambda\sigma_k, E) = 0\,. \tag{3.3}$$

It follows then immediately that if the conditions (3.3) are to be fulfilled then the corresponding integration paths $\gamma_j(\infty_k)$ and $\gamma_k(\infty_j)$ cannot be chosen to be canonical since with the latter choice $\chi_j(\infty_k, \lambda\sigma_j, E)$ and $\chi_k(\infty_j, \lambda\sigma_k, E)$ have to approach unity when $\lambda \to +\infty$. For the corresponding Stokes graph itself (3.3) means that the graph is in a critical position *i.e.* some of its turning points are linked by a Stokes line so that the canonical communication of the sectors D_j and D_k is broken.

However, it is always possible to continue (3.1) to two independent sectors, say p and q, along a system of canonical paths [27–28] so that $C_{jk}(\lambda)$ can be eliminated from the condition (3.1). Then, the above condition can be given, as a rule, the following x-independent form:

$$F\Big(\chi_{j\to p}(\lambda, E), \cdots, \chi_{k\to q}(\lambda, E), \exp\Big(\lambda\oint_{C_1} q^{\frac{1}{2}}\Big), \cdots, \exp\Big(\lambda\oint_{C_r} q^{\frac{1}{2}}\Big)\Big) = 0, \tag{3.4}$$

where F is a multilinear function of some primitive canonical coefficients $\chi_{j\to p}(\lambda, E), \ldots$ *etc.* and of the typical phase coefficients $\exp(\lambda\oint_{C_1} q^{1/2}), \ldots$ *etc.* with no other additional dependence on λ and E. The integrals in the phase coefficients are closed contour integrals with some pair of turning points (real or complex) inside the contours. These integrals are nothing but the values of the classical action corresponding to the different solutions, both real and complex, of the relevant (one-dimensional) classical equation of motion when the corresponding periodic movements are bounded between different pairs of turning points (real or complex). The presence of contributions coming also from complex trajectories is permanent and characteristic property of our approach. Their role in the semiclassical description of quantum phenomena have been also noticed and appreciated in the Feynman path integral approach [16–20]. It will be shown in the next sections that the complex classical trajectories are those which determine the singularity structure of the Borel plane corresponding to different quantities and, consequently, their subdominant asymptotic behaviour when $\lambda \to \infty$ (see (1.4) in Section 1).

In general there is a number of equivalent equations of the type (3.4) in each particular case of the quantization condition. This variety of equivalent equations arises due to the existence of the corresponding number of identities involving some suitable phase coefficients as well as all possible primitive canonical coefficients which can be defined for each particular case of the Stokes graph (see Appendix B). These identities have also the form (3.4).

Since $\psi(x, \lambda, E)$ in (3.1) as well as the coefficients $\chi_{j \to p}(\lambda, E) \ldots$ *etc.* in (3.3) and (3.4) can be continued analytically in the λ-plane then (3.3) and (3.4) define the energy levels E as a function of the complex variable λ. Since, further, the phase coefficients are entire functions of λ and are singular at the same points in the E-plane at which the primitive canonical coefficients are then a solution $E(\lambda)$ as defined implicitly by (3.3) and (3.4) has to have the following typical properties:

1. it is a holomorphic function of λ everywhere except the following possible points:
   a) $\lambda = 0$, where the primitive canonical coefficient are singular;
   b) the points λ for which $E(\lambda)$ becomes a singular point for the primitive canonical coefficients or for the phase coefficients;
   c) the points λ for which:

$$\frac{\partial F(\lambda, E)}{\partial E} = 0 \tag{3.5}$$

2. it has to behave in a definite way in some sectors of the λ-plane when $\lambda \to \infty$; this behaviour can be read off from the known asymptotic behaviour of the phase- and the primitive canonical coefficients entering the corresponding quantization condition (3.4).

The latter circumstance is of a special importance since it decides about the possible Laplace transformation of $E(\lambda)$ and of the Borel summability of its asymptotic series expansion. As it follows from [40] (Appendix 1) for the Laplace (or Borel) transform $E(s)$ to exist it is necessary for $E(\lambda)$ to be analytic in the relevant sector (or strip) only asymptotically *i.e.* to be analytic there for λ large enough. Therefore, we shall use the condition (3.5) (defining the positions of possible singularities of $E(\lambda)$) only asymptotically *i.e.* we will check whether this equation is satisfied or not only for $\lambda \to \infty$ under the condition that (3.4) is satisfied in the same limit.

As it has been shown by Bender and Wu [30] the analytical properties of $E(\lambda)$ in the λ-plane are determined by the following factors:

1° the existence of several physical channels associated with the potential considered and realized by analytic continuations in the λ- and x-planes;

2° the energy level structures in each physical channel of the problem considered; and
3° the symmetries of the investigated Hamiltonian.

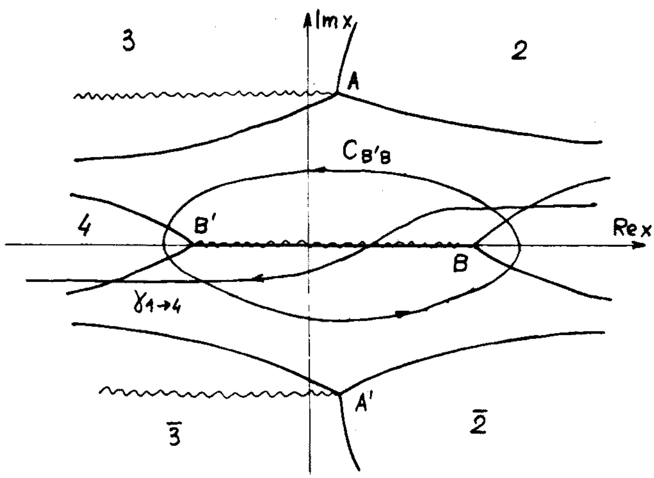

Fig. 6. The Stokes graph corresponding to the single-well channel of the cubic-quartic oscillator

However, a common property of each energy level independent of the channel considered is its typical asymptotic behaviour when $\lambda \to \infty$. Namely, each energy level is quantized in this limit in the deepest well with the behaviour: $E(\lambda) = V_0 + O(\lambda^{-1})$, where V_0 is the absolute minimum of the potential [27]. It means that the corresponding quantum object is settling down at the bottom of the deepest well and this property is visualized in the Stokes graph as the collapsing of these pairs of the real turning points which correspond to the considered energies. Thus, for example, for the single-well channel of Fig. 6 the collapsing pair is (B, B'). In the case of the double-well channel of Fig. 7 the collapsing pair is (A, B). Such an asymptotic behaviour of energy levels we shall call a normal asymptotic behaviour and the sector of the λ-plane where such a behaviour takes place we shall call a normal sector. We will show in the next section, considering the anharmonic oscillator as a typical example, that on the Riemann surface of the energy level $E(\lambda)$ the normal sectors dominate *i.e.* there are only isolate directions along which the asymptotic behaviour of $E(\lambda)$ is different from the normal one.

In fact, our conclusions about the normal asymptotic behaviour follow directly from the quantization condition (3.4) (or its equivalents) if we make use of the following asymptotic properties of the primitive canonical coefficients (which on their own follow from (2.3)):

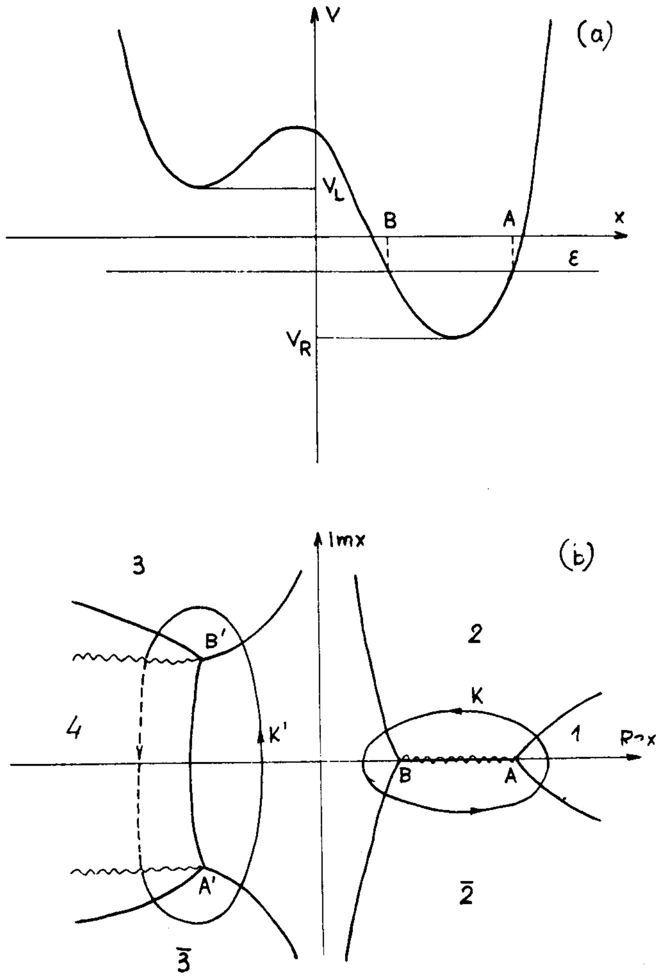

Fig. 7. The double-well channel of the cubic-quartic oscillator and the corresponding Stokes graph

if λ goes to the infinity along a direction such that both the energy $E(\lambda)$ stays finite and the integration paths defining the primitive coefficients in (3.4) stay canonical then these coefficients have to approach the unity and both their partial derivatives (with respect to λ and E) have to vanish.

Since we are interested rather in an asymptotic analytic properties of energy levels we shall assume the following strategy in our investigations of the problem:

1. for a given physical channel fix the energy parameter E at V_0 — the

absolute minimum of the potential in this channel; this corresponds to take the limit $\lambda \to \infty$ in the following asymptotic series expansion of $E(\lambda)$ in the considered channel:

$$E(\lambda) \sim V_0 + \sum_{n \geq 0} E_n \lambda^{-n-1} \tag{3.6}$$

2. determine the normal sectors for all the primitive canonical coefficients entering equivalent quantization conditions using the Stokes graph corresponding to the case considered;
3. take a common part of the normal sectors corresponding to the primitive canonical coefficients entering the same quantization condition;
4. take a union of all the common parts obtained in the previous step; if such a union is connected it constitutes an asymptotically normal sector for $E(\lambda)$ of a maximal angular size *i.e.* a maximal normal sector for $\lambda \to \infty$.

The normal sector obtained in the way described above ensures that: *(i)* the energy $E(\lambda)$ satisfies some of the quantization conditions (3.4) and behaves asymptotically according to (3.6); and *(ii)* the condition (3.5) fails to be fulfilled. (In fact, if $\lambda \to \infty$ inside the normal sector of $E(\lambda)$ then $\partial F(\lambda, E(\lambda))/\partial E \sim \lambda$ as one can easily checked by the direct differentiation of the left hand side of (3.5).) The latter fact means that $E(\lambda)$ is a holomorphic function inside its normal sector for λ large enough.

Following the procedure described in the points 1.–4. above we have to deform suitably the Stokes graph pattern according to varying λ. One of the indications that the corresponding deformations are performed properly is the impossibility to communicate canonically the sectors D_j and D_k (corresponding to the solutions matched) in any stage of such deformations.

Now we could apply the procedure described above to any polynomial potential. However, instead of diving into a discussion of a general case we shall consider the relevant asymptotic behaviour of $E(\lambda)$ in the λ-plane using rather a particular example of such a potential, namely this of the cubic-quartic anharmonic oscillator. The relevant results obtained for the latter case are typical enough for the problem considered, so that their generalization as well as a generalization of the methods applied will appear to be obvious both for any polynomial potential and for nonpolynomial ones. We will do it in Section 5.

4. Analytic properties and Borel summability of energy levels of the cubic–quartic single- and double-well oscillators

The case of the cubic-quartic oscillator in its single- and double-well variants is particularly interesting because of the following reasons:

1. it is the simplest non-trivial case (in comparison with the harmonic one) of the bounding potential;
2. it is the case which properties we are most interested in are best examined [4–5, 25–26, 30–34];
3. it is a good illustration of the techniques we are going to apply investigating the analytic properties of $E(\lambda)$ and its asymptotic behaviour when $\lambda \to \infty$; and
4. its results can be easily generalized (see the next section).

We shall consider a general quartic potential of the form

$$\beta^{-1}V(\beta^{\frac{1}{2}}x) = \beta^{-1}(\beta x^2 + \frac{1}{4})^2 - b\beta^{\frac{1}{2}}x^3 .$$

Making the following change of scales: $x \to x\beta^{-1/2}$ and $E \to \varepsilon\beta^{-1}$, in the corresponding Schrödinger equation (2.1) we are left with:

$$\psi''(x,\lambda,b,\varepsilon) - \lambda^2[(x^2+\tfrac{1}{4})^2 - bx^3 - \varepsilon]\psi(x,\lambda,b,\varepsilon) = 0 , \qquad (4.1)$$

where $\lambda^2 = 2m/(\hbar\beta)^2$. The Hamiltonian in (4.1) is invariant under the following joint reflection transformations in the complex x- and b-planes: $(x,b) \to (-x,-b)$. A name of parity invariance shall be adopted for this symmetry.

If x,λ,ε,b are all real with $\varepsilon > 1/16$ and $b < 4/3$ then (4.1) represents the problem of the single–well asymmetric anharmonic oscillator with the corresponding Stokes graph shown in Fig. 6. Let $\psi_k(x,\lambda,b,\varepsilon)$ with $\lambda > 0$ be a fundamental solution corresponding to sector k in Fig. 6, $k = 1,2,\bar{2},3,\bar{3},4$. Because of the parity invariance we can assume that $\psi_4(x,\lambda,b,\varepsilon) \equiv \psi_1(-x,\lambda,-b,\varepsilon)$ as well as $\psi_3(x,\lambda,b,\varepsilon) \equiv \psi_{\bar{2}}(-x,\lambda,-b,\varepsilon)$ and $\psi_{\bar{3}}(x,\lambda,b,\varepsilon) \equiv \psi_2(-x,\lambda,-b,\varepsilon)$. Besides, $\psi_k(x,\lambda,b,\varepsilon)$ coincide, up to some constant $C_k(\lambda,b,\varepsilon)$, with $\psi_1(x,\lambda,b,\varepsilon)$ rotated in the λ-plane by, say, $\pi, 2\pi, 3\pi$ for $k = 2,3,4$ and by $-\pi,-2\pi$ for $k = \bar{2},\bar{3}$, respectively.

4.1. Quantization of energy levels

The quantization condition (3.3) in the case of Fig. 6 reads then:

$$\chi_{1\to4}(\lambda,b,\varepsilon) = 0 , \qquad (4.2)$$

where $\chi_{1\to4}(\lambda,b,\varepsilon)$ is given by (2.3) with the integration path $\gamma_{1\to4}(\lambda)$ shown in Fig. 6. The parity invariance and the reality of $\varepsilon(\lambda,b)$ for $\lambda,b > 0$ enforce the following relations satisfied by the latter:

$$\bar{\varepsilon}(\lambda,b) \equiv \varepsilon(\bar{\lambda},\bar{b}) \qquad \text{and} \qquad \varepsilon(\lambda,b) \equiv \varepsilon(\lambda,-b) \qquad (4.3)$$

Since the path $\gamma_{1\to4}(\lambda)$ is not canonical it is more convenient to substitute (4.2) by the following equivalent condition:

$$\chi_{2\to4}(\lambda,b,\varepsilon)\cdot\chi_{1\to3}(\lambda,b,\varepsilon)+\exp[-\lambda\oint_{C_{B'B}} q(x,\lambda,b,\varepsilon)^{\frac{1}{2}}dx]=0\,, \qquad (4.4)$$

where $q(x,\lambda,b,\varepsilon)$ is defined by (4.1) and $C_{B'B}$ is a closed anticlockwise oriented contour around the cut (B',B) (see Fig. 6). Eq. (4.4) follows from matching the solutions $\psi_4(x,\lambda,b,\varepsilon)$ and $\psi_1(x,\lambda,b,\varepsilon)$ in sector 2 and next in sector 3. Note, that the complex conjugation of (4.4) provides a quantization condition equivalent to (4.4) with $\chi_{2\to4}(\lambda,b,\varepsilon)$ and $\chi_{1\to3}(\lambda,b,\varepsilon)$ replaced by $\chi_{\bar{2}\to4}(\lambda,b,\varepsilon)$ and $\chi_{1\to\bar{3}}(\lambda,b,\varepsilon)$, correspondingly.

Any continuation of (4.4) in the λ- and b-planes defines $\varepsilon(\lambda,b)$ at complex values of λ and b. In particular, as it follows from (4.1) and Fig. 6, rotating each solution $\psi_k(x,\lambda,b,\varepsilon)$, $k=1,\ldots,4$, first in the λ-plane by the angle $-3\pi/2$ (the Stokes graph in Fig. 6 rotates then anticlockwise), then in the x-plane by $+\pi/2$, next in the b-plane by $+\pi/2$, and, finally, continuing the solutions in the ε-plane from the region $\varepsilon>1/16$ to the region $V_R<\varepsilon<V_L$ (above the point $\varepsilon=1/16$ — this places the point B at the new real axis to the left from the point A) we arrive at the Stokes graph of Fig. 7b *i.e.* at the graph corresponding to the double–well "channel" of the considered quartic potential with the unequal depth V_R and V_L of the corresponding wells (see Fig. 7a). Therefore, taking again into account the parity invariance we obtain the following relations between the energy levels of the single- (s-w) and double-well (d-w) channels:

$$\varepsilon_R^{\text{d-w}}(\lambda,b)\equiv\varepsilon^{\text{s-w}}\left(\lambda e^{3\pi i/2},\, be^{i\pi/2}\right)$$

or

$$E_R^{\text{d-w}}(\lambda,b)\equiv iE^{\text{s-w}}\left(\lambda e^{3\pi i/2},\, be^{i\pi/2}\right). \qquad (4.5)$$

(Note that we can also rotate in the λ- and b-planes in the opposite direction *i.e.* by $+3\pi/2$ and $-\pi/2$, respectively, arriving at a result obtained from (4.5) by applying the properties (4.3) and corresponding again to Fig. 7. However, rotating in the b-plane by $-\pi/2$ instead of $\pi/2$ as in (4.5) we would exchange only the wells in Fig. 7 *i.e.* we would have $V_L<V_R$.)

In the case when $b=0$ the parity invariance implies also that starting with any λ in (4.1) and rotating in the λ-plane by $\pm3\pi$ we arrive again at the same Stokes graph pattern *i.e.* we have to have:

$$\varepsilon(\lambda e^{\pm3\pi},0)=\varepsilon(\lambda,0)\,. \qquad (4.6)$$

Eq. (4.6) means that $\lambda=0$ is a branch point for $\varepsilon(\lambda)$ ($\equiv\varepsilon(\lambda,0)$) such that $\varepsilon(\lambda)\sim\lambda^{4/3}$ for $\lambda\to0$, so that $E(\lambda)\sim\lambda^{1/3}$ and is finite in the limit

$\lambda \to 0$ since the latter corresponds to the limit $\beta \to \infty$ which makes the anharmonic potential purely quartic.

In the double-well channel of Fig. 7 the primitive coefficients in both the conditions (4.2) and (4.4) are no longer canonical (they lost their canonicity whilst continued from the single- to the double-well channel). The corresponding condition with the canonical coefficients is the following:

$$\chi_{2\to4}(\lambda,b,\varepsilon)\exp\Big(\tfrac{1}{2}\lambda\oint_K q^{\frac{1}{2}}dx\Big) + \chi_{\bar{2}\to4}(\lambda,b,\varepsilon)\exp\Big(-\tfrac{1}{2}\lambda\oint_K q^{\frac{1}{2}}dx\Big) = 0\,, \tag{4.7}$$

where $q(x,\lambda,b,\varepsilon) = (x^2-1/4)^2 - bx^3 - \varepsilon$ with the closed contour K shown in Fig. 7b. The condition (4.7) can be obtained by matching ψ_1 and ψ_4 in sector 2 and next in sector $\bar{2}$ (see Fig. 7b).

The analytic properties of the solution $\varepsilon(\lambda,b)$ to (4.3) and its asymptotic behaviour in the λ-plane can now be read off from the conditions (4.4) and (4.7) and their equivalences complex conjugated to them. Following the prescription of Section 3.2 (points 1.–4.) we shall determine the normal sectors for $\varepsilon(\lambda,b)$ corresponding to the single-well channel as well as to the double-well one. The sectors should be different since they correspond to different asymptotic expansions (3.6) of $\varepsilon(\lambda,b)$.

4.2. Energy level normal sector: the single-well case

Consider first the single-well case corresponding to Fig. 6. When $\varepsilon = 1/16$ (*i.e.* ε is equal to absolute depth of the well) then the points B and B' coincide and the graph in Fig. 6 takes on a form shown in Fig. 8. The normal sectors for $\chi_{2\to4}$, $\chi_{1\to\bar{3}}$, $\chi_{1\to3}$ and $\chi_{\bar{2}\to4}$ follow directly from Fig. 8. They are determined by the critical angles corresponding to the coefficients considered (see Section 2.2.) *i.e.* by these values of $\arg\lambda$ crossing over of which causes these coefficients to be no longer canonical.

Consider, for example, the coefficient $\chi_{2\to4}$. One of its critical angles is achieved when the graph in Fig. 8 is rotated anticlockwise (around its turning points) so that it causes the slashed Stokes lines in Fig. 8 to coincide. This happens when $\mathrm{Re}\left[\lambda\int_B^A q^{1/2}dx\right] = 0$. Since the integral in parentheses is real and negative $\arg\lambda$ which satisfies the last condition has to be equal to $-\pi/2$ (λ rotates clockwise in the λ-plane). On the other hand one can easily check that rotating the graph clockwise by $\arg\lambda = +\pi$ (this causes only the cyclic permutation of the sectors in Fig. 8 *i.e.* $1 \to \bar{2} \to \bar{3} \to 4 \to 3 \to 2 \to 1$)

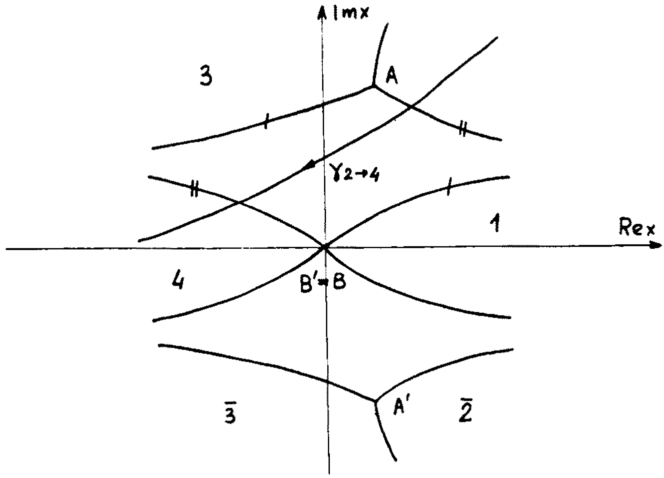

Fig. 8. The Stokes graph corresponding to the particle setlled down at the well's bottom of the single-well case.

no coincidence of the Stokes lines can break the canonicity of $\chi_{2\to 4}$. However, rotating still further again by $+\pi/2$ we cause the Stokes lines slashed double in Fig. 8 to coincide. This breaks ultimately the canonical contact between the sectors 2 and 4. Therefore, we can conclude that the limiting angular size of the $\chi_{2\to 4}$-normal sector when $\lambda \to \infty$ is the following:

$$-\frac{\pi}{2} < \arg\lambda < +\frac{3\pi}{2}\,. \tag{4.8}$$

Exactly the same normal sector is obtained when $\chi_{1\to\bar{3}}$ is analyzed. On the other hand the angular size of the normal sectors corresponding to $\chi_{\bar{2}\to 4}$ and $\chi_{1\to 3}$ can be readily obtained from (4.8) by the symmetry arguments to be:

$$-\frac{3\pi}{2} < \arg\lambda < +\frac{\pi}{2}\,. \tag{4.9}$$

Concluding from (4.8) and (4.9) we obtain, therefore, the following maximal angular size of the normal sector for $\varepsilon(\lambda, b)$:

$$-\frac{3\pi}{2} < \arg\lambda < +\frac{3\pi}{2}\,. \tag{4.10}$$

In fact, it is the last size which has been proved by Simon and Dicke [26] to be the maximal normal sector for $\varepsilon(\lambda, b)$ for the case considered.

4.3. *Energy level normal sector: the double-well case*

This case can be considered in a way completely analogous to the previous one. Analysing the corresponding graph in Fig. 9 we obtain for the normal sector of $\chi_{2\to4}$:

$$-\pi < \arg\lambda < +\frac{\pi}{2} - \varphi_- \tag{4.11}$$

and for the normal sector of $\chi_{\bar{2}\to4}$:

$$-\frac{\pi}{2} + \varphi_- < \arg\lambda < +\pi\,, \tag{4.12}$$

where $-\pi/2 < \varphi_- < 0$ is defined by $\tan\varphi_- = \mathrm{Im}\,\xi(A,B')/\,\mathrm{Re}\,\xi(A,B')$.

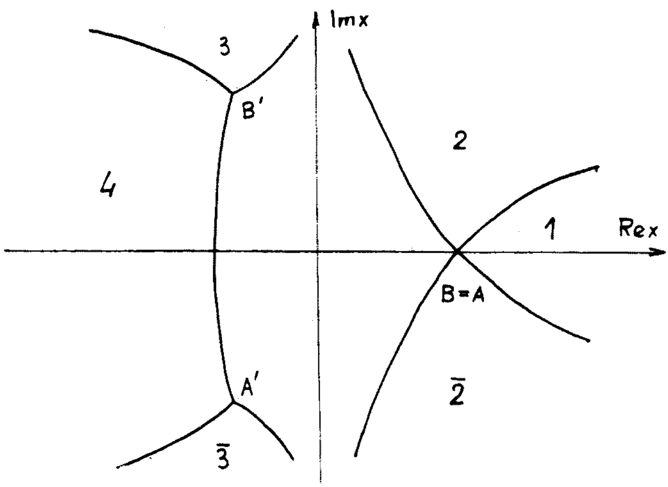

Fig. 9. The Stokes graph corresponding to the particle setlled down at the well's bottom of the double-well case.

Therefore, the maximal normal sector for $\varepsilon(\lambda,b)$ is given correspondingly by:

$$-\frac{\pi}{2} + \varphi_- < \arg\lambda < +\frac{\pi}{2} - \varphi_-\,. \tag{4.13}$$

In the next section we shall show that the asymptotic series (3.6) are Borel summable to $\varepsilon(\lambda,b)$ in both considered channels.

4.4. *Borel summability of energy level asymptotic series*

In showing the Borel summability of the $\varepsilon(\lambda,b)$ asymptotic series we shall follow the Bender–Wu method [5]. However, we are not going to repeat the calculations of Bender and Wu but rather to use the method in a way which makes it effective and efficient. The use of the identities (A.2) of Appendix A plays a crucial role in our application of the method.

4.4.1. The single-well case

The normal sector for $\varepsilon(\lambda, b)$ is given in this case by (4.10). Since the quantities $\varepsilon(\lambda e^{i\pi}, b)$ and $\varepsilon(\lambda e^{-i\pi}, b)$ have in the sector (4.10) the same asymptotic expansion (3.6) their difference $D\varepsilon(\lambda, b) = \varepsilon(\lambda e^{i\pi}, b) - \varepsilon(\lambda e^{-i\pi}, b)$ has to vanish exponentially when $\lambda \to \infty$. Therefore, we can apply the Bender–Wu formula [5] to estimate the large order behaviour of the coefficient $\varepsilon_n(b)$ in the series (3.6). For this, however, it is necessary to know the precise asymptotic behaviour of $D\varepsilon(\lambda, b)$. We will determine it by constructing a function having the same asymptotic series expansion (3.6) as each of the functions $\varepsilon(\lambda e^{i\pi}, b)$ and $\varepsilon(\lambda e^{-i\pi}, b)$ and being in a definite relations with the latter. To this end let us note that such a function (we denote it by $\varepsilon_{2\bar{2}}(\lambda, b)$) can be obtained as a solution to the following equation:

$$\chi_{2\to\bar{2}}(\lambda, b, \varepsilon) = 0\,, \tag{4.14}$$

which is nothing but the result of matching the fundamental solutions $\psi_2(x, \lambda, b, \varepsilon)$ and $\psi_{\bar{2}}(x, \lambda, b, \varepsilon)$ (see Fig. 6). Matching these solutions in sector 4 the following equivalence to (4.14) is obtained:

$$\chi_{2\to 4}(\lambda, b, \varepsilon) \exp\Big(\tfrac{1}{2}\lambda \oint_{C_{B'B}} q^{\frac{1}{2}}\, dx\Big) + \chi_{\bar{2}\to 4}(\lambda, b, \varepsilon) \exp\Big(-\tfrac{1}{2}\lambda \oint_{C_{B'B}} q^{\frac{1}{2}}\, dx\Big) = 0\,, \tag{4.15}$$

which shows that $\varepsilon_{2\bar{2}}(\lambda, b)$ has the same asymptotic expansion when $\lambda \to \infty$ as $\varepsilon(\lambda, b)$ has. From (4.15) it follows also that the normal sector for $\varepsilon_{22}(\lambda, b)$ is given by:

$$-\frac{\pi}{2} < \arg\lambda < +\frac{\pi}{2}\,. \tag{4.16}$$

Therefore, within the boundaries of this sector $\chi_{2\to\bar{2}}(\lambda, b, \varepsilon)$ fails to fulfil the condition (3.5) asymptotically, that is:

$$\frac{\partial \chi_{2\to\bar{2}}(\lambda, b, \varepsilon_{2\bar{2}}(\lambda, b))}{\partial \varepsilon_{2\bar{2}}} \neq 0\,. \tag{4.17}$$

Let us now note further that the following identities corresponding to Fig. 6 take place:

$$\chi_{2\to 4}(\lambda, b, \varepsilon)\cdot\chi_{\bar{2}\to 3}(\lambda, b, \varepsilon) = \chi_{2\to\bar{2}}(\lambda, b, \varepsilon) + \chi_{\bar{2}\to 4}(\lambda, b, \varepsilon) \exp\Big(2\lambda \int_{B'}^{A} q^{\frac{1}{2}}\, dx\Big)$$

and

$$\chi_{\bar{2}\to 4}(\lambda,b,\varepsilon)\cdot\chi_{2\to\bar{3}}(\lambda,b,\varepsilon)=\chi_{2\to\bar{2}}(\lambda,b,\varepsilon)+\chi_{2\to 4}(\lambda,b,\varepsilon)\exp\left(2\lambda\int\limits_{B'}^{A} q^{\frac{1}{2}}dx\right) \tag{4.18}$$

and the coefficients $\chi_{2\to\bar{3}}(\lambda,b,\varepsilon)$ and $\chi_{\bar{2}\to 3}(\lambda,b,\varepsilon)$ obey the following equations:

$$\chi_{2\to\bar{3}}(\lambda,b,\varepsilon(\lambda e^{-i\pi},b))=0 \quad \text{and} \quad \chi_{\bar{2}\to 3}(\lambda,b,\varepsilon(\lambda e^{i\pi},b))=0\,. \tag{4.19}$$

The latter are nothing but the condition (4.2) rotated in the λ-plane by $\pm\pi$, respectively. Making now the following substitutions: $\varepsilon\to\varepsilon(\lambda e^{i\pi},b)$ into the first of Eqs (4.18) and $\varepsilon\to\varepsilon(\lambda e^{-i\pi},b)$ into the second one and next expanding $\chi_{2\to\bar{2}}(\lambda,b,\varepsilon)$ in these equations around the point $\varepsilon=\varepsilon_{2\bar{2}}$ and then using (4.15) and truncating the Taylor series at the first nonvanishing terms (see the condition (4.17)) we get in the limit $\lambda\to+\infty$:

$$\begin{aligned}\Delta_{+}(\lambda,b)&=\varepsilon(\lambda e^{i\pi},b)-\varepsilon_{2\bar{2}}(\lambda,b)\\ &\sim -\chi_{\bar{2}\to 4}(\lambda,b,\varepsilon(\lambda e^{i\pi},b))\frac{\exp\left(2\lambda\int_{B'}^{A}q^{\frac{1}{2}}dx\right)}{\frac{\partial}{\partial\varepsilon_{2\bar{2}}}\chi_{2\to\bar{2}}\left(\lambda,b,\varepsilon_{2\bar{2}}(\lambda,b)\right)};\end{aligned}$$

and

$$\begin{aligned}\Delta_{-}(\lambda,b)&=\varepsilon(\lambda e^{-i\pi},b)-\varepsilon_{2\bar{2}}(\lambda,b)\\ &\sim -\chi_{2\to 4}(\lambda,b,\varepsilon(\lambda e^{-i\pi},b))\frac{\exp\left(2\lambda\int_{B'}^{A'}q^{\frac{1}{2}}dx\right)}{\frac{\partial}{\partial\varepsilon_{2\bar{2}}}\chi_{2\to\bar{2}}\left(\lambda,b,\varepsilon_{2\bar{2}}(\lambda,b)\right)}\,.\end{aligned} \tag{4.20}$$

Note that the Taylor series truncations made above are justified since $\Delta_{\pm}$ are exponentially small quantities in the limit $\lambda\to+\infty$. Therefore, using the Bender–Wu formula we obtain the following result for the large order behaviour ($n\to\infty$) of $\varepsilon_n(b)$ in (3.6):

$$\begin{aligned}\varepsilon_n(b)&\sim-\frac{1}{2\pi i}\int\limits_{0}^{\infty}\left(\Delta_{+}(\lambda,b)-\Delta_{-}(\lambda,b)\right)(-\lambda)^n d\lambda\\ &\sim\frac{(-1)^{m+n}}{\pi^2\sqrt{2}\left(\frac{\sqrt{2}(m+1/2)}{e}\right)^{m+\frac{1}{2}}}\left(-2\int\limits_{B'(\infty)}^{A(\infty)}q^{\frac{1}{2}}dx\right)^{-n-m-\frac{1}{2}}\\ &\times C_{14}(b,m)\,\mathrm{Im}\left(\frac{A(\infty)}{C_A(m)}\right)^{2m+1}\Gamma(m+n+\tfrac{1}{2})\left(1+O\left(\frac{1}{m+n+\frac{1}{2}}\right)\right),\end{aligned} \tag{4.21}$$

where m is the number of the level considered and the integration limits in (4.21) (*i.e.* the points $B'(\infty)$, $A(\infty)$) correspond to Fig. 8. The subtleties of these calculations as well as the definitions of the constants $C_{14}(b,m)$ and $C_A(m)$ are given in Appendix B. It can be checked that when $b = 0$ (4.21) reduces to the well known result of Bender and Wu [5] with $\Gamma(m+1)$ given in a form of Stirling's formula.

As a consequence of (4.21) we obtain that the Borel functions $\tilde{\varepsilon}_B(s,b)$ corresponding to the series (3.6) exist and are holomorphic in the circle $|s| < |\xi(B'(\infty), A(\infty))|$ with a singularity at $s_0 = -\xi(B'(\infty), A(\infty))$ lying on the positive real axis.

Therefore, it is confirmed in this way that the normal sector for $\varepsilon(\lambda, b)$ is given by (4.10) since its size predicts that the Laplace transforms $\varepsilon_L(s.b)$ of $\varepsilon(\lambda, b)$ should be holomorphic in the s-plane cut along the positive real axis. Our calculations confirm this prediction showing additionally that the cut begin at the point $s_0 = -\xi(B'(\infty), A(\infty))$ and that $\tilde{\varepsilon}_B(s,b) \equiv \tilde{\varepsilon}_L(s,b)$.

4.4.2. The double-well case

Our investigations of the case will be to a large extent similar to the previous ones. As before we shall construct auxiliary functions (two of them this time) which together with $\varepsilon(\lambda, b)$ will allow us to estimate the large order behaviour of $\varepsilon_n(b)$ in the asymptotic series (3.6) corresponding to the case. These two functions will have the following properties:

(i) the asymptotic expansions (3.6) identical with that of $\varepsilon(\lambda, b)$;

(ii) the sizes of their normal sectors larger than π *i.e.* allowing to take the Laplace transformations of them;

(iii) disjoint holomorphicity sectors in the Borel plane being simultaneously the neighbors of the sector corresponding to $\varepsilon(\lambda, b)$.

Let us denote these functions by $\varepsilon_{13}(\lambda, b)$ and $\varepsilon_{1\bar{3}}(\lambda, b)$. Each of them arises as a solution to a suitable "quantization" condition resulting when matching relevant pairs of the fundamental solutions corresponding to Fig. 7b. Namely, matching ψ_1 with ψ_3 and ψ_1 with $\psi_{\bar{3}}$ we get the following "quantization" conditions:

$$\chi_{\bar{2}\to 3}(\lambda, b, \varepsilon) + \exp\Big(\lambda \oint_K q^{\frac{1}{2}}\, dx\Big) = 0$$

and

$$\chi_{2\to \bar{3}}(\lambda, b, \varepsilon) + \exp\Big(-\lambda \oint_K q^{\frac{1}{2}}\, dx\Big) = 0 \tag{4.22}$$

defining the functions $\varepsilon_{13}(\lambda, b)$ and $\varepsilon_{1\bar{3}}(\lambda, b)$, respectively and making the property *(i)* above obvious. The corresponding normal sector sizes which

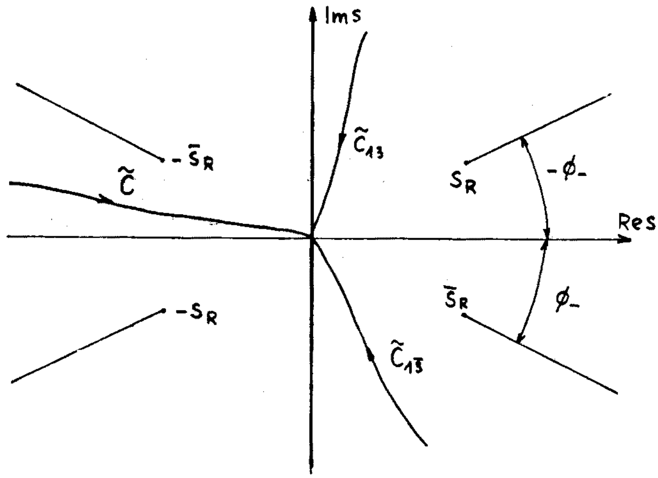

Fig. 10. The Borel plane corresponding to $\varepsilon(\lambda, b)$ in the double-well case

follow from (4.22) are:

$$-\frac{\pi}{2} - \varphi_- < \arg\lambda < \frac{3\pi}{2} + \varphi_- \quad \text{for} \quad \varepsilon_{13}(\lambda, b)$$

and

$$-\frac{3\pi}{2} - \varphi_- < \arg\lambda < \frac{\pi}{2} + \varphi_- \quad \text{for} \quad \varepsilon_{1\bar{3}}(\lambda, b)\,, \tag{4.23}$$

where φ_- is the angle defined in (4.11). The sectors (4.23) are complex conjugated due to the relation:

$$\bar{\varepsilon}_{13}(\lambda, b) \equiv \varepsilon_{1\bar{3}}(\bar{\lambda}, \bar{b}) \tag{4.24}$$

which follows from (4.22). From (4.13) and (4.23) it follows further that the energy $\varepsilon(\lambda, b)$ and the functions $\varepsilon_{13}(\lambda, b)$ and $\varepsilon_{1\bar{3}}(\lambda, b)$ can be Laplace transformed from their normal sectors and their corresponding transforms $\tilde{\varepsilon}(s, b)$, $\tilde{\varepsilon}_{13}(s, b)$ and $\tilde{\varepsilon}_{1\bar{3}}(s, b)$ are holomorphic in the following sectors of the s-plane:

$$\begin{aligned}
\widetilde{D} &= \{s : \pi + \varphi_- < \arg s < \pi - \varphi_-\,, \qquad s > 0\} \\
\widetilde{D}_{13} &= \{s : -\varphi_- < \arg s < \pi + \varphi_-, \qquad s > 0\} \\
\widetilde{D}_{1\bar{3}} &= \{s : \pi - \varphi_- < \arg s < 2\pi + \varphi_-, \qquad s > 0\}\,,
\end{aligned} \tag{4.25}$$

respectively.

We now show that the asymptotic series (3.6) common to all the functions $\varepsilon(\lambda, b)$, $\varepsilon_{13}(\lambda, b)$ and $\varepsilon_{1\bar{3}}(\lambda, b)$ is Borel summable to each of these functions with the Borel integration contours $\widetilde{C}$, $\widetilde{C}_{13}$ and $\widetilde{C}_{1\bar{3}}$ taken in the corresponding sectors (4.25) and shown in Fig. 10. To this end we make use of the following three identities:

$$\chi_{2\to 4}(\lambda, b, \varepsilon) \cdot \chi_{\bar{2}\to 3}(\lambda, b, \varepsilon) - \chi_{\bar{2}\to 4}(\lambda, b, \varepsilon) - \chi_{2\to \bar{2}}(\lambda, b, \varepsilon) \exp\left(-2\lambda\xi(B, B')\right) \equiv 0$$

$$\chi_{\bar{2}\to 4}(\lambda, b, \varepsilon) \cdot \chi_{2\to \bar{3}}(\lambda, b, \varepsilon) - \chi_{2\to 4}(\lambda, b, \varepsilon) - \chi_{2\to \bar{2}}(\lambda, b, \varepsilon) \exp\left(-2\lambda\xi(B, A')\right) \equiv 0$$

and

$$\chi_{2\to \bar{3}}(\lambda, b, \varepsilon) \equiv \chi_{\bar{2}\to 3}(\lambda e^{i\pi}, b, \varepsilon). \tag{4.26}$$

The following relations are particular applications of the last identity:

$$\chi_{2\to \bar{3}}(\lambda, b, \varepsilon_{13}(\lambda e^{i\pi}, b)) \equiv \chi_{\bar{2}\to 3}(\lambda e^{i\pi}, b, \varepsilon_{13}(\lambda e^{i\pi}, b))$$

and

$$\chi_{\bar{2}\to 3}(\lambda, b, \varepsilon_{1\bar{3}}(\lambda e^{-i\pi}, b)) \equiv \chi_{2\to \bar{3}}(\lambda e^{-i\pi}, b, \varepsilon_{1\bar{3}}(\lambda e^{-i\pi}, b)). \tag{4.27}$$

Eqs (4.27) and the conditions (4.22) show that we have:

$$\chi_{2\to \bar{3}}(\lambda, b, \varepsilon_{13}(\lambda e^{i\pi}, b)) + \exp\left(-\lambda \oint_K q^{\frac{1}{2}}(x, b, \varepsilon_{13}(\lambda e^{i\pi}, b))\right) = 0$$

and

$$\chi_{\bar{2}\to 3}(\lambda, b, \varepsilon_{1\bar{3}}(\lambda e^{-i\pi}, b)) + \exp\left(+\lambda \oint_K q^{\frac{1}{2}}(x, b, \varepsilon_{1\bar{3}}(\lambda e^{-i\pi}, b))\right) = 0 \tag{4.28}$$

Consider now the functions:

$$\varepsilon_{13}^{\text{sub}}(\lambda, b) = \varepsilon_{13}(\lambda, b) - V_R$$

and

$$\varepsilon_{1\bar{3}}^{\text{sub}}(\lambda, b) = \varepsilon_{1\bar{3}}(\lambda, b) - V_R$$

and write for them the following Cauchy integrals:

$$\varepsilon_{1\bar{3}}^{\text{sub}}(\lambda, b) = \frac{1}{2\pi i} \int_{C_{1\bar{3}}} \frac{\varepsilon_{1\bar{3}}^{\text{sub}}(\lambda', b)}{\lambda' - \lambda} d\lambda'$$

and

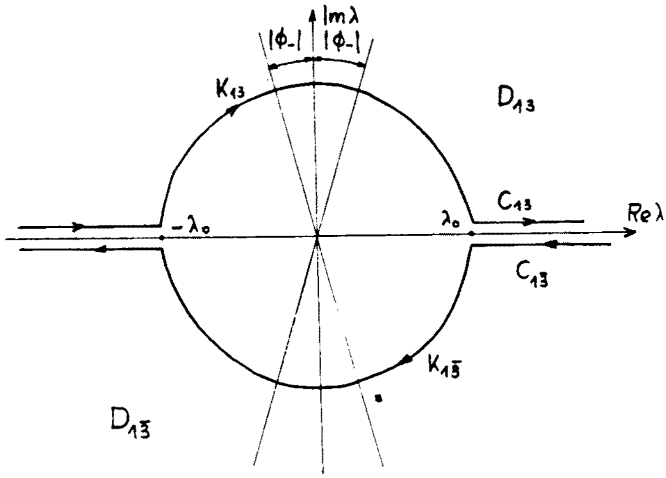

Fig. 11. The integration contours in the λ-plane used in the formulae (4.29)

$$0 = \frac{1}{2\pi i}\int C_{13}\frac{\varepsilon_{13}^{\rm sub}(\lambda',b)}{\lambda'-\lambda}d\lambda', \tag{4.29}$$

where the point λ and the integration contours $C_{1\bar{3}}$ and C_{13} are shown in Fig. 11. The radius λ_0 of the semicircles $K_{1\bar{3}}$ and K_{13} is chosen to be large enough in order to make it sure that the functions $\varepsilon_{1\bar{3}}$ and ε_{13} are holomorphic in their corresponding domains $D_{1\bar{3}}$ and D_{13} having $C_{1\bar{3}}$ and C_{13} as their respective boundaries. Subtracting Eqs (4.29) we get:

$$\begin{aligned}\varepsilon_{1\bar{3}}^{\rm sub}(\lambda,b) = &\frac{1}{2\pi i}\int_{\lambda_0}^{\infty}\Big(\varepsilon_{1\bar{3}}(\lambda' e^{-i\pi},b)-\varepsilon_{13}(\lambda' e^{i\pi},b)\Big)\frac{d\lambda'}{\lambda'+\lambda}\\ &-\frac{1}{2\pi i}\int_{\lambda_0}^{\infty}\Big(\varepsilon_{1\bar{3}}(\lambda',b)-\varepsilon_{13}(\lambda',b)\Big)\frac{d\lambda'}{\lambda'-\lambda}\\ &+\frac{1}{2\pi i}\int_{K_{1\bar{3}}}\varepsilon_{1\bar{3}}^{\rm sub}(\lambda',b)\frac{d\lambda'}{\lambda'-\lambda}\\ &+\frac{1}{2\pi i}\int_{K_{13}}\varepsilon_{13}^{\rm sub}(\lambda',b)\frac{d\lambda'}{\lambda'-\lambda}.\end{aligned} \tag{4.30}$$

From (4.30) the following asymptotic Bender–Wu representation for $\varepsilon_n(b)$

emerges immediately:

$$\varepsilon_n(b) \sim \frac{1}{2\pi i}\int_0^\infty \Big((\Delta'_{1\bar{3}} - \Delta'_{13})(-\lambda)^n + (\Delta_{1\bar{3}} - \Delta_{13})\lambda^n\Big)d\lambda\,, \tag{4.31}$$

where, by the way, we have introduced another four functions:

$$\begin{aligned}
\Delta'_{13}(\lambda,b) &= \varepsilon_{13}(\lambda e^{i\pi},b) - \varepsilon(\lambda,b)\\
\Delta'_{1\bar{3}}(\lambda,b) &= \varepsilon_{1\bar{3}}(\lambda e^{-i\pi},b) - \varepsilon(\lambda,b)\\
\Delta_{13}(\lambda,b) &= \varepsilon_{13}(\lambda,b) - \varepsilon(\lambda,b)\\
\Delta_{1\bar{3}}(\lambda,b) &= \varepsilon_{1\bar{3}}(\lambda,b) - \varepsilon(\lambda,b)\,.
\end{aligned} \tag{4.32}$$

The asymptotic behaviour of Δ's in (4.32) when $\lambda \to +\infty$ can be now easily obtained in the following way. Substitute into the first two of the identities in (4.26) $\varepsilon = \varepsilon(\lambda, b)$ and use (4.7) to get:

$$\begin{aligned}
&\chi_{\bar{2}\to 3}(\lambda,b,\varepsilon) + \exp\Big(\lambda\oint_K q^{\frac{1}{2}}(x,b,\varepsilon)dx\Big)\\
&\qquad = \chi_{2\to\bar{2}}(\lambda,b,\varepsilon)\frac{\exp\big(-2\lambda\xi(B,B')\big)}{\chi_{2\to 4}(\lambda,b,\varepsilon)}\\
&\chi_{2\to\bar{3}}(\lambda,b,\varepsilon) + \exp\Big(-\lambda\oint_K q^{\frac{1}{2}}(x,b,\varepsilon)dx\Big)\\
&\qquad = \chi_{2\to\bar{2}}(\lambda,b,\varepsilon)\frac{\exp\big(-2\lambda\xi(B,A')\big)}{\chi_{\bar{2}\to 4}(\lambda,b,\varepsilon)}\,.
\end{aligned} \tag{4.33}$$

Next expand the left-hand sides of (4.33) into the Taylor series around the points: $\varepsilon = \varepsilon_{13}(\lambda, b)$ and $\varepsilon = \varepsilon_{1\bar{3}}(\lambda e^{-i\pi}, b)$ — for the first of Eqs (4.33) and around the points: $\varepsilon = \varepsilon_{1\bar{3}}(\lambda, b)$ and $\varepsilon = \varepsilon_{13}(\lambda e^{-i\pi}, b)$ — for the second one. Use the conditions (4.22) and (4.28) and truncate the series at their first nonvanishing terms. In the limit $\lambda \to \infty$ one obtains for the leading terms successively:

$$\begin{aligned}
\Delta_{13} &\approx \frac{\chi_{2\to\bar{2}}(\lambda,b,\varepsilon)\exp\Big(-2\lambda\int_B^{B'} dx q^{\frac{1}{2}}(x,b,\varepsilon)\Big)}{\chi_{2\to 4}(\lambda,b,\varepsilon)\frac{\partial}{\partial\varepsilon_{13}}\Big(\chi_{\bar{2}\to 3}(\lambda,b,\varepsilon_{13}) + \exp\big(\lambda\oint_K q^{\frac{1}{2}}(x,b,\varepsilon_{13})dx\big)\Big)}\\
&\sim \frac{1}{\pi i\lambda}C_{2\bar{2}}(b,m)\sqrt{\frac{V''_R}{2}}\exp\Big(-2\lambda\int_{A(\infty)}^{B'(\infty)} dx q^{\frac{1}{2}}\Big)\left(\frac{(B'(\infty)-A(\infty))\sqrt{\lambda}}{C_{B'}\sqrt{e\sqrt{2}(m+\frac{1}{2})}}\right)^{2m+1}\\
\Delta'_{1\bar{3}} &\sim \Delta_{13}\\
\Delta_{1\bar{3}} &\sim -\frac{1}{\pi i\lambda}C\exp\Big(-2\lambda\int_{A(\infty)}^{A'(\infty)} q^{\frac{1}{2}}dx\Big)\left(\frac{\big(A'(\infty)-A(\infty)\big)\sqrt{\lambda}}{\tilde{C}_B\sqrt{e\sqrt{2}(m+\frac{1}{2})}}\right)^{2m+1}\\
\Delta'_{13} &\sim \Delta_{1\bar{3}}\,,
\end{aligned} \tag{4.34}$$

where $C = C_{2\bar{2}}(b,m)(V''_R/2)$ and $C_{2\bar{2}}(b,m) = \lim_{\lambda\to\infty} \chi_{2\to\bar{2}}(\lambda, b, \varepsilon(\lambda, b))$ and where the integrals in the exponentials correspond to Fig. 9. $C_{2\bar{2}}(b,m)$ is given again by (B.6) and $C_{B'}$ — by (B.3) where, additionally, the following substitutions should be made: $B(\infty) \to A(\infty)$, $A(\infty) \to B'(\infty)$ — $A(\infty)$, $V'_R \to V'_{B'}$ and $(m+1/2) \to (m+1/2)\sqrt{V''_R}$. Substituting (4.34) to (4.31) we get finally for $n \to \infty$:

$$\begin{aligned}\varepsilon_n(b) \sim \frac{-2C}{(2\pi)^2}\Bigg\{(-1)^n\Bigg[&\left(2\int_{A(\infty)}^{B'(\infty)} q^{\frac{1}{2}}dx\right)^{-n-m-\frac{1}{2}}\left(\frac{B'(\infty)-A(\infty)}{C_{B'}\sqrt{e\sqrt{2}(m+\frac{1}{2})}}\right)^{2m+1}\\ &+\left(2\int_{A(\infty)}^{A'(\infty)} q^{\frac{1}{2}}dx\right)^{-n-m-\frac{1}{2}}\left(\frac{A'(\infty)-A(\infty)}{\bar{C}_{B'}\sqrt{e\sqrt{2}(m+\frac{1}{2})}}\right)^{2m+1}\Bigg]\\ &+\left(2\int_{A(\infty)}^{A'(\infty)} q^{\frac{1}{2}}dx\right)^{-n-m-\frac{1}{2}}\left(\frac{A'(\infty)-A(\infty)}{\bar{C}_{B'}\sqrt{e\sqrt{2}(m+\frac{1}{2})}}\right)^{2m+1}\\ &+\left(2\int_{A(\infty)}^{B'(\infty)} q^{\frac{1}{2}}dx\right)^{-n-m-\frac{1}{2}}\left(\frac{B'(\infty)-A(\infty)}{C_{B'}\sqrt{e\sqrt{2}(m+\frac{1}{2})}}\right)^{2m+1}\Bigg\}\Gamma(n+m+\tfrac{1}{2}).\end{aligned} \tag{4.35}$$

The result (4.35) proves that the Borel function $\varepsilon(s,b)$ corresponding to Eq. (3.6) exists and is holomorphic in the circle $|s| < |s_R|$ where $s_R = \int_A^{A'} q^{1/2}(x,b,V_R)dx$ with four singularities at its boundary, namely, at $s = s_R,\ \bar{s}_R,\ -s_R,\ -\bar{s}_R$ (see Fig. 10). The Borel function $\varepsilon(s,b)$ is, therefore, holomorphic in the s-plane cut as it is shown in Fig. 10.

4.5. A dependence of $\varepsilon(\lambda,b)$ and $\tilde{\varepsilon}(s,b)$ on their arguments. A limiting case — the symmetric double-well

The results obtained in the previous section assumed that for a given $b > 0$ the variable λ has changed within the sector (4.13) with $|\lambda| > \lambda_0(b)$ and with the latter number chosen large enough to ensure that $\varepsilon(\alpha_0,b) < V_L(b)$. Therefore, the Borel transformation:

$$\varepsilon(\lambda,b) = 2\int_{\tilde{C}} e^{2\lambda s}\tilde{\varepsilon}(s,b)ds \tag{4.36}$$

(with $\tilde{C}$ shown in Fig. 10) is expected to reconstruct $\varepsilon(\lambda,b)$ in the sector (4.13) and for $|\lambda| > \lambda_0(b)$. Since the radius $\lambda_0(b)$ has to increase without bound when $b \to 0_+$ then the formula (4.36) is not expected to work in this limit. In fact, in the limit $b \to 0_+$ the singularities $-s_R$ and $-\bar{s}_R$ in Fig. 10 tend to coincide pinching in this way the integration contour $\widetilde{C}$ in (4.36), so that the latter formula becomes singular.

On the other hand in the limit $b \to 0_+$ the potential becomes the symmetric double-well with the energy levels given by:

$$\varepsilon(\lambda) \equiv \varepsilon(\lambda, 0). \tag{4.37}$$

The above equality allows us to regard the energy levels of the symmetric double-well as the Borel summable quantities, despite the fact that the corresponding Borel transformation (4.37) cannot be performed. (Note, however, that $\tilde{\varepsilon}(s, 0)$ is the well defined Borel function which can be used to obtain, for example, $\varepsilon_{13}(\lambda, 0)$ (defined by (4.22)) by the regular Borel transformation (4.36) taken along $\widetilde{C}_{13}$ shown in Fig. 10. A relevant discussion how to obtain explicitly $\varepsilon(\lambda)$ in terms of the Borel summable quantities $\varepsilon_{13}(\lambda, 0)$ and $\varepsilon_{1\bar{3}}(\lambda, 0)$ is postponed to Section 4.7).

Eq. (4.37) can be easily justified in the following way. If λ in $\varepsilon(\lambda, b)$, being kept fixed and positive, and b are chosen initially so that $\varepsilon(\lambda, b) < V_L$ (*i.e.* $|\lambda| > \lambda_0(b)$) then decreasing b (*i.e.* shallowing the right well) we can always cause the level to be pushed out above $V_L(b)$ *i.e.* above the bottom of the higher (left) well. In such a case the level satisfies the following quantization condition:

$$\chi_{2\to\bar{3}} \exp\left(\frac{\lambda}{2}\Big(\oint_K - \oint_{K'}\Big) q^{\frac{1}{2}} dx\right) + \chi_{\bar{2}\to 3} \exp\left(-\frac{\lambda}{2}\Big(\oint_K - \oint_{K'}\Big) q^{\frac{1}{2}} dx\right)$$
$$+ \exp\left(\frac{\lambda}{2}\Big(\oint_K + \oint_{K'}\Big) q^{\frac{1}{2}} dx\right) + \exp\left(-\frac{\lambda}{2}\Big(\oint_K + \oint_{K'}\Big) q^{\frac{1}{2}} dx\right) = 0, \tag{4.38}$$

which corresponds to Fig. 12. However, the above condition is also exactly equivalent to (4.7) if the energy $\varepsilon(\lambda, b)$ is smaller than V_L and the corresponding Stokes graph has rather the form shown in Fig. 7b. A continuation of (4.38) with respect to b from the form corresponding to the latter graph to the form corresponding to the one in Fig. 12 is, therefore, an analytic operation so that the level $\varepsilon(\lambda, b)$ behaves analytically with b crossing over the value V_L. This is certainly true if λ is fixed initially at sufficiently large value so that the left-hand side of (4.38) fails to fulfil (3.5) during the continuation. We conclude, therefore, that under the above condition the point $b = 0$ is regular for $\varepsilon(\lambda, b)$. When b becomes negative the roles of the wells are revised: the left well becomes deeper ($V_L < V_R$) and when $\lambda \to +\infty$ the level $\varepsilon(\lambda, b)$ is settling down just at the bottom of the

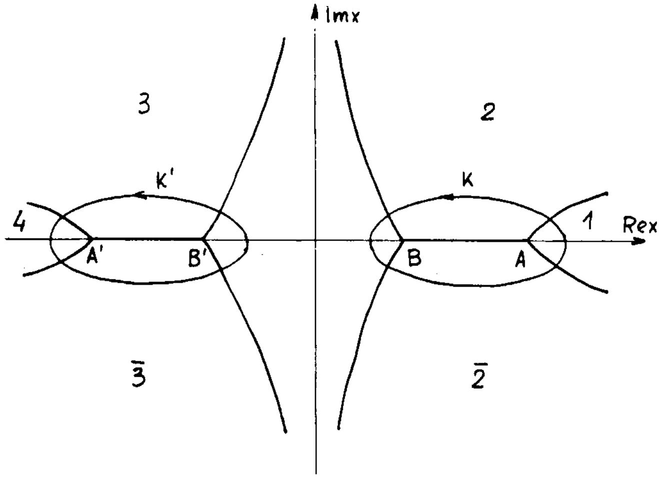

Fig. 12. The Stokes graph corresponding to the symmetric double-well

left well *i.e.* its asymptotic series expansion is determined according to the Bohr–Sommerfeld quantization corresponding to this well [27].

Summarizing, the behaviour of $\varepsilon(\lambda, b)$ is the following. Choosing initially $\lambda \gg 1$ such that $V_R < \varepsilon(\lambda, b) < 0 < V_L$, fixing λ and decreasing b to zero we cause $\varepsilon(\lambda, b)$ to increase up to the value $\varepsilon(\lambda, 0)$ *i.e.* to the corresponding energy level of the symmetric double-well. Decreasing b further $\varepsilon(\lambda, b)$ is also decreased taking the same values for the same absolute values of b *i.e.* $\varepsilon(\lambda, b) = \varepsilon(\lambda, -b)$. The point $b = 0$ is distinguished not only as the extremal point of $\varepsilon(\lambda, b)$ but also as the point where the Riemann surface of $\varepsilon(\lambda, b)$ splits into two disjoint parts each carrying different values of parity and corresponding to different functions $\varepsilon^{\pm}(\lambda)$ *i.e.* to two independent solutions to (4.38). On the other hand, as long as $b \neq 0$ all energy levels are only different branches of the same ramified function $\varepsilon(\lambda, b)$.

We have already mentioned that $\varepsilon(\lambda, b)$ cannot be obtained for $b = 0$ by the Borel transformation (4.36) of $\varepsilon(s, b)$ since the transformation becomes then singular. This problem cannot be avoided by any deformation of the contour $\widetilde{C}$ in Fig. 10 since for $b = 0$ the singularities of $\tilde{\varepsilon}(s, b)$ at $-s_R$ and at $-\bar{s}_R$ have to coincide. One can suppose, however, that decreasing b and passing by the point $b = 0$ from the left or from the right (note, that b moves along the imaginary axis in the b-plane) one can obtain from (4.36) an energy level in the left, now deeper, well. However, it is not the case. To see this, let us note that the pattern of the Riemann surface sheet shown in Fig. 10 changes with b from this in the figure when b is small and positive to that in Fig. 13 when b becoming negative passes by the origin $b = 0$, say, from the left. But, as it follows from (4.3), we should also have $\tilde{\varepsilon}(s, -b) = \tilde{\varepsilon}(s, b)$. Therefore, the pattern in Fig. 13 proves that the Borel transformation (4.36) cannot be defined on the sheet shown in the figure if we want to obtain $\varepsilon(\lambda, b)$ for negative b since the proper form of the corresponding sheet should be the same as in Fig. 10. We can conclude, therefore, that the relevant transformation is defined on some other sheet of the Riemann surface which for positive b looks like in Fig. 13 and for the negative one — like in Fig. 10. This means that it is the Riemann surface as a whole rather than its particular sheets which remains invariant under the reflection $b \to -b$.

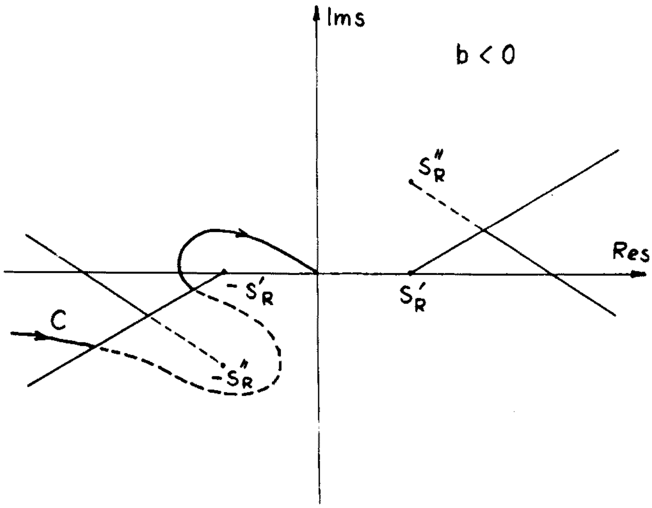

Fig. 13. The Borel plane corresponding to $\varepsilon(\lambda, b)$ in the double-well case after changing b from $b > 0$ to $b < 0$

4.6. Application of the modified Borel transformation

The problem of the singular behaviour of the Borel transformation in the symmetric double-well case can be avoided by applying the modified Borel transformation (2.15). To see this, let us note that, by definition, the singularities at $-s_R$ and at $-\bar{s}_R$ in Fig. 10 are transformed when $b \neq 0$ to lie on the second sheet of the Riemann surface corresponding to the modified Borel function $\tilde{\varepsilon}_{\rm mod}(s, b)$ (see Fig. 14) with the integration contour lying entirely in the first sheet and encircling the negative half of the real axis. An integration along this contour is not disturbed by the coincidence of the singularities at $-s_R$ and $-\bar{s}_R$ when $b \to 0$ since in the latter limit these singularities do not coincide at all on the considered Riemann surface. Therefore, the relevant integration is regular and can be performed also for $b = 0$.

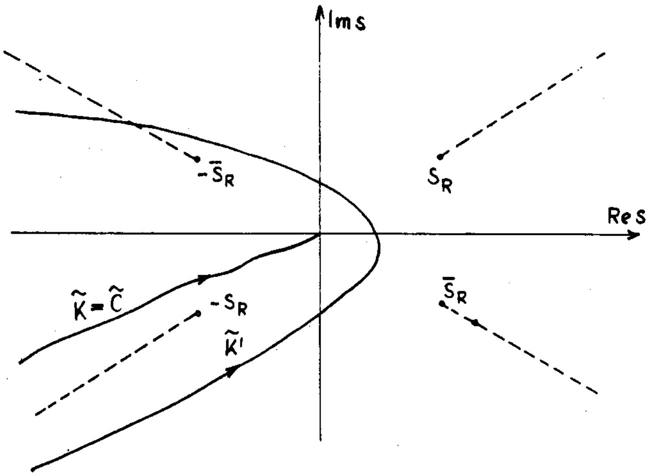

Fig. 14. The integration contour K' and the cut K in the Borel plane corresponding to the modified Borel transformation

Nevertheless, we shall show below, using the conventional Borel resumption methods, that energy levels in the symmetric double-well, lying below the top of the barrier dividing the wells (the levels are, therefore, essentially non-Borel summable quantities) can be constructed as a sum of some definitely Borel summable terms and of exponentially small explicitly non-Borel summable ones.

4.7. Energy levels in the symmetric double-well potential

For simplicity, we assume rather that the relevant potential is given by (4.1) where we have to put $b = 0$ and $x^2 \to -x^2$. (Note, however, that a general case of the symmetric double-well can be also considered in a similar way developed below not disturbing basic conclusions.) Because of the parity invariance we can also put $\psi_4(x,\lambda,\varepsilon) \equiv \psi_1(-x,\lambda,\varepsilon)$ (see Fig. 12 for the corresponding Stokes graph). Therefore, in the energy quantization condition:

$$\psi_4(x,\lambda,\varepsilon) = C(\lambda,\varepsilon)\psi_1(-x,\lambda,\varepsilon) \tag{4.39}$$

$C(\lambda,\varepsilon) \equiv \pm 1$ depending on the level parity (+1 for the even parity and −1 for the odd one). Matching the solutions ψ_1 and ψ_4 in sector 3 we obtain:

$$\chi_{\bar{2}\to 3}(\lambda,\varepsilon^C(\lambda)) + \exp\left(\lambda \oint_K q^{\frac{1}{2}} dx\right)$$
$$= iC\chi_{2\to\bar{2}}(\lambda\varepsilon^C(\lambda)) \exp\left(\left(-\int_B^{B'} + \oint_K\right) q^{\frac{1}{2}} dx\right). \tag{4.40}$$

Now, we want to relate the solution $\varepsilon^C(\lambda)$ to (4.40) to the Borel summable quantities $\varepsilon_{13}(\lambda,0))$ and $\varepsilon_{1\bar{3}}(\lambda)$ ($\equiv \varepsilon_{1\bar{3}}(\lambda,0))$ defined by the conditions (4.22). We want to do it for λ sufficiently large since only then all the functions $\varepsilon^C(\lambda)$ ans $\varepsilon_{1\bar{3}}(\lambda)$ can differ by exponentially small amounts. Note, that $\varepsilon_{13}(\lambda)$ and $\varepsilon_{1\bar{3}}(\lambda)$ are mutually complex conjugated being two different branches of the same quantity.

To achieve our goal let us expand the l.h.s. of (4.40) into the Taylor series around the point $\varepsilon_{13}(\lambda)$ and truncate the series on the first nonvanishing term. Taking into account the identities (4.22) we get:

$$\begin{aligned}&\left(\varepsilon^C(\lambda)-\varepsilon_{13}(\lambda)\right)\frac{\partial\left(\chi_{\bar{2}\to 3}(\lambda,\varepsilon_{13}(\lambda))+\exp\left(\lambda\oint_K q^{\frac{1}{2}}dx\right)\right)}{\partial\varepsilon_{13}(\lambda)}\\&=iC\chi_{2\to\bar{2}}\left(\lambda,\varepsilon^C(\lambda)\right)\exp\left(\left(-\int_B^{B'}+\oint_K\right)q^{\frac{1}{2}}dx\right).\end{aligned}\tag{4.41}$$

From (4.41) we obtain finally

$$\begin{aligned}\varepsilon^C(\lambda)&=\operatorname{Re}\varepsilon_{13}(\lambda)\\&-\operatorname{Im}\varepsilon_{13}(\lambda)\frac{\operatorname{Im}\frac{\partial}{\partial\varepsilon_{13}(\lambda)}\left[\chi_{\bar{2}\to 3}(\lambda,\varepsilon_{13}(\lambda))+\exp\left(\lambda\oint_K\sqrt{q}(x,\varepsilon_{13}(\lambda))dx\right)\right]}{\operatorname{Re}\frac{\partial}{\partial\varepsilon_{13}(\lambda)}\left(\chi_{\bar{2}\to 3}(\lambda,\varepsilon_{13}(\lambda))+\exp\left(\lambda\oint_K\sqrt{q}dx\right)\right)}\\&-C\frac{\chi_{2\to\bar{2}}(\lambda,\varepsilon^C(\lambda))\sin\left(\frac{\lambda}{i}\oint_K\sqrt{q}(x,\varepsilon^C(\lambda))dx\right)\exp\left(-\lambda\int_B^{B'}\sqrt{q}(x,\varepsilon^C(\lambda))dx\right)}{\operatorname{Re}\frac{\partial}{\partial\varepsilon_{13}(\lambda)}\left[\chi_{\bar{2}\to 3}(\lambda,\varepsilon_{13}(\lambda))+\exp\left(\lambda\oint_K\sqrt{q}(x,\varepsilon_{13}(\lambda))dx\right)\right]}.\end{aligned}\tag{4.42}$$

Note, that $\operatorname{Re}\frac{\partial}{\partial\varepsilon_{13}(\lambda)}\left[\chi_{\bar{2}\to 3}(\lambda,\varepsilon_{13}(\lambda))+\exp\left(\lambda\oint_K\sqrt{q}(x,\varepsilon_{13}(\lambda))dx\right)\right]\neq 0$ for λ large enough because of (4.22). Further, the second term in the r.h.s. of Eq. (4.42) has to be exponentially small due to the factor $\operatorname{Im}\varepsilon_{13}(\lambda)$ since the latter does not contribute to the asymptotic series expansion of $\varepsilon_{13}(\lambda)$. (This expansion coincides with the one corresponding to $\varepsilon^C(\lambda)$ and, therefore, has real coefficients.) In fact, it is shown in Appendix D that $\operatorname{Im}\varepsilon_{13}(\lambda)$ behaves as $\exp\left(-2\lambda\int_B^{B'}q^{1/2}(x,\operatorname{Re}\varepsilon_{13}(\lambda))dx\right)$ when $\lambda\to+\infty$, so that the discussed term should be neglected since it is outside the scope of the assumed accuracy with which $\varepsilon^C(\lambda)$ can be defined by (4.41). (To take this term into account we should also have to keep up the second and the third non-vanishing terms in the Taylor series expansion of the l.h.s. of (4.40).)

We would like to compare the result (4.42) with the corresponding well known textbook formula for the symmetric double-well energy splitting [3]. To this end let us estimate (4.42) in the limit $\lambda \to +\infty$ under the assumption that $\mathrm{Re}\,\varepsilon_{13}(\lambda)$ is kept fixed. (It is the assumption under which the energy splitting formula is discussed.) In this limit we have:

$$\chi_{2\to\bar{2}}(\lambda,\varepsilon^C(\lambda)) \sim 1$$

$$\sin\left(\frac{\lambda}{i}\oint_K q^{\frac{1}{2}}(x,\varepsilon^C(\lambda))dx\right) \sim \frac{1}{2i\lambda}\oint_K \omega(x,\mathrm{Re}\,\varepsilon_{13}(\lambda))dx$$

and

$$\mathrm{Re}\,\partial(\chi_{\bar{2}\to 3}(\lambda,\varepsilon_{13}(\lambda)) + \exp\frac{\lambda\oint_K q^{\frac{1}{2}}(x,\varepsilon_{13}(\lambda))dx}{\partial\varepsilon_{13}(\lambda)}$$

$$\sim -\frac{1}{4}\oint_K \omega(x,\mathrm{Re}\,\varepsilon_{13}(\lambda))dx\cdot\oint_K q^{-1/2}(x,\mathrm{Re}\,\varepsilon_{13}(\lambda))dx\,, \qquad (4.43)$$

where ω is given by (2.4). The first limit in (4.43) is obvious since under our assumptions the cut between the points B and A in Fig. 14 does not shrink to the point and $\chi_{2\to\bar{2}}$ behaves like a typical canonical coefficient. The second limit in (4.43) follows as a result of the Bohr–Sommerfeld quantization condition for $\varepsilon^C(\lambda)$ (see formulae (4.48) below). To obtain the last limit in (4.43) we have to estimate the behaviour of the quantity $\frac{\partial}{\partial\varepsilon_{13}(\lambda)}\left[(\chi_{\bar{2}\to 3}(\lambda,\varepsilon_{13}(\lambda))+\exp\left(\lambda\oint_K\sqrt{q}(x,\varepsilon_{13}(\lambda))dx\right)\right]$ more carefully. First, we can use its asymptotic form [27]:

$$\chi^{as}_{\bar{2}\to 3}(\lambda,\mathrm{Re}\,\varepsilon_{13}) = \exp\left(\int_{\infty_{\bar{2}}}^{\infty_3}\chi^-(x,\lambda,\mathrm{Re}\,\varepsilon_{13})dx\right), \qquad (4.44)$$

where $\chi^-(x,\lambda,\mathrm{Re}\,\varepsilon_{13})$ is given by the following asymptotic series [27]:

$$\chi^-(x,\lambda,\mathrm{Re}\,\varepsilon_{13}) = \sum_{s\geq 0}\lambda^{-n-1}\chi_{n+2}(x,\lambda,\mathrm{Re}\,\varepsilon_{13})\,. \qquad (4.45)$$

However, due to the simple form of the symmetric double-well potential being considered we have:

$$\int_{\infty_{\bar{2}}}^{\infty_3}\chi^-(x,\lambda,\mathrm{Re}\,\varepsilon_{13})dx = -\oint_K\chi^-(x,\lambda,\mathrm{Re}\,\varepsilon_{13})dx \qquad (4.46)$$

and this quantity is purely imaginary, as $\oint_K q^{1/2}(x, \mathrm{Re}\,\varepsilon_{13}(\lambda))dx$ is. Therefore, we have:

$$\mathrm{Re}\,\frac{\partial\left[\chi_{2\to 3}(\lambda, \varepsilon_{13}(\lambda)) + \exp\left(\lambda \oint_K q^{\frac{1}{2}}(x, \mathrm{Re}\,\varepsilon_{13})dx\right)\right]}{\partial\,\mathrm{Re}\,\varepsilon_{13}(\lambda)}$$
$$= -\frac{1}{i}\oint\limits_K \frac{\partial\chi^-}{\partial\,\mathrm{Re}\,\varepsilon_{13}}dx \sin\left(\frac{1}{i}\oint\limits_K \chi^- dx\right) + \frac{\lambda}{2i}\oint\limits_K q^{-1/2}dx \sin\left(\frac{\lambda}{i}\oint\limits_K q^{\frac{1}{2}}dx\right). \tag{4.47}$$

The first term in the r.h.s. of (4.47) behaves, according to (4.45), as

$$\frac{1}{\lambda^2}\oint\limits_K \chi_2^- dx \oint\limits_K \frac{\partial\chi_2^-}{\partial\,\mathrm{Re}\,\varepsilon_{13}}dx \quad \text{in the limit} \quad \lambda \to +\infty\,.$$

To estimate the corresponding behaviour of the second term let us note that we have in the limit $\lambda \to +\infty$:

$$-\frac{\lambda}{i}\oint\limits_K q^{\frac{1}{2}}(x, \mathrm{Re}\,\varepsilon_{13})dx - \frac{1}{i}\oint\limits_K \chi^-(x, \lambda, \mathrm{Re}\,\varepsilon_{13}) = (2n+1)\pi\,. \tag{4.48}$$

This is nothing but the Bohr–Sommerfeld condition which follows from (4.22) (see, for example, [27]). Therefore, in the limit $\lambda \to +\infty$ we get:

$$\begin{aligned}
&\frac{\lambda}{2i}\oint\limits_K q^{-1/2}(x, \mathrm{Re}\,\varepsilon_{13})dx \sin\left(\frac{\lambda}{i}\oint\limits_K q^{\frac{1}{2}}(x, \mathrm{Re}\,\varepsilon_{13})dx\right)\\
&= \frac{\lambda}{2i}\oint\limits_K q^{-1/2}(x, \mathrm{Re}\,\varepsilon_{13})dx \sin\left(\frac{1}{i}\oint\limits_K \chi^-(x, \lambda, \mathrm{Re}\,\varepsilon_{13})dx\right)\\
&\sim -\frac{1}{2}\oint\limits_K q^{-1/2}(x, \mathrm{Re}\,\varepsilon_{13})dx \oint\limits_K \chi_2^-(x, \mathrm{Re}\,\varepsilon_{13})dx\\
&= -\frac{1}{4}\oint\limits_K q^{-1/2}(x, \mathrm{Re}\,\varepsilon_{13})dx \oint\limits_K \omega(x, \mathrm{Re}\,\varepsilon_{13})dx
\end{aligned} \tag{4.49}$$

so proving the correctness of the last asymptotic in (4.43). The final form of (4.42) is, therefore:

$$\varepsilon^C(\hbar) = \mathrm{Re}\,\varepsilon_{13}(\lambda) - C\frac{\hbar}{T}\exp\left(\frac{1}{\hbar}\int\limits_B^{B'} v(x, \mathrm{Re}\,\varepsilon_{13})dx\right), \tag{4.50}$$

where T and v are the classical quantities: the period of the classical movement between the turning points A and B, and the velocity (defined by $2q = e^{-i\pi}v^2$), respectively. We have also substituted λ by $\sqrt{2/\hbar}$ in (4.50).

The splitting of opposite parity energy levels in the symmetric double-well which follows from (4.50) is, therefore:

$$\Delta\varepsilon(\hbar) = \varepsilon^-(\hbar) - \varepsilon^+(\hbar) \sim \frac{2\hbar}{T} \exp\left(\frac{1}{\hbar}\int_B^{B'} v(x, \mathrm{Re}\,\varepsilon_{13})dx\right). \tag{4.51}$$

The above formula coincides with the conventional one [3] proving that the asymptotic limit of $\Delta\varepsilon(\hbar)$ is independent of the summation method used to sum the asymptotic series expansion corresponding to $\varepsilon^{\pm}(\hbar)$ being, therefore, in a full accordance with the general philosophy related to the case [48]. This formula can be completed in two ways. First, the asymptotic value of the pre-exponential coefficient in (4.51) can be substituted by the actual value of this coefficient given by (4.42). Secondly, other subdominant terms can be added to the r.h.s. of (4.51) (with the help of the so called topological expansion [41]). Since one of such terms should be proportional to C^3 then it should be also proportional to the third power of the exponential factor itself present in (4.51). These new subdominant terms can be again calculated from (4.40) with its l.h.s. Taylor expanded around ε_{13} up to the third non-vanishing term.

5. Generalizations: λ-dependent nonpolynomial potentials and perturbations admitting semiclassical treatment

The results and methods developed in the previous Sections can be generalized at least in two directions:

1. to admit nonpolynomial potentials; and
2. to admit λ-dependent potentials.

The latter extension is closely related to the possibility of treating some perturbation expansions in the semiclassical manner. This possibility was used in the previous Section due to the scaling operation allowed by the anharmonic perturbation considered there. Perturbing potentials of this type form, however, only a limited class among all of them. Nevertheless, this class is distinctive in that that its confining potentials produce energy levels perturbation series, coefficients of which grow factorially. Perturbing potentials contained in this class are characterized in Section 5.2, below. Energy levels corresponding to perturbing potentials living outside the discussed class can show different behaviours [35–39].

Obviously, the above two directions of generalizations are interrelated and, in fact, the first one can be considered as a part of the second. Let us characterize, therefore, a class of λ-dependent potentials whose properties cannot change seriously the methods and results obtained in the previous sections.

5.1. λ-dependent potentials still admitting semiclassical treatment

If $V(x,\lambda)$ is such a potential then it should have the following "regular" properties:

1° $V(x,\lambda)$ is real for x real and λ real and positive;

2° $V(x,\lambda)$ is meromorphic in the x-plane for $|\lambda| > \lambda_0$ with λ_0 sufficiently large and independent of x and with at most square root branch point at the infinity in the λ-plane so that the domain $|\lambda| > \lambda_0$ lies on the λ-Riemann surface consisting of two sheets;

3° $V(x,\lambda)$ has the following asymptotic expansion for $\lambda \to \infty$, uniform in the λ-plane:

$$V(x,\lambda) \sim V_0(x) + \sum_{r=0,1}\sum_{n\geq 1} \lambda^{-n+\gamma r} V_{n,r}(x) \tag{5.1}$$

with $\gamma_0 = 0$ and $\gamma_1 = 1/2$ and with $V_0(x)$ and $V_{n,r}(x)$, $n \geq 1$, $r = 0,1$, meromorphic in the x-plane and fulfilling there the reality condition (condition 1° above) each;

4° for any fixed λ ($|\lambda| > \lambda_0$) the Stokes graph $S(\lambda,E)$ corresponding to $q(x,\lambda,E)$ ($\equiv V(x,\lambda) - E$) is normal *i.e.* for any anti-Stokes line $\mathrm{Im}\int_{x_p}^{x} q^{1/2}dx = 0$ emerging from (any) root x_p ($\equiv x_p(\lambda,E)$) of $q(x,\lambda,E)$ the domains $\mathrm{Re}\int_{x_p}^{x} q^{1/2}dx > 0$ or $\mathrm{Re}\int_{x_p}^{x} q^{1/2}dx < 0$ which contain this anti-Stokes line do not contain any other zero of $q(x,\lambda,E)$ for $|x|$ large enough;

5° each zero (or pole) of $q(x,\lambda,E)$ fulfils:

$$\lim_{\lambda\to\infty} x_p(\lambda,E) = x_p(E)\,, \tag{5.2}$$

where $x_p(E)$ is a zero (or pole) of $q_0(x,E)$ ($\equiv V_0(x) - E$); and

6° all zeros (poles) of $V(x,\lambda)$ are simple and do not coincide when the limit (5.2) is taken *i.e.* zeros and poles of $V_0(x)$ are also simple.

The latter limitation is not very serious and is of rather technical importance since each multifold zero (pole) can be always splitted into several simple ones by introducing suitable splitting parameters. This technical assumption allows to utilize in the most effective way the canonical properties of the Stokes graphs.

The properties 1° – 6° together allow to repeat almost literally the reasonings of the previous sections applying the same method and technique as used there.

The reality condition 1° is not related directly to the subject considered and can be released if necessary. It is included for definiteness.

The properties 2° – 3° ensure that the asymptotic calculus we have developed in our earlier paper [27] can be applied here as well.

The property 4° allows to maintain the use of the notions of sectors, canonical paths, canonical domains, fundamental solutions, canonical coefficients *etc.* as well as to write quantization conditions in the usual manner demonstrated in the previous sections.

The property 5° guarantees the asymptotic properties of the relevant quantities (wave functions, canonical coefficients, energy levels, matrix elements *etc.*) to be determined by the potential $V_0(x)$ with rest of the potential coefficients in (5.1) playing a role of asymptotic corrections.

In particular if $V_0(x)$ is a single-well potential then it is easy to show with the methods of Section 4 that the large order coefficients of the asymptotic expansion corresponding to an energy level $E(\lambda)$ in the potential $V(x,\lambda)$ have the factorial growth (see Appendix D for details) *i.e.* such a rate of growth appears to be a rule.

5.2. Perturbations allowing semiclassical treatment

The last statement of the previous section seems to be in a contradiction with the well known fact that the rate of growth of the levels $E(\lambda)$ corresponding to the potential $U(x,\beta) = x^2 + \beta x^{2m}$ is faster (being ruled by the $(mn)!$ law) if the asymptotic perturbative calculation are developed with respect to β around $\beta = 0$. However, such cases of perturbative calculations can be easily included in the semiclassical scheme of the asymptotic calculations described above by suitable rescaling procedure. In general, such a procedure can be performed with potentials from a limited class only. This class can be characterized as follows.

Let $U(x,\beta)$ be a holomorphic function of β at $\beta = 0$, so that the series:

$$U(x,\beta) = \sum_{n\geq 0} \beta^n U_n(x) \tag{5.3}$$

is convergent in some circle $|\beta| < \beta_0$. We shall say that the perturbation:

$$U_{\text{pert}}(x,\beta) = \sum_{n\geq 1} \beta^n U_n(x) \tag{5.4}$$

admits semiclassical treatment if there is a function $f(\lambda)$ holomorphic for $|\lambda| > \lambda$ in the two-sheeted Riemann surface and vanishing at its infinity and such that in the relation:

$$U(x\sqrt{\lambda}, f(\lambda)) \equiv \lambda V(x,\lambda) \tag{5.5}$$

the new potential $V(x,\lambda)$ have all the properties 1°–6° enumerated above.

It should be noted that the transformation: $x \to x\sqrt{\lambda}$ if performed in the Schrödinger equation is unitary. Therefore, the energy spectrum $E_U(\lambda)$ of the Hamiltonian $H_U = p^2/2 + U(x,\beta)$ is determined by the spectrum $E_V(\lambda)$ of the Hamiltonian $H_V = p^2/2 + V(x,\lambda)$ according to the relation:

$$E_U(f(\lambda)) = \lambda E_V(\lambda)\,. \tag{5.6}$$

As a consequence of (5.5) and (5.6) and in agreement with our previous discussion, large order coefficients of the asymptotic series expansion in λ of the level $E_U(f(\lambda))$ have the factorial rate of growth.

A simple illustration to these considerations is a potential $U(x,\beta) = x^2 + \beta P_{2m}(x)$, where $P_{2m}(x)$ is a polynomial of degree $2m$ (with $a_{2m} > 0$). We can take for this case $f(\lambda) = 1/(a_{2m}\lambda^{m-1})$ to obtain $V(x,\lambda)$ in the form:

$$V(x,\lambda) = x^2 + V_{2m}(x,\lambda)\,, \tag{5.7}$$

where $V_{2m}(x,\lambda)$ is a polynomial with respect to x and $\lambda^{-1/2}$ such that:

$$\lim_{\lambda\to\infty} V_{2m}(x,\lambda) = x^{2m}\,. \tag{5.8}$$

An example of a potential with an exponential growth in the x-plane and still admitting semiclassical treatment is the following one:

$$U(x,\beta) = x^2 + \beta P_{2m}(x)\exp(\beta^{\alpha}P_n(x)) \qquad \text{with} \qquad \alpha \geq \frac{n}{2m-1}\,.$$

As a counterexample can serve the potential $U(x\beta) = x^2 + \beta e^x$ which does not admit semiclassical treatment so that the large order coefficients of its energy level perturbation series (constructed with respect to β) can deviate from the factorial rule of growth. In fact, these coefficients grow faster with n than $(\gamma n)!$ with any $\gamma > 0$ [35–39]. A particular application of the theory developed above to the supersymmetric potential (with broken supersymmetry) can be found in [41].

6. Summary and conclusions

We have constructed in this paper the consistent and rigorous semiclassical theory of the one-dimensional Schrödinger equation based on the Balian–Bloch representation. In this theory the properties of the fundamental solutions and the primitive canonical coefficients play essential roles ensuring:

1. the existence of their Laplace transforms *i.e.* their Balian–Bloch representations;

2. the existence of their semiclassical series expansions; and
3. the existence of their corresponding Borel functions and Borel sums.

Since the sets of the fundamental solutions are sufficient to solve any quantum mechanical one-dimensional problem one can conclude that the physical quantities which are determined by the primitive canonical coefficients can be analyzed in a definite semiclassical way. In particular, it has been shown that some properties seem to be universal for any semiclassical series expansion. To this class belongs the large order behaviour of the series coefficients which is factorial independently of the quantity considered. As a consequence of this is the common Borel summability of the semiclassical series expansions substituted if necessary by the modified Borel summability.

An important application of the semiclassical theory developed in this paper is the perturbation theory where a large class of perturbing potentials exists admitting semiclassical treatment. The existence of such a class of potentials allows to treat uniformly many perturbing potentials a separate investigation of which can show apparently different behaviours of corresponding perturbation series they generate.

The theory developed in the present paper can be completed by construction of explicit representations for both the Borel functions and the corresponding Borel transformations of the wave functions and of the primitive canonical coefficients as well. It is done elsewhere [41]. Their existence provides a constructive proof that the relevant semiclassical expansions in the one-dimensional quantum mechanics are Borel summable.

Let us note, finally, that the approach to the semiclassical theory of the one-dimensional Schrödinger equation developed in this paper can be extended effectively to n-dimensions promising a hope for the semiclassical quantization of the chaotic classical mechanical systems [43]. As such it can be an alternative to the approach developed recently and based on the semiclassical limit of the Feynman path integral and on the Riemann–Siegel formula (see [44–47], for example).

It is my great pleasure thank to dr P. Kosiński for many fruitful discussions as well as for the final, careful reading of the manuscript.

Appendix A

Let K_ρ be for a given $\arg\lambda$ and E the pairwise communicated canonical domains corresponding to fundamental solutions $\psi_\rho(x,\lambda,E)$, $\rho = i,j,k,r$. Then, the canonical coefficients $\alpha_{i/j\to k}(\lambda,E)$, $\alpha_{i/j\to r}(\lambda,E),\ldots$ *etc.* fulfil the following identities:

$$\alpha_{i/j\to k}(\lambda,E) = \alpha_{i/j\to r}(\lambda,E) + \alpha_{i/r\to j}(\lambda,E)\alpha_{r/j\to k}(\lambda,E) \qquad \text{(A.1)}$$

with any perturbation of the indices i, j, k, r. The identities (A.1) follow as a result of the existence of several equivalent linear relations between each triad of the solutions ψ_ρ. These identities if expressed in terms of the primitive canonical coefficients $\chi_{i\to j}(=\chi_{j\to i}), \ldots$ *etc.* provide a unique relation between them of the form:

$$\chi_{i\to K}(\lambda, E)\chi_{r\to j}(\lambda, E) = \chi_{i\to r}(\lambda, E)\chi_{j\to k}(\lambda, E) + \chi_{i\to j}(\lambda, E)\chi_{k\to r}(\lambda, E)\exp\left(\sigma\lambda \oint_K q^{\frac{1}{2}} dx\right), \quad (A.2)$$

where $\exp\left(\sigma\lambda \oint_K q^{1/2} dx\right)$ $(\sigma = \pm 1)$ is some phase coefficient which vanishes when $|\lambda| \to \infty$ *i.e.* $\mathrm{Re}\left(\sigma\lambda \oint_K q^{1/2} dx\right) \to -\infty$ in this limit. The contour K is closed with some pair of turning points inside it. The identity (A.2) can be continued analytically with λ and E still preserving the form (A.2) *i.e.* the primitive coefficients $\chi_{i\to j}(\lambda, E)\ldots$ *etc.* in (A.2) which lost their canonicity during the continuation can be substituted by the canonical ones multiplied by some new phase coefficients so that after simple algebraical manipulations the resulting identity takes on again the form (A.2) with asymptotically ($|\lambda| \to \infty$) vanishing phase coefficient.

Appendix B

Since $\lim_{\lambda\to+\infty} \chi_{2\to 4}(\lambda, b, \varepsilon(\lambda e^{i\pi}, b)) = \lim_{\lambda\to+\infty} \chi_{\bar{2}\to 4}(\lambda, b, \varepsilon(\lambda e^{i\pi}, b)) = 1$ in the formulae (4.20) then only the presence of the remaining factors in (4.21) needs some explanation. A part of them come from the integrals $\int_{B'}^{A} q^{1/2} dx$ and $\int_{B'}^{A'} q^{1/2} dx$ in (4.20). Namely, since $B'(\lambda e^{\pm i\pi}) = \pm i(2\varepsilon_0/\lambda)^{1/2}$, $A(\lambda e^{\pm i\pi}) \sim A(\infty) - \varepsilon_0/[q'(A(\infty))\lambda]$ and $A'(\lambda e^{\pm i\pi}) \sim A'(\infty) - \varepsilon_0/[q'(A'(\infty))\lambda]$ then for λ large enough we have (up to the order $O(\lambda^{-1})$):

$$2\lambda \int_{B'(\lambda e^{i\pi})}^{A(\lambda e^{i\pi})} \sqrt{q(x, b, \varepsilon(\lambda e^{i\pi}))}\, dx = 2\lambda \int_{B'(\infty)}^{A'(\infty)} \sqrt{q(x, b, 1/16)}\, dx$$
$$+ (2m+1)\log\sqrt{\lambda} + (2m+1)\log\left(\frac{A(\infty)}{C_A(m)}\right) + \frac{1}{2}\left(m+\frac{1}{2}\right)\log 2$$
$$- \log\left(m+\frac{1}{2}\right)^{m+\frac{1}{2}} - \left(m+\frac{1}{2}\right)(1+i\pi), \quad (B.1)$$

where $B'(\infty)=0$ and $A'(\infty)=[b+i(2-b^2)^{1/2}]/2$ (so that $|A'(\infty)|=1/\sqrt{2}$) and ε_0 is given by the following asymptotic expansion for the m-th energy level $\varepsilon\big(\lambda e^{\pm i\pi},b\big)$ [27]:

$$\varepsilon\big(\lambda e^{\pm i\pi},b\big)\sim\frac{1}{16}-\frac{2(m+\frac{1}{2})}{\lambda}+\ldots \tag{B.2}$$

so that $\varepsilon_0=\sqrt{2}(m+\frac{1}{2})$ (*cf.* (4.6)). The constant $C_A(m)$ in (B.1) is given by:

$$\begin{aligned}\log C_A(m)=\sqrt{2}\Big(&\big[2\big(A(\infty)V'(A(\infty))\big)^{-1}\big)\\&+\mathrm{Re}\,\big(A(\infty)V'(A(\infty))\big)^{-1}\big]\int\limits_{B'(\infty)}^{A(\infty)}q^{\frac{1}{2}}(x,b,\tfrac{1}{16})dx\\&+2i\,\mathrm{Im}\,\big(A(\infty)V'(A(\infty))\big)^{-1}\int\limits_{B'(\infty)}^{A(\infty)}x^2\Big(\frac{x-A(\infty)}{x-A'(\infty)}\Big)^{\frac{1}{2}}dx\\&+2\int\limits_{B'(\infty)}^{A(\infty)}dx\frac{\Big(\frac{x-A(\infty)}{x-A'(\infty)}\Big)^{\frac{1}{2}}-\frac{1}{\sqrt{2}}}{x}.\end{aligned} \tag{B.3}$$

Another factor coming out from taking the limit of the derivative

$$\frac{\partial\chi_{2\to\bar{2}}\big(\lambda,b,\varepsilon_{2\bar{2}}(\lambda,b)\big)}{\partial\varepsilon_{2\bar{2}}(\lambda,b)}\quad\text{when}\quad\lambda\to+\infty$$

can be estimated using the following identity:

$$\begin{aligned}\chi_{1\to4}(\lambda,b,\varepsilon)\chi_{2\to\bar{2}}(\lambda,b,\varepsilon)&=\chi_{2\to4}(\lambda,b,\varepsilon)\\&+\chi_{\bar{2}\to4}(\lambda,b,\varepsilon)\exp\Big(-\lambda\oint_K q^{\frac{1}{2}}(x,b,\varepsilon)dx\Big),\end{aligned} \tag{B.4}$$

from which it follows that:

$$\begin{aligned}\frac{\partial\chi_{2\to\bar{2}}(\lambda,b,\varepsilon_{2\bar{2}}(\lambda,b))}{\partial\varepsilon_{2\bar{2}}(\lambda,b)}&=\frac{1}{\chi_{1\to4}(\lambda,b,\varepsilon_{2\bar{2}}(\lambda,b))}\\&\times\frac{\partial}{\partial\varepsilon}\Big(\chi_{2\to4}(\lambda,b,\varepsilon)+\chi_{\bar{2}\to4}(\lambda,b,\varepsilon)\exp\Big(-\lambda\oint_K q^{\frac{1}{2}}(x,b,\varepsilon)dx\Big)\Big)\Big|_{\varepsilon=\varepsilon_{2\bar{2}}(\lambda,b)}\\&\sim\frac{\lambda}{2}\frac{1}{C_{14}(b,m)}\int_K q^{-\frac{1}{2}}(x,b,\varepsilon_{2\bar{2}}(\lambda,b))dx\sim\frac{\pi i\sqrt{2}\lambda}{C_{14}(b,m)},\end{aligned} \tag{B.5}$$

where $C_{14}(b,m) = \lim_{\lambda\to\infty} \chi_{1\to4}(\lambda, b, \varepsilon_{2\bar{2}}(\lambda, b))dx$ has been determined in our earlier paper [27] being given by the following asymptotic series:

$$\log C_{14}(b,m) = 2\pi\sqrt{2}\sum_{p\geq 0}\left(-4\sqrt{2}(m+\tfrac{1}{2})\right)^{-p-1}\sum_{r+2\nu=m_p} 2^{-\frac{r}{2}}e^{-\frac{i\pi r}{2}}$$
$$\times\frac{f_{pr\nu}(B'(\infty))}{\Gamma(p+1+\frac{r}{2}+\frac{1}{2})}\sum_{q=0}^{r}\binom{r}{q}(-2)^q\frac{\Gamma(2p+2+r-q)}{\Gamma(p+\frac{3}{2}+\frac{r}{2}-q)} \quad (B.6)$$

with the coefficients $f_{pr\nu}$ defined in [27] (see Appendix 2 there).

A similar estimation of the integral $\int_{b'}^{A'} q^{1/2}dx$ results in changing $A(\infty)$ into $A'(\infty)$ $(= \bar{A}(\infty))$ in (B.1) so that together with (B.1) and (B.5) and with the equality $\int_{B'(\infty)}^{A(\infty)} q^{1/2}dx = \int_{B'(\infty)}^{A'(\infty)} q^{1/2}dx$ the final formula (4.21) is obtained.

Appendix C

We shall show below that the results obtained in Section 4 for the cubic-quartic oscillator can be generalized to the potentials $V(x,\lambda)$ discussed in Section 5.1. For simplicity we limit our considerations to the case of the single-well potentials.

The relevant considerations need to take into account three types of the corresponding Stokes graph patterns. Two of them resemble the Stokes line configurations shown in Figs 6 and 7. The third one can be obtained as a variation of the Fig. 7 when additional turning points are placed pairwise inside sector 1 developing in this way a pattern similar to the left part of the Fig. 7. Since the relevant considerations repeat to large extent the discussions performed in Section 4 we shall examine only the case shown in Fig. 15.

According to our assumptions about the properties of $V(x,\lambda)$ made in Section5 there exist sector 2 and $p-1$ (therefore, also $\bar{2}$ and $\overline{p-1}$) being the nearest neighbours to the sector 1 and p, correspondingly, as well as there are roots $A_1, \bar{A}_1, A_1', \bar{A}_1'$ with the Stokes lines emerging from them being the nearest to the corresponding Stokes lines bounding the sectors 1 and p, respectively. The fundamental solutions corresponding to the latter sectors are to be matched. All sectors $2, \bar{2}, p-1$ and $\overline{p-1}$ communicate canonically with the sectors 1 and p so that the quantization condition reads:

$$\chi_{2\to p}(\lambda,E)\chi_{1\to\overline{p-1}}(\lambda,E) + \exp\Big(-\lambda\oint_{C_{B'B}} q^{\frac{1}{2}}dx\Big) = 0\,. \quad (C.1)$$

Note that the positions of roots $A_1, \ldots$ *etc.* (and possible poles) depend now on λ. However, if we take λ sufficiently large, then the Stokes graph in Fig. 15 will almost coincide with that corresponding to the term $V(x)$ in the expansion (5.1) in Section 5. Putting as usual $E = V_{\rm min}$ with $V_{\rm min}$ being the absolute minimum of $V(x, \lambda)$) we arrive at the graph in Fig. 16. The next step is to establish the normal sectors of $\chi_{2\to p}(\lambda, V_{\rm min})$ and $\chi_{1\to\overline{p-1}}(\lambda, V_{\rm min})$ following from the graph in Fig. 16. It should be clear that these sectors differ only slightly from those given by the inequalities (4.8) and (4.9) in Section 4 due to exactly the same reasoning as done in the case of cubic-quartic oscillator. Namely, rotating the graph in Fig. 16 anticlockwise (argλ revolves clockwise) the canonical contact of the sectors 2 and p is interrupted as soon as the single slashed Stokes lines in Fig. 16 coincide *i.e.* after the rotation by $\pi/2+\varphi$ ($\arg\lambda = -\pi/2 - \varphi$), where $\varphi = \arg(\pm \int_{B'}^{A_1} q^{1/2} dx)$ (a choice of the sign is determined by the condition $|\varphi| < \pi/2$). Conversely, this canonical contact does not interrupt if we rotate the graph in Fig. 16 clockwise up to the following sector substitutions: $2 \to 1 \to \bar{2}$ and $\overline{p-1} \to p \to p-1$ *i.e.* up to the rotation of the graph by $-\pi + \phi$ ($\arg\lambda = \pi - \varphi$). Only the further rotation by $-\pi/2$ which leads to coincidence of the Stokes lines (being now in the new configuration) slashed double breaks ultimately the canonical communication of the considered sectors. Therefore, the normal sector for $\chi_{2\to p}(\lambda, V_{\rm min})$ is given by:

$$-\frac{\pi}{2} - \varphi < \arg\lambda < \frac{3\pi}{2} - \varphi. \qquad \text{(C.2)}$$

In a completely similar way we obtain for the normal sector of $\chi_{1\to\overline{p-1}}(\lambda, V_{\rm min})$;

$$-\frac{\pi}{2} - \varphi' < \arg\lambda < \frac{3\pi}{2} - \varphi', \qquad \text{(C.3)}$$

where $\varphi' = \arg(\pm \int_{B'}^{\bar{A}_1'} q^{1/2} dx)$, $|\varphi'| < \pi/2$. Taking also into account the sectors complex conjugated to (C.2) and (C.3) we obtain for the normal sector of energy $E(\lambda)$:

$$|\arg\lambda| < \frac{3\pi}{2} - \varphi_{\rm max}, \qquad \text{(C.4)}$$

where $\varphi_{\rm max} = \max(|\varphi|, |\varphi'|) < \pi/2$.

There is no problem in extending all the further constructions of Section 4 corresponding to the considered case. Again, we can match the fundamental solutions ψ_2 and $\psi_{\bar{2}}$ to construct an auxiliary function $E_{2\bar{2}}(\lambda)$ with the same asymptotic series expansion as $E(\lambda)$ has, and we can repeat all the further reasonings of Section 4.4.1 leading to the result (4.21) *i.e.* to the conclusion that $E(\lambda)$ is the Borel summable quantity.

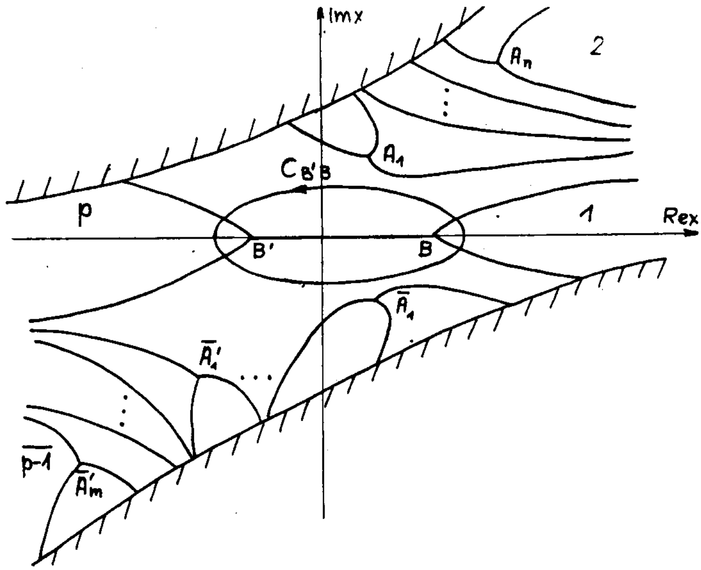

Fig. 15. The Stokes graph corresponding to the general case of the single-well potential

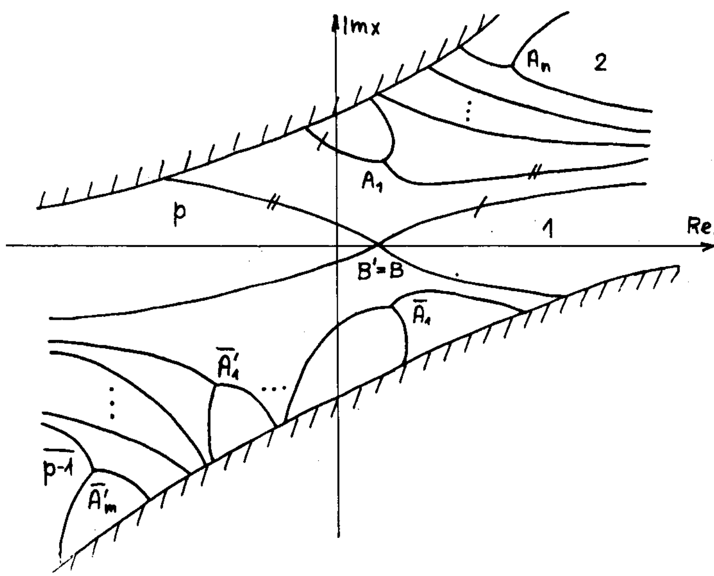

Fig. 16. The Stokes graph corresponding to the particle setlled down at the well's bottom of the general single-well case

There should also be no doubt that the remaining two cases can be treated in a similar manner copying the corresponding results of the anharmonic oscillator.

Appendix D

We shall show below that $\operatorname{Im}\varepsilon_{13}(\lambda)$ is exponentially small for $\lambda \to +\infty$. To this end let us note that the asymptotic forms $\chi^{as}_{2\to\bar{3}}(\lambda, \varepsilon^{as}(\lambda))$ and $\chi^{as}_{\bar{2}\to 3}(\lambda, \varepsilon^{as}(\lambda))$ of $\chi_{2\to\bar{3}}(\lambda, \varepsilon(\lambda))$ and $\chi_{\bar{2}\to 3}(\lambda, \varepsilon(\lambda))$, correspondingly, (given by (4.44) for $\chi_{\bar{2}\to 3}(\lambda, \varepsilon(\lambda))$ and by an analogous formula for $\chi_{2\to\bar{3}}(\lambda, \varepsilon(\lambda))$) fulfil the following relation:

$$\chi^{as}_{\bar{2}\to 3}(\lambda, \varepsilon^{as}(\lambda))\chi^{as}_{2\to\bar{3}}(\lambda, \varepsilon^{as}(\lambda)) = 1. \tag{D.1}$$

Since, further, the holomorphic domains of the Borel functions corresponding to $\chi_{\bar{2}\to 3}(\lambda, \varepsilon(\lambda))$ and $\chi_{2\to\bar{3}}(\lambda, \varepsilon(\lambda))$ are given by the last two equations in (4.25) (with $\varphi_- = 0$) then summing a la Borel both sides of (D.1) along the contour $\widetilde{C}_{13}$ lying in $\widetilde{D}_{13}$ (see Section 4.4.2. for definitions) with $\operatorname{Re}\widetilde{C}_{13} \leq 0$ we get:

$$\chi_{\bar{2}\to 3}(\lambda, \varepsilon_{13}(\lambda))\chi^{\widetilde{C}_{13}}_{2\to\bar{3}}(\lambda, \varepsilon_{13}(\lambda)) = 1, \tag{D.2}$$

where, according to the rules of the Borel resummations (see [40], for example):

$$\begin{aligned}\chi^{\widetilde{C}_{13}}_{2\to\bar{3}}(\lambda, \varepsilon_{13}(\lambda)) &= \int\limits_{\widetilde{C}_{13}} \chi^{*}_{2\to\bar{3}}(s, \tilde{\varepsilon}(s))e^{2\lambda s}ds \\ &= \int\limits_{\widetilde{C}_{13}} \widetilde{\chi}_{2\to\bar{3}}\Big(s, \operatorname{Re}\varepsilon_{13} + \int\limits_{\widetilde{C}_{13}} \widetilde{\varepsilon}(s')e^{2\lambda s'}ds'\Big)e^{2\lambda s}ds. \qquad (D.3)\end{aligned}$$

However, the Borel sums $\chi^{*}_{2\to\bar{3}}(s, \widetilde{\varepsilon}(s))$ and $\widetilde{\varepsilon}(s)$ are holomorphic in the cut s-plane with two cuts along the real axis at

$$\Big(-\infty, -\int\limits_{B}^{B'} q^{\frac{1}{2}}(x, \operatorname{Re}\varepsilon_{13})dx\Big) \quad \text{and at} \quad \Big(+\int\limits_{B}^{B'} q^{\frac{1}{2}}(x, \operatorname{Re}\varepsilon_{13})dx, +\infty\Big).$$

The last statement follows directly from (4.25). Therefore, deforming the contour $\widetilde{C}_{13}$ from the half-plane $\widetilde{D}_{13}$ into the $\widetilde{D}_{1\bar{3}}$ one, we obtain:

$$\chi_{2\to\bar{3}}(\lambda, \varepsilon_{13}(\lambda)) = \chi_{2\to\bar{3}}(\lambda, \varepsilon_{1\bar{3}}(\lambda)) + \int\limits_{\widetilde{C}} \chi^{*}_{2\to\bar{3}}(s, \tilde{\varepsilon}(s))e^{2\lambda s}ds$$

$$= \chi_{2\to\bar{3}}(\lambda, \varepsilon_{1\bar{3}}(\lambda)) + \int_{\widetilde{C}} \tilde{x}_{2\to\bar{3}}\Big(s, \mathrm{Re}\,\varepsilon_{13} + \int_{\widetilde{C}} \tilde{\varepsilon}(s')e^{2\lambda s'}ds'\Big)e^{2\lambda s}ds, \tag{D.4}$$

where $\widetilde{C} = \widetilde{C}_{13} - \widetilde{C}_{1\bar{3}}$ is the contour rounding the cut $(-\infty, -\int_B^{B'} q^{1/2}(x, \mathrm{Re}\,\varepsilon_{13})dx)$ clockwise. Substituting (D.4) into (D.2) and using (4.22) we get:

$$\exp\left(\lambda \oint_K \Big(q^{\frac{1}{2}}(x,\varepsilon_{13}) - q^{\frac{1}{2}}(x,\varepsilon_{1\bar{3}})\Big)dx\right) = 1$$
$$-\chi_{2\to\bar{3}}(\lambda, \varepsilon_{1\bar{3}}(\lambda)) + \int_{\widetilde{C}} \widetilde{\chi}_{2\to\bar{3}}\Big(s, \mathrm{Re}\,\varepsilon_{13} + \int_{\widetilde{C}} \tilde{\varepsilon}(s')e^{2\lambda s'}ds'\Big)e^{2\lambda s}ds. \tag{D.5}$$

Since $\varepsilon_{1\bar{3}}(\lambda) = \bar{\varepsilon}_{13}(\bar{\lambda})$ and

$$\int_{\widetilde{C}} \tilde{\varepsilon}(s)e^{2\lambda s}ds \sim \exp\Big(-2\lambda \int_B^{B'} q^{\frac{1}{2}}(x, \mathrm{Re}\,\varepsilon_{13})dx\Big)$$

then from (D.5) we obtain finally:

$$\mathrm{Im}\,\varepsilon_{13}(\lambda) \sim \frac{\chi_{\bar{2}\to 3}(\lambda, \mathrm{Re}\,\varepsilon_{13})}{\lambda \oint_K q^{-1/2}(x, \mathrm{Re}\,\varepsilon_{13})dx} \int_{\widetilde{C}} \widetilde{\chi}_{2\to\bar{3}}(s, \mathrm{Re}\,\varepsilon_{13})e^{2\lambda s}ds. \tag{D.6}$$

Eq. (D.6) shows that, indeed, $\mathrm{Im}\,\varepsilon_{13}(\lambda) \sim \exp\big(-2\lambda \int_B^{B'} q^{1/2}(x, \mathrm{Re}\,\varepsilon_{13}dx)\big)$ because the integral along $\widetilde{C}$ in (D.6) has this property.

REFERENCES

[1] R. Balian, C. Bloch, *Ann. Phys.* (N.Y.) **108**, 514 (1974).
[2] R. Balian, G. Parisi, A. Voros, in: Lecture Notes in Physics *Feynman Path Integrals*, **106**, 337, Springer Verlag, Berlin 1979.
[3] L.D. Landau, E.M. Lifshitz, *Quantum Mechanics. Nonrelativistic Theory*, Pergamon Press, Oxford, New York 1965.
[4] S. Graffi, V. Grecchii, B. Simon, *Phys. Lett.* **32B**, 631 (1970).

[5] C.M. Bender, T.T. Wu, *Phys. Rev.* **D7**, 1620 (1973).
[6] L.N. Lipatov, *Zh. Eksp. Teor. Fiz. Lett.* **25**, 104 (1977).
[7] G. Parisi, *Phys. Lett.* **66B**, 167 (1977).
[8] E. Brezin, J.-C. Le Guillou, J. Zinn-Justin, *Phys. Rev.* **D15**, 1544 (1977).
[9] E. Brezin, J.-C. Le Guillou, J. Zinn-Justin, *Phys. Rev.* **D15**, 1558 (1977).
[10] E. Brezin, G. Parisi, J. Zinn-Justin, *Phys. Rev.* **D16**, 408 (1977).
[11] J. Zinn-Justin, Salamanca 1977, Lecture Notes in Physics 77, **126**, ed. J.A. Ascerraga, Springer Verlag, Berlin 1977.
[12] J. Zinn-Justin, *Phys. Rep.* **70**, 109 (1981).
[13] T. Spencer, *Commun. Math. Phys.* **74**, 273 (1980).
[14] S. Breen, *Commun. Math. Phys.* **92**, 179 (1983).
[15] J. Magnen, V. Rivasseau, *Commun. Math. Phys.* **102**, 59 (1985).
[16] R.N. Carlitz, D.A. Nicole, *Ann. Phys.* (N.Y.) **164**, 411 (1985).
[17] R.N. Carlitz, Pitsburgh preprint PITT-19-84 (1984).
[18] D.A. Nicole, P.J. Walters, *J. Phys.* **A21**, 2351 (1988).
[19] P.A. Millard, *Nucl. Phys.* **B259**, 266 (1985).
[20] J.C. Garrison, E.M. Wright, *Phys. Lett.* **108A**, 129 (1985).
[21] A.V. Koudinov, M.A. Smondyrev, *Teor. Mat Fiz.* **56**, 357 (1983) in Russian.
[22] E.B. Bogomolny, *Yad. Fiz.* (*J. Nucl. Phys.*) **40**, 915 (1984).
[23] W.C. Popov, W.M. Weinberg, W.D. Mur, *Yad. Fiz. (J. Nucl. Phys.)* **44**, 1103 (1986).
[24] W.F. Crutchfield II, *Phys. Rev.* **D19**, 2370 (1979).
[25] A. Voros, *Ann. Inst. Henri Poincaré* **A39**, 211 (1983).
[26] B. Simon, A. Dicke, *Ann. Phys.* (N.Y.) **58**, 76 (1970).
[27] S. Giler, *J. Phys.* **A22**, 2965 (1989).
[28] S. Giler, *Acta Phys. Pol.* **B21**, 675 (1990).
[29] A. Voros, Preprint NSF-ITP-89-104.
[30] C.M. Bender, T.T. Wu, *Phys. Rev.* **184**, 1231 (1969).
[31] J.J. Loeffel, A. Martin, B. Simon, A.S. Wightman, *Phys. Lett.* **30B**, 656 (1969).
[32] W.Y. Crutchfield II, *Phys. Lett.* **77B**, 109 (1978).
[33] P. Achuthan, H.J.W. Müller-Kirsten, A. Wiedemann, *Fortschr. Phys.* **38**, 77 (1990).
[34] P.E. Shanley, *Phys. Lett.* **141B**, 331 (1989).
[35] A.D. Dolgov, V.S. Popov, *Phys. Lett.* **B79**, 403 (1978).
[36] M. Maioli, *J. Math. Phys.* **22**, 1952 (1981).
[37] V. Grecchi, M. Maioli, *Ann. Inst. Henri Poincaré* **A41**, 37 (1984).
[38] V. Grecchi, M. Maioli, *J. Math. Phys.* **25**, 3439 (1984).
[39] S. Breen, *Ann. Inst. Henri Poincaré* **A46**, 155 (1987).
[40] S. Giller, preprint IFUŁ-11(40)/90.
[41] S. Giller, preprint IFUŁ-12(41)/90.
[42] I.S. Gradstein, I.M. Rhyzhyk, *Tables of Integrals, Sums, Series and Products*, Nauka, Moskva 1971 in Russian.
[43] S. Giller, in progress.
[44] M.C. Gutzwiller, *Physica* **5D**, 183 (1982).

[45] M.C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, Springer Verlag, New York 1990.
[46] M.V. Berry, J.P. Keating, *J. Phys. A: Math. Gen.* **23**, 4839 (1990).
[47] E.B. Bogomolny, *Comments At. Mol. Phys.* **25**, 67 (1990).
[48] S. Giler, P. Kosiński, J. Rembieliński, P. Maślanka, *J. Phys. A* **22**, 647 (1989).