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# ALGEBRAICALLY DEFORMED BOSON FIELD AND POSSIBLE SATURATION OF ITS ENERGY MODES

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*Dedicated to Wiesław Czyż in honour of his 65th birthday*

An algebraically deformed three-dimensional harmonic oscillator is described. It is an exactly solvable quantum-dynamical system, although its coordinates as well as its momenta are non-Abelian. Nevertheless, its angular momentum generates the usual, non-deformed rotation group. A possibility of its infinite-dimensional generalization towards a new, algebraically deformed boson field is sketched. For such a field a new phenomenon of energy-mode saturation may appear.

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Thinking about the simple but, perhaps, unexpected phenomenon of energy-mode saturation, possibly caused by a hypothetical algebraic deformation of field harmonic oscillators, gave me a lot of enjoyment, though such an effect, if it existed, would be very small. It is a great honour and real pleasure for me to dedicate the paper on this new possible phenomenon to my friend Wiesław Czyż on the happy occasion of his $65^{\text{th}}$ birthday.

Our starting point will be an algebraically deformed three-dimensional harmonic oscillator, a hypothetical physical system related to the deformed three-dimensional annihilation and creation operators defined in Eqs (1) and (2) later on. In a very similar way, an algebraically deformed rotator may be connected with the deformed angular momentum defined through the algebra of deformed (or "quantum") rotation group [1]. The algebraically deformed harmonic oscillator will get its natural, infinite-dimensional generalization in the form of an algebraically deformed boson field.

Unless the algebraically deformed harmonic oscillator and/or the algebraically deformed rotator is only a purely mathematical being, its existence in Nature should manifest itself in some (fine) experiments. Especially exciting would be the option, where real physical harmonic oscillators and/or physical rotators should display (very small) algebraic deformations determined by some (two/one) new fundamental constants of Nature. A natural experimental area, where one might look for such (tiny) deviations of physical harmonic oscillators and/or physical rotators, could be provided by lasers in highly excited states and/or nuclei of very high spins, respectively.

It may happen that a very small algebraic deformation of physical harmonic oscillators is a real effect, while physical rotators are not algebraically deformed. At any rate, the algebraically deformed three-dimensional harmonic oscillator defined in this paper turns out to have the non-deformed orbital angular momentum.

Let us define the algebraically deformed three-dimensional harmonic oscillator by the normal Hamiltonian (*cf.* Eq. (8) later on) and the following commutation relations

$$\left[a_k, a_l^+\right] = \delta_{kl} + \left(\lambda^2 - 1\right)a_l^+ a_k \tag{1}$$

and

$$\left[a_k, a_l\right] = 0 = \left[a_k^+, a_l^+\right], \tag{2}$$

where $k, l = 1, 2, 3$ and $\lambda^2 > 0$ is a fixed parameter whose deviation from 1 measures the deformation. For $\lambda^2 = 1$ the quantum-dynamical system is the usual, non-deformed harmonic oscillator. The relation (1) can be also rewritten as

$$a_k a_l^+ - \lambda^2 a_l^+ a_k = \delta_{kl} . \tag{3}$$

This is a generalization to three dimensions of the commutation relation

$$a a^+ - \lambda^2 a^+ a = 1 \tag{4}$$

characterizing an algebraically deformed one-dimensional harmonic oscillator [2] that becomes recently pretty popular [3], though it plays as yet a rather technical role in physics. The parameter $\lambda^2$ in Eq. (4) is usually denoted by $q$ ($q$ for "quantum" deformation)[1].

Since, due to Eqs (2) and (3), the operators

$$N_k = a_k^+ a_k \tag{5}$$

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[1] The hypothesis (1) is isotropic in space, but, if valid nontrivially (*i.e.*, with $\lambda^2 \neq 1$, what we leave here as an open question), it is pretty drastic, leading to non-Abelian coordinates and momenta (*cf.* Eqs (10) and (11)), though the implied orbital angular momentum is non-deformed(*cf.* Eq. (16)). Less drastic is the anisotropic hypothesis

$$\left[a_k, a_l^+\right] = \delta_{kl}\left[1 + (\lambda^2 - 1)a_l^+ a_l\right] \tag{1F}$$

satisfy the relations

$$a_k N_l - \lambda^2 N_l a_k = \delta_{kl} a_l\,, \quad N_k a_l^+ - \lambda^2 a_l^+ N_k = \delta_{kl} a_l^+ \tag{6}$$

and commute with each other, one can demonstrate that their simultaneous exact eigenvalues have the form

$$N_k^{(n_k)} = \frac{\lambda^{2n_k} - 1}{\lambda^2 - 1}\,, \quad n_k = 0, 1, 2, \ldots \tag{7}$$

(*cf.* the case of one-dimensional oscillator in Ref. [2]). Note that $N_k^{(n_k)} \to \infty$ or $N_k^{(n_k)} \to 1/(1 - \lambda^2)$ as $n_k \to \infty$, if $\lambda^2 > 1$ or $\lambda^2 < 1$, respectively. So, in the latter case there is a new, non-familiar saturation phenomenon for the spectrum $N_k^{(n_k)}$. If $\lambda^2 \to 1$ then $N_k^{(n_k)} \to n_k$, consistently with the deformation of harmonic oscillator, disappearing in this case. Notice that two first terms of the spectrum $N_k^{(n_k)}$ are always such as those for non-deformed harmonic oscillator: $N_k^{(0)} = 0$ and $N_k^{(1)} = 1$[2].

The Hamiltonian of our algebraically deformed harmonic oscillator (in the isotropic case) is

$$H = \sum_k \tfrac{1}{2}\left(p_k^2 + \omega^2 q_k^2\right) = \sum_k \tfrac{1}{2}\left(a_k^+ a_k + a_k a_k^+\right)\hbar\omega\,, \tag{8}$$

where

$$q_k = \sqrt{\hbar}\frac{1}{\sqrt{\omega}}\frac{a_k + a_k^+}{\sqrt{2}}\,, \quad p_k = \sqrt{\hbar}\sqrt{\omega}\frac{a_k - a_k^+}{i\sqrt{2}} \tag{9}$$

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(no summation over $l$), equivalent to [4]

$$a_k a_l^+ - \lambda^{2\delta_{kl}} a_l^+ a_k = \delta_{kl} \tag{3F}$$

as $\delta_{kl}(\lambda^2 - 1) = \lambda^{2\delta_{kl}} - 1$. This gives another possible generalization of Eq. (4) to three dimensions. Together with the assumption (2) Eq. (1F) leads to Abelian coordinates and momenta (*cf.* Eqs (10F) and (11F) in footnote 3), though it implies a deformed orbital angular momentum (*cf.* Eq. (16F) in footnote 4).

[2] In the case of hypothesis (1F) in footnote 1 and assumption (2), Eq. (6) should be replaced by

$$a_k N_l - \lambda^{2\delta_{kl}} N_l a_k = \delta_{kl} a_l, \quad N_k a_l^+ - \lambda^{2\delta_{kl}} a_l^+ N_k = \delta_{kl} a_l^+ \tag{6F}$$

(no summation over $l$), but Eq. (7) still holds (with all its consequences, of course).

are its canonical variables. Due to Eqs (1) and (2), they satisfy the following Heisenberg deformed commutation relations:

$$[q_k, p_l] = i\hbar\left(1 - \frac{\lambda^2 - 1}{\lambda^2 + 1}\right)\delta_{kl} + i\frac{\lambda^2 - 1}{\lambda^2 + 1}\frac{1}{2}\left(\frac{1}{\omega}\{p_k, p_l\} + \omega\{q_k, q_l\}\right) \quad (10)$$

and

$$\omega[q_k, q_l] = \frac{1}{\omega}[p_k, p_l] = -i\frac{\lambda^2 - 1}{\lambda^2 + 1}\frac{1}{2}\left(\{q_k, p_l\} - \{q_l, p_k\}\right). \quad (11)$$

In the latter relation

$$\tfrac{1}{2}\left(\{q_k, p_l\} - \{q_l, p_k\}\right) = q_k p_l - q_l p_k \quad (12)$$

since $[q_k, p_l] = [q_l, p_k]$, according to the symmetry of Eq. (10) in the indices $k, l$. Thus, the coordinates $q_k$ as well as the momenta $p_k$ of the algebraically deformed harmonic oscillator do not commute for different $k$ (are non-Abelian), though the annihilation operators $a_k$ as well as the creation operators $a_k^+$ commute, what was postulated in Eq. (2). Using Eqs (8), (3) and (5), one can write

$$H = \sum_k \left(\frac{\lambda^2 + 1}{2} N_k + \frac{1}{2}\right)\hbar\omega\,, \quad (13)$$

and hence readily find the exact energy spectrum

$$E^{(n_1 n_2 n_3)} \equiv \sum_k E_k^{(n_k)} = \sum_k \left(\frac{\lambda^2 + 1}{2} N_k^{(n_k)} + \frac{1}{2}\right)\hbar\omega \quad (14)$$

with $N_k^{(n_k)}$ as given in Eq. (7). So, the quantum-dynamical system is exactly solvable. If $\lambda^2 < 1$, the new saturation phenomenon for the energy spectrum (14) appears[3].

[3] In contrast to non-Abelian Eqs (10) and (11), the hypothesis (1F) in footnote 1 together with the assumption (2) gives Abelian commutation relations

$$[q_k, p_l] = i\hbar\delta_{kl}[1 + (\lambda^2 - 1)N_l] \quad (10F)$$

(no summation over $l$) and

$$[q_k, q_l] = 0 = [p_k, p_l]\,, \quad (11F)$$

where $N_l = a_l^+ a_l = (\lambda^2 + 1)^{-1}[\hbar\omega^{-1}(p_l^2 +^2 q_l^2) - 1]$. So, then, coordinates and momenta are Abelian, though they still are deformed. The Hamiltonian (8) or (13) and its spectrum (14) (with all consequences of the latter) does not change.

The operator of the form

$$\vec{J} = -i\hbar\vec{a}^{+} \times \vec{a} = \left( -i\epsilon_{klm} a_l^{+} a_m \right) \tag{15}$$

satisfies, due to Eqs (1) and (2), the algebra of the usual, non-deformed rotation group

$$[J_k, J_l] = i\hbar\epsilon_{klm} J_m \,. \tag{16}$$

Making use of Eq. (9) and of the symmetry of Eq. (10) in the indices $k, l$ (implying $\vec{q} \times \vec{p} = -\vec{p} \times \vec{q}$), one can show that

$$\vec{J} = \vec{q} \times \vec{p} - i\frac{1}{2}\left(\frac{1}{\omega}\vec{p} \times \vec{p} + \omega\vec{q} \times \vec{q}\right), \tag{17}$$

where by means of Eqs (11) and (12)

$$\omega\vec{q} \times \vec{q} = \frac{1}{\omega}\vec{p} \times \vec{p} = -i\frac{\lambda^2 - 1}{\lambda^2 + 1}\vec{q} \times \vec{p}. \tag{18}$$

Thus, the operator (15) equal to

$$\vec{J} = \left(1 - \frac{\lambda^2 - 1}{\lambda^2 + 1}\right)\vec{q} \times \vec{p} \tag{19}$$

is the (non-deformed) orbital angular momentum of our algebraically deformed harmonic oscillator (described by the deformed position $\vec{q}$ and deformed momentum $\vec{p}$)[4]. If antisymmetrized with respect to $\vec{a}^{+}$ and $\vec{a}$ (*i.e.*, multiplied by $\frac{1}{2}(\lambda^2 + 1)$), it becomes $\vec{q} \times \vec{p}$ and then satisfies Eq. (16) whose r.h.s. is multiplied by $\frac{1}{2}(\lambda^2 + 1)$.

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[4] The anisotropic hypothesis (1F) in footnote 1 and the assumption (2) imply the deformed orbital angular momentum satisfying the commutation relations

$$[J_k, J_l] = i\hbar\epsilon_{klm}[1 + (\lambda^2 - 1)N_m]J_m \tag{16F}$$

(summation over $m$) with $\vec{J}$ as given in Eq. (15), though it takes now the usual form

$$\vec{J} = \vec{q} \times \vec{p}. \tag{19F}$$

Note that for $N = \sum_l N_l = \vec{a}^{+} \cdot \vec{a}$

$$[J_k, N] = i\hbar\epsilon_{klm}(\lambda^2 - 1)a_l^{+}(N_l - N_m)a_m \neq 0$$

(summation over $l$ and $m$), while in the case of isotropic hypothesis (1) one gets $[J_k, N] = 0$.

Notice that $\vec{J}$ commutes with $N \equiv \sum_k N_k = \vec{a}^+ \cdot \vec{a}$ and $H = \frac{1}{2}[(\lambda^2+1)N + 3]\hbar\omega$ (but not with $N_k$, though $N$ and $H$ commute with $N_k$). So, eigenstates of $H$ can be alternately labelled by $N$, $j$, $m$, while its eigenvalues depend only on $N$, $E^{(N)} = \frac{1}{2}[(\lambda^2+1)N + 3]\hbar\omega$, where $N$ denotes now eigenvalues of the operator $N$ (here, spectrum $N$ = spectrum $\sum_k N_k^{(n_k)}$ with $N_k^{(n_k)}$ as given in Eq. (7)). Calculating $\vec{J}^2$ from Eqs (15), (3) and (6), one can deduce that

$$N(N+1) = j(j+1) + \lambda^2 \langle Njm|\vec{a}^{+2}\vec{a}^2|Njm\rangle \,, \tag{20}$$

where $j = 0, 1, 2, \ldots$, and hence

$$2N_{\text{radial}} \equiv N - j = N + \tfrac{1}{2} - \left[\left(N+\tfrac{1}{2}\right)^2 - \lambda^2\langle Njm|\vec{a}^{+2}\vec{a}^2|Njm\rangle\right]^{1/2} \geq 0 \,. \tag{21}$$

In the case of $\lambda^2 = 1$ one gets $N \equiv n = 0, 1, 2, \ldots$ and $2N_{\text{radial}} = 2n_{\text{radial}} \equiv n - j = 0, 2, 4, \ldots$ (of course, for $n = 0, 1$ only $j = n$ is possible). In general, from Eq. (20) it follows that

$$\lambda^2\langle Njm|\vec{a}^{+2}\vec{a}^2|jm\rangle = (N-j)(N+j+1) \,. \tag{22}$$

Two first terms of the spectrum $N$ are always $N = 0$ and $N = 1$ (for them $j = N$ only, as the l.h.s. of Eq. (22) vanishes for $N = 0, 1$). Further on, we shall put $\hbar = 1$ (and still $c = 1$, of course).

The algebra of deformed annihilation and creation operators $a_k$ and $a_k^+$, defined by their commutation relations (1) and (2) or (3) and (2), can be generalized to an arbitrary number of dimensions, finite or even infinite (while different frequencies $\omega_k$ may be ascribed to different dimensions). In the infinite case one may speak of an algebraically deformed boson field.

For instance, the algebraically deformed real scalar field (in the free case)

$$\varphi(\vec{r}, t) = \varphi_+(\vec{r}, t) + \varphi_-(\vec{r}, t) \tag{23}$$

with

$$\varphi_-(\vec{r}, t) = \varphi_+^+(\vec{r}, t) \tag{24}$$

may be defined by the explicit formula

$$\varphi_+(\vec{r}, t) = \left[\frac{\frac{1}{2}(\lambda^2+1)}{L}\right]^{3/2} \sum_{\vec{k}} \frac{1}{\sqrt{(\lambda^2+1)\omega_k}} a_{\vec{k}} \exp\left(i\tfrac{1}{2}(\lambda^2+1)(\vec{k}\cdot\vec{r} - \omega_k t)\right), \tag{25}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$. Here, the field commutation relations have the form

$$a_{\vec{k}} a_{\vec{k}'}^+ - \lambda^2 a_{\vec{k}'}^+ a_{\vec{k}} = \delta_{\vec{k}\vec{k}'} \tag{26}$$

and

$$\left[a_{\vec{k}},\, a_{\vec{k}'}\right] = 0 = \left[a_{\vec{k}}^{+},\, a_{\vec{k}'}^{+}\right], \tag{27}$$

following from Eqs (3) and (2) through their extension to the index $\vec{k}$ given by $\vec{k} = (2\pi/L)\,\vec{\nu}$ with $\vec{\nu}$ being a running vector of integral components (in the box approximation). Extending Eq. (8) one gets also the field Hamiltonian

$$H = \frac{1}{2}\sum_{\vec{k}} \left(a_{\vec{k}}^{+} a_{\vec{k}} + a_{\vec{k}} a_{\vec{k}}^{+}\right)\omega_k = \sum_{\vec{k}} \left(\frac{\lambda^2+1}{2} N_{\vec{k}} + \frac{1}{2}\right)\omega_k, \tag{28}$$

where the commuting operators $N_{\vec{k}} = a_{\vec{k}}^{+} a_{\vec{k}}$ have the simultaneous eigenvalues $N_{\vec{k}}^{(n_{\vec{k}})} = \left(\lambda^{2n_{\vec{k}}} - 1\right)/\left(\lambda^2 - 1\right)$ with $n_{\vec{k}} = 0, 1, 2, \ldots$ which arise from the extension of Eq. (7). The zero-point energy will be rejected in Eq. (28). It can be done by normal-ordering of the operator (28), where $: a_{\vec{k}} a_{\vec{k}'}^{+} := \lambda^2 a_{\vec{k}'}^{+} a_{\vec{k}}$.

Due to the formulae (25) and (24) one obtains the free field equation

$$i\dot{\varphi}_{\pm}(\vec{r}, t) = \pm\sqrt{-\Delta + \tfrac{1}{4}\left(\lambda^2+1\right)^2 m^2}\,\varphi_{\pm}(\vec{r}, t) \tag{29}$$

or

$$\left[\Box - \tfrac{1}{4}\left(\lambda^2+1\right)^2 m^2\right]\varphi_{\pm}(\vec{r}, t) = 0\,. \tag{30}$$

On the other hand, Eqs (28), (26) and (27) imply that

$$a_{\vec{k}} H - \lambda^2 H a_{\vec{k}} = \tfrac{1}{2}\left(\lambda^2+1\right)\omega_k a_{\vec{k}}\,. \tag{31}$$

Hence, making use of the formula (25), one can infer that

$$\varphi_{+}(\vec{r}, t) H - \lambda^2 H \varphi_{+}(\vec{r}, t) = i\dot{\varphi}_{+}(\vec{r}, t)\,. \tag{32}$$

Thus, the evolution law for the free, algebraically deformed boson field $\varphi_{+}(\vec{r}, t)$ is given by the biunitary transformation

$$\begin{aligned} \varphi_{+}(\vec{r}, t) &= \exp\left(i\lambda^2 H t\right)\varphi_{+}(\vec{r}, 0)\exp(-iHt) \\ &= \exp\left(-i\sqrt{-\Delta + \tfrac{1}{4}\left(\lambda^2+1\right) m^2 t}\right)\varphi_{+}(\vec{r}, 0) \end{aligned} \tag{33}$$

which reduces to the unitary transformation only in the non-deformed case of $\lambda^2 = 1$. Note that

$$\exp\left(i\lambda^2 H t\right) a_{\vec{k}} \exp(-iHt) = a_{\vec{k}} \exp\Big(-i\tfrac{1}{2}(\lambda^2+1)\omega_k t\Big), \tag{34}$$

but the products $a^+_{\vec{k}} a_{\vec{k}'}$ (that appear in field observables) transform unitarily (and conventionally). Finally, the formulae (25), (26) and (27) lead to the explicit field commutation relations

$$\varphi_+(\vec{r}, t)\dot{\varphi}^+_+(\vec{r}\,', t) - \lambda^2 \dot{\varphi}^+_+(\vec{r}\,', t)\varphi_+(\vec{r}, t) = i\tfrac{1}{2}\delta^3(\vec{r} - \vec{r}\,') \tag{35}$$

$$\text{and} \quad [\varphi_+(\vec{r}, t), \varphi_+(\vec{r}\,', t')] = 0 = [\varphi^+_+(\vec{r}, t), \varphi^+_+(\vec{r}\,', t')]\,. \tag{36}$$

The two last can be differentiated separately with respect to $t$ and $t'$. From Eqs (23), (24) and (25) one can also deduce that the field Hamiltonian (28) may be rewritten as

$$H = \tfrac{1}{2}\int d^3\vec{r}\Big\{[\dot{\varphi}(\vec{r}, t)]^2 + [\vec{\partial}\varphi(\vec{r}, t)]^2 + \tfrac{1}{4}(\lambda^2+1)^2 m^2[\varphi(\vec{r}, t)]^2\Big\}, \tag{37}$$

what is its explicit field form. To get rid of the zero-point energy operator (37) should be normal-ordered.

If $\lambda^2 < 1$, modes $\vec{k}$ of an algebraically deformed boson field would display a new, non-familiar phenomenon of saturation of their effective occupation numbers $N^{(n_{\vec{k}})}_{\vec{k}} \to 1/(1-\lambda^2)$ and, consequently, their energies $E^{(n_{\vec{k}})}_{\vec{k}} \to \tfrac{1}{2}[(1+\lambda^2)/(1-\lambda^2)+1]\omega_{\vec{k}}$ when $n_{\vec{k}} \to \infty$. Such a phenomenon could be a subtle but characteristic feature of the deformed modes, provided $\lambda^2 < 1$ [5].

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[5] In the case of hypothesis (1F) in footnote 1 and assumption (2), both extended to the index $\vec{k} = (2\pi/L)\vec{\nu}$ the real scalar field given by Eqs (23), (24) and (25) satisfies the commutation relations

$$[\varphi(\vec{r}, t), \dot{\varphi}(\vec{r}\,', t)] = i[\delta^3(\vec{r} - \vec{r}\,') + (\lambda^2 - 1)N(\vec{r} - \vec{r}\,')] \tag{35F}$$

$$\text{and} \quad [\varphi(\vec{r}, t), \varphi(\vec{r}\,', t)] = 0 = [\dot{\varphi}(\vec{r}, t), \dot{\varphi}(\vec{r}\,', t)]\,. \tag{36F}$$

Here, $N(\vec{r} - \vec{r}\,') = \left[\frac{\frac{1}{2}(\lambda^2+1)}{L}\right]^3 \sum_{\vec{k}} N_{\vec{k}} \exp\left(i\tfrac{1}{2}(\lambda^2+1)\vec{k}\cdot(\vec{r} - \vec{r}\,')\right)$,

while $N_{\vec{k}} = a^+_{\vec{k}} a_{\vec{k}}$ and has the space-reflection symmetry $N_{-\vec{k}} = N_{\vec{k}}$ implying $N(\vec{r}\,' - \vec{r}) = N(\vec{r} - \vec{r}\,')$. For $[\varphi_+(\vec{r}, t), \dot{\varphi}^+_+(\vec{r}\,', t)]$ the equation of the form (35F) holds, but with the factor 1/2 on its r.h.s. The Hamiltonian (28) and (37) and its spectrum do not change, so if $\lambda^2 < 1$ the saturation phenomenon still exists in the case of Eqs (1F) and (2). However, Eq. (31) should be replaced by

$$[a_{\vec{k}}, H] = \tfrac{1}{2}(\lambda^2+1)\omega_{\vec{k}}[1 + (\lambda^2 - 1)N_{\vec{k}}]a_{\vec{k}}\,, \tag{31F}$$

what makes the counterpart of the Eq. (32) involved,

$$[\varphi_+(\vec{r}, t), H] = i\int d^3\vec{r}\,'[\delta^3(\vec{r} - \vec{r}\,') + (\lambda^2 - 1)N(\vec{r} - \vec{r}\,')]\dot{\varphi}_+(\vec{r}\,', t)\,, \tag{32F}$$

though the simple equation (29) is still satisfied (due to the definition (25) of $\varphi_+(\vec{r}, t)$).

*A priori*, any boson fields appearing in Nature, say the electromagnetic field, might be actually algebraically deformed (very slightly, of course). Then, the case of $\lambda^2 \neq 1$ (but $\lambda^2 \cong 1$) would be realistic. Such a possibility seems to deserve some (probably long-term) experimental investigations, including very fine experimentation with laser which can provide us with practically monochromatic modes of the electromagnetic field in highly excited coherent states (corresponding to large $n_k$'s with single $k$'s). The case of $\lambda^2 < 1$ would be most interesting.

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