KINEMATICS AND DYNAMICS IN THE ANGULAR DECAY DISTRIBUTIONS OF PARTICLES AND RESONANCES*

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Dedicated to Wiesław Czyż in honour of his 65th birthday

The dependence of the angular decay distributions of particles (resonances) on the decay dynamics is discussed. The simplest cases, where this dynamics either is irrelevant, or affects only one constant parameter are described. A simple method of calculating the parameters entering the angular decay distribution is presented for the case, when the decay dynamics is known.

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1. Introduction

The study of the decay dynamics of particles (here and in the following particle means particle or resonance) is an important branch of particle physics. Consider a generic decay

$$X \to a_1 + \ldots + a_n. \tag{1}$$

Its decay distribution is shaped by three factors:

- The spin state of X, which is determined by the production mechanism. This spin state can be described by the corresponding spin density matrix ρ_{mn}, or equivalently by the statistical tensors (*cf.* [1] and references contained there)

$$T^J_M = \sum_{mm'} (-1)^{s+m-J} \langle s, -m; s, m' | j, M \rangle \rho_{mm'}. \tag{2}$$

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Here $\langle \ldots \rangle$ are Clebsch–Gordan coefficients and s is the spin of X.

- The decay kinematics, which explains (for a given spin state of X) much of the observed decay distribution. In the next section we present an example, where kinematics yields a complete description of the decay distribution. In such cases, of course, the decay distribution teaches us nothing about the decay dynamics.
- The decay dynamics, which is the subject of the present paper. Our aim is to separate it from the previous two factors shaping the decay distributions.

The final state of the decay (1) is described by giving the momenta p_i (here and in the following momentum means three-momentum unless explicitly stated to the contrary) and the spin states (*e.g.* the helicities λ_i) of all the decay products. The full decay distribution is a distribution in all these variables. Here we shall consider only the angular decay distributions, *i.e.* the distributions summed over the spin states λ_i and integrated over all the invariant masses $(p_i - p_k)^2$ for $i, k = 1, \ldots, n$, where p are four-vectors. The reasons for these two summations are different. The summation over the spin states corresponds to the usual experimental situation, where the particle polarizations are not measured. The invariant masses, on the other hand, are measured. *E.g.* for $n = 3$ their distribution is presented on the Dalitz plot. We do not discuss these distributions, because their kinematics (phase space, threshold factors *etc.*) is rather trivial, while their dynamics is little understood. We shall denote the independent parameters constructed from these masses by $\omega_1, \cdots, \omega_n$. In general for $n > 2$ there are $\nu = 3n - 7$ such parameters[1] and for $n = 2$ none. After these summations the final state can be described by specifying the three Euler angles ϕ, θ, ψ. We define ϕ and θ as the spherical angles of the *analyser* *i.e.* of a vector or pseudovector constructed from the momenta of the decay products. We shall study the angular decay distribution $W(\theta, \phi)$ defined in the rest frame of the decaying particle, integrated over ψ and normalized by the condition

$$\int W(\theta, \phi) d\Omega = 1. \tag{3}$$

For $n = 2$ the only possible choice of the analyser is the momentum of one of the decay products. This is a simple example of a vector analyser. For $n = 3$ it is usual to choose as analyser the normal to the decay plane defined in the rest frame of particle X. This normal can be defined as the vector product of two momenta and is, therefore, a simple example of a pseudovector analyser.

[1] $3n$ momentum components minus the four constraints from four-momentum conservation minus the three Euler angles corresponding to the rigid rotations of the final state.

For the definition of the decay distribution we still need the reference frame to which the angles θ and ϕ refer. We shall always assume that this frame is defined in the rest frame of particle X, besides that, however, our results will be valid for any choice of frame.

The plan of this paper is as follows. In the next section we consider two simple examples illustrating the fact that a decay distribution may, but does not have to, depend on the dynamics of the decay. The general formula for the angular decay distribution is given in Section 3. In this formula the decay dynamics affects at most the coefficients $F(J)$. In Sections 4 we discuss the case, when these coefficients can be predicted without assumptions about the decay dynamics and the case when one parameter dependent on the dynamics is necessary and sufficient. In Section 5 we show a simple method of obtaining the values of the necessary dynamical parameters, when the decay dynamics is known. Section 6 contains some generalizations of the results described in the preceding sections.

2. Two examples

Let us consider two textbook cases as examples. We begin with the decay of a spin one particle polarized along the z-axis ($m_z = 1$) into two spin zero particles. For definiteness we choose the decay $\rho^0 \to \pi^+\pi^-$. From angular momentum conservation the two-pion final state has both the angular momentum and its projection on the z-axis equal one. Moreover, from the definitions of spin and of the angular momentum, the projection of the angular momentum on the direction of the momentum of the π^+ vanishes. Therefore, the amplitude of the final state is [2,3]

$$A(\theta, \phi) = ND^{1*}_{1,0}(\phi, \theta, 0). \tag{4}$$

Here D is the D-function known from the theory of angular momentum and/or as the wave function of the symmetric top, ϕ and θ are the spherical angles of the momentum of the π^+ and N is a normalization factor. The value zero for the third argument of the D-function is a matter of convention. Taking the squared modulus of both sides of (4) and substituting the expression of the D-function in terms of trigonometric functions, we find the angular decay distribution

$$W(\theta, \phi) = \frac{3}{8\pi} \sin^2 \theta, \tag{5}$$

where the normalization (3) has been used. This example illustrates the case, when the decay angular distribution is completely determined by kinematics and consequently yields no information about the decay dynamics.

As the second example consider the decay of a spin one-half particle polarized along the z-axis ($m_z = 1/2$) into a spin one-half particle and a spin zero particle. For definiteness we choose the decay $\Lambda \to p\pi^-$. In the final state both the total angular momentum and its projection on the z-axis equal one-half. There are, however, two possible helicities of the proton: $\lambda_p = \pm 1/2$. Consequently two amplitudes contribute

$$A_+(\theta,\phi) = N_+ D^{\frac{1}{2}*}_{\frac{1}{2},\frac{1}{2}}, \qquad A_-(\theta,\phi) = N_- D^{\frac{1}{2}*}_{\frac{1}{2},-\frac{1}{2}}, \tag{6}$$

and the angular decay distribution depends on one dynamical parameter characterizing the decay:

$$W(\theta,\phi) = \frac{1}{4\pi}\left[1 + \frac{N_+ - N_-}{N_+ + N_-}\cos\theta\right]. \tag{7}$$

Higher spins usually imply more independent dynamical parameters. On the other hand symmetries decrease their number. *E.g.* for the decay $N^* \to p\pi$ parity conservation implies $N_+ = N_-$ and formula (7) reduces to an absolute prediction.

3. General formula

The general formula for the angular decay distribution $W(\theta,\phi)$ can be obtained by a simple extension of the method described in the preceding section. This, however, leads to formulae quadratic in D-functions and much trigonometry is necessary to simplify them. Fortunately, using Clebsch–Gordan coefficients it is possible to transform in the general case the expression quadratic in D-functions into an expression linear in spherical harmonics (*cf.* [1] and references contained there). One finds

$$W(\theta,\phi) = \sum_{J,M} F(J) T^J_M Y^{J*}_M(\theta,\phi). \tag{8}$$

Let us make some remarks about the angular part of this formula. Note that this is an expansion in complex conjugates of the spherical harmonics. Since the distribution W is real, one could take the complex conjugation of both sides of (8) and go over to an expansion in spherical harmonics. This is not done, however, because the choice (8) leads to the convenient identity

$$\langle Y^J_M(\theta,\phi)\rangle = F(J) T^J_M, \tag{9}$$

where $\langle\ldots\rangle$ denotes averaging over all the events in the sample. After expressing the spherical harmonics in terms of trigonometric functions, one may be tempted to simplify the resulting expression. For instance to replace

$$a + b(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}) \tag{10}$$

by

$$\alpha + \beta\cos^2\theta, \tag{11}$$

where $\alpha = a - b/2$ and $\beta = 3b/2$. It is recommended to resist this temptation for two closely related reasons. In formula (10) the normalization condition (3) fixes a, while in formula (11) it fixes $\alpha + \beta/3$ and the determination of either parameter cannot be decoupled from fitting. This makes the evaluation of experimental errors more difficult. Moreover, when formula (10) is extended by adding the term with the next Legendre polynomial ($cP_4(\cos\theta)$) the coefficients a and b are not affected, while, when a term $\gamma\cos^4\theta$ is added to the right hand side of formula (11), the coefficients α and β have to be recalculated.

The coefficients T^J_M are given by formula (2) and depend only on the production mechanism. Thus, the information about the decay dynamics is contained only in the coefficients $F(J)$. These coefficients are given by the formula [4,5]

$$F(J) = N\sqrt{\frac{4\pi}{2s+1}}\sum_{\mu}|R_\mu|^2\langle s,\mu;s,0|s,\mu\rangle, \tag{12}$$

where N is a normalization constant. For $n > 2$:

$$|R_\mu|^2 = \sum_{\lambda_1,\cdots,\lambda_n}\int d\omega_1\cdots d\omega_n|M_\mu(\omega_1,\cdots,\omega_n;\lambda_1,\cdots,\lambda_n)|^2, \tag{13}$$

where μ is the projection of the total angular momentum on the analyser and M are the decay amplitudes. For $n = 2$: R_μ are the decay constants $M(\lambda_1, \lambda_2)$ and choosing the momentum of particle one as analyser we have $\mu = \lambda_1 - \lambda_2$ [1]. Note that the coefficients $F(J)$ do not depend on the spin projection m_z of the decaying particle. This is a consequence of the Eckart–Wigner theorem.

4. Simplest cases

By the simplest cases we understand the cases with the smallest number of dynamical parameters necessary to predict the decay angular distribution. We shall discuss in detail the case, when an absolute prediction can be made, *i.e.* a prediction without information about the decay dynamics, and the case, when one dynamical parameter is necessary. The elaboration of more complex cases can be done along similar lines. Parity conservation and symmetries with respect to exchanges of particles can sometimes be used to reduce the number of necessary independent dynamical parameters. In such cases the number of parameters is understood as the number of independent parameters after the symmetry has been used.

The decay distribution (8) depends on the dynamical parameters $|R_\mu|^2$ for $\mu = -s, \cdots, +s$ *via* the parameters $F(J), J = 0, \ldots, 2s$ and N. The normalization factor N can be eliminated by using the normalization condition (3). This yields

$$\sqrt{4\pi} F(0) T_0^0 = 1. \tag{14}$$

Substituting the definitions (2) and (12), then using the identities

$$\langle s, \mu; 0, 0 | s, \mu \rangle = 1,$$
$$\langle s, -m; s, m | 0, 0 \rangle = (-1)^{s+m} / \sqrt{2s+1}$$

and the condition that the trace of the spin density matrix equals one, one finds the relation between the parameters $|R_\mu|^2$ and the normalization factor N:

$$N \sum_\mu |R_\mu|^2 = \frac{2s+1}{4\pi}. \tag{15}$$

Using the formulae (12) and (15) one derives the (trivial) absolute predictions that the angular decay distribution for any spin zero particle into any decay channel is spherically symmetrical and that for a spin one-half particle the decay distribution can be expressed in terms of one dynamical parameter, *e.g.* of the asymmetry

$$\alpha = \frac{|R_\mu|^2 - |R_{-\mu}|^2}{|R_\mu|^2 + |R_{-\mu}|^2}, \tag{16}$$

where, of course, in this case $\mu = {}^1\!/_2$. We go over now to less trivial predictions.

When $|R_\mu|^2 \neq 0$ for one value of μ only, formulae (12) and (15) yield

$$F(J) = \sqrt{\frac{2s+1}{4\pi}} \langle s, \mu; J, 0 | s, \mu \rangle. \tag{17}$$

For two-body decays with the momentum of the first particle chosen as analyser $\mu = \lambda_1 - \lambda_2$. Thus for decays into two spin zero particles $\mu = 0$ is the only possibility and (17) holds. An example is the decay $\rho^0 \to \pi^+\pi^-$ presented in Section 2. Another application is to decays *via* the $V - A$ coupling into a massless lepton and an antineutrino. Choosing the lepton momentum as analyser one has $\mu = \lambda_l - \lambda_{\bar{\nu}} = -1$. Similarly for the corresponding decays into a massless antilepton and a neutrino $\mu = \lambda_{\bar{l}} - \lambda_\nu = +1$. The case of massive leptons will be discussed later.

When $|R_\mu|^2$ is different from zero for one value of $|\mu|$ only, one can use the identity

$$\langle s, \mu; J, 0|s, \mu\rangle = (-1)^J \langle s, -\mu; J, 0|s, -\mu\rangle \tag{18}$$

to derive

$$F(J) = \sqrt{\frac{2s+1}{4\pi}} \frac{|R_\mu|^2 + (-1)^J |R_{-\mu}|^2}{|R_\mu|^2 + |R_{-\mu}|^2} \langle s, \mu; J, 0|s, \mu\rangle. \tag{19}$$

Thus, in this case we get absolute predictions for the parameters $F(J)$ with J even, while the parameters $F(J)$ with J odd are proportional to the asymmetry (16). Formula (19) applies with $\mu = 1/2$ to two-body decays into a spin zero and a spin one-half particle like the decay $\Lambda \to p\pi^-$ presented in Section 2. When parity conservation holds, $|R_\mu|^2 = |R_{-\mu}|^2$ and $\alpha = 0$. Then we get an absolute prediction for the angular distribution. Another application is to decays into a spin zero particle and a photon. Then $|\mu| = 1$ and again in the general case the decay angular distribution depends on the asymmetry α, but this drops out when the decay conserves parity.

Situations with one dynamical parameter in the decay distribution occur whenever for exactly two values of μ the parameters $|R_\mu|^2$ are different from zero. Examples not reducing to the previous case of opposite values of μ include the decays *via* the $V - A$ coupling into a massive lepton and an antineutrino ($\mu = 0, -1$) and the corresponding decays into a massive antilepton and a neutrino ($\mu = 1, 0$). For parity conserving decays we may add decays into a spin zero particle and a spin $3/2$ particle, or a spin zero and a spin one massive particle.

For decays into three particles we choose the normal to the decay plane as analyser. For decays into three spin zero particles parity conservation implies that μ must be even (odd) when the product of internal parities of the four particles (further denoted η) is $+1$ (-1). Thus for parity conserving decays with $s = 1$, $\eta = +1$, like $\omega \to 3\pi$, only $\mu = 0$ is possible and again formula (17) with no dynamical parameters holds. For $\eta = -1$ and $s = 1, 2$, we have $\mu = \pm 1$ and one can use formula (19) with one dynamical parameter. An additional reduction occurs, when the final state is symmetric or antisymmetric with respect to the exchange of two out of the three final state particles. Then the vector product of the momenta of these two particles can be chosen as analyser and $\alpha = 0$ leading again to an absolute prediction. Examples are the decays $a_1^+ \to \pi^+\pi^+\pi^-$ and $a_2^+ \to \pi^+\pi^+\pi^-$, where the symmetry with respect to exchanges of identical particles can be used. With exchange symmetry one gets absolute predictions for decays (not necessarily parity conserving) of a spin one-half particle into any final channel, while one dynamical parameter is enough to predict the decay angular distributions for parity conserving decays into three spin zero particles for $s = 2$, $\eta = +1$, or $s = 3$, $\eta = \pm 1$ or $s = 4$ and $\eta = -1$. The same holds for arbitrary parity conserving decays of spin three halves particles. In all these cases it is assumed that the vector product of the momenta of the particles related by exchange symmetry is used as analyser.

5. Evaluation of the decay constants

When the decay dynamics is known, or when a model is assumed, it is possible to calculate the dynamical parameters occurring in the formula for the decay angular distribution. One can derive either linear or quadratic expressions for the decay amplitudes. Here we present the calculation leading to linear formulae. As an example we calculate the decay constants for an off-shell W-boson of mass $\sqrt{q^2}$ decaying into a massive antilepton and a neutrino. The analogous calculation using trace and tensor methods in order to get quadratic formulae can be found in Ref. [6].

In our approach the decay amplitude

$$A = \bar{u}_\nu \gamma^\mu (1 - \gamma^5) v_{\bar{l}} \tag{20}$$

is compared with the same amplitude written in terms of the decay constants

$$A = M_s(\lambda_{\bar{l}}, \lambda_\nu) D^{s*}_{0\mu}(\phi, \theta, 0). \tag{21}$$

Here the subscript s refers to the spin of the W-boson. On shell the W-boson is a vector particle, in semileptonic decays, however, the W is far off-shell and both the $s = 0$ and the $s = 1$ states contribute. The amplitudes (21) refer to decays with the spin projection on the z-axis $m_z = 0$. The analysis of the $s = 1$, $m_z = \pm 1$ amplitudes would give nothing more. Note that the overall normalization is here unimportant, because any overall constant factor can be absorbed into the normalizing factor N.

The necessary D-functions are

$$D^0_{00}(\phi, \theta, 0) = 1, \tag{22}$$

$$D^1_{01}(\phi, \theta, 0) = \frac{\sin\theta}{\sqrt{2}}, \tag{23}$$

$$D^1_{00}(\phi, \theta, 0) = \cos\theta. \tag{24}$$

For the Dirac γ matrices we use the Dirac representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma^z = \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix}; \quad \gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{25}$$

The necessary Dirac bispinors are

$$u_\nu^+ = (\phi_\nu^+, -\phi_\nu^+); \qquad v_{\bar{1}} = \begin{pmatrix} \mp\sqrt{\gamma-1}\phi_{\bar{1}} \\ \sqrt{\gamma+1}\phi_{\bar{1}} \end{pmatrix}. \tag{26}$$

Here γ is the Lorentz factor of the antilepton in the rest frame of the W-boson, the $\mp$ sign is minus the sign of the helicity of the antilepton and the ϕ's (the spinors corresponding to the spin states of the particles) are:

$$\phi_\nu = \begin{pmatrix} e^{-\frac{i}{2}\phi}\cos\frac{\theta}{2} \\ e^{+\frac{i}{2}\phi}\sin\frac{\theta}{2} \end{pmatrix}, \; \phi_{\bar{1}+} = -\begin{pmatrix} e^{-\frac{i}{2}\phi}\sin\frac{\theta}{2} \\ -e^{+\frac{i}{2}\phi}\cos\frac{\theta}{2} \end{pmatrix}, \; \phi_{\bar{1}-} = -\begin{pmatrix} e^{-\frac{i}{2}\phi}\cos\frac{\theta}{2} \\ e^{+\frac{i}{2}\phi}\sin\frac{\theta}{2} \end{pmatrix}, \tag{27}$$

where θ and ϕ are the polar angles of the antilepton momentum. The polar angles of the neutrino momentum are $\pi - \theta$ and $\phi + \pi$. The third Euler angle, which is a matter of convention, has been put zero for the antilepton and π for the neutrino. When evaluating the amplitude (20) one uses the identities

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\wedge\vec{b}), \tag{28}$$

where the dot denotes the scalar and the wedge the vector product, and

$$(\sqrt{\gamma+1} \pm \sqrt{\gamma-1})^2 = 2\sqrt{2\epsilon}^{\mp 1}, \tag{29}$$

where

$$\epsilon = \frac{m_{\bar{1}}^2}{2q^2} \tag{30}$$

is in practice usually a small parameter.

The procedure is to evaluate expression (20) for $s = 1$, $m_z = 0$ (*i.e.* for $\mu = z$), $\lambda_{\bar{1}} = \pm^1\!/_2$, $\lambda_\nu = -^1\!/_2$ and for $s = 0$, (*i.e.* $\mu = 0$), $\lambda_{\bar{1}} = -^1\!/_2$, $\lambda_\nu = -^1\!/_2$, to compare the results with formulae (21) and to extract the decay constants. One finds

$$M_1(\tfrac{1}{2}, -\tfrac{1}{2}) = \frac{4}{\sqrt[4]{2\epsilon}}, \tag{31}$$

$$M_1(-\tfrac{1}{2}, -\tfrac{1}{2}) = -\sqrt{\epsilon}M_1(\tfrac{1}{2}, -\tfrac{1}{2}), \tag{32}$$

$$M_0(-\tfrac{1}{2}, -\tfrac{1}{2}) = \sqrt{\epsilon}M_1(\tfrac{1}{2}, -\tfrac{1}{2}). \tag{33}$$

The relative sign of the decay constants (32) and (33) must be minus, while their overall sign and the sign of the amplitude (31) are a matter of convention.

Note that in the present case the normalization condition (3) does not imply the normalization condition (15), because now two spins of the decaying particle are involved. We discuss this more complicated case, as well as some other extensions, in the following section. Since, however, the decay constants (31), (32) and (33) contain all the dynamical information about the decay process considered, they must be sufficient also for the generalizations of the present analysis. In the next section we show that this is indeed the case.

6. Generalizations

Formula (8) holds for the decay of one particle with a well-defined spin, when (for non-two-body decays) the decay distribution has been integrated over the third Euler angle ψ. It is possible to remove these restrictions.

For decays of pairs of resonances one finds the joint decay distribution [1]

$$W(\theta_1,\phi_1;\theta_2,\phi_2) = \sum_{J_1 J_2 M_1 M_2} F_1(J_1)F_2(J_2)T^{J_1 J_2}_{M_1 M_2} Y^{J_1 *}_{M_2}(\theta_1,\phi_1)Y^{J_2 *}_{M_2}(\theta_2,\phi_2). \quad (34)$$

Note that this formula does not factorize, so that it contains all the possible correlations between the two single particle angular decay distributions. The double statistical tensors

$$T^{J_1 J_2}_{M_1 M_2} = \sum_{m,m',n,n'} (-1)^{s_1+s_2+m+n-J_1-J_2} \langle s_1,-m,;s_1,m'|J_1 M_1\rangle\langle s_2,-n;s_2,n'|J_2 M_2\rangle \rho^{mm'}_{nn'}, \quad (35)$$

where $\rho^{mm'}_{nn'}$, the joint spin density matrix of the two decaying objects, depends only on the production dynamics. The decay dynamics affects only the single particle coefficients $F(J)$ defined by (12). For instance, the six joint angular decay distributions obtained for pairs of vector mesons in [7] are special cases of formula (34) and can be obtained by substituting into it the various sets of coefficients $F(J)$ calculated in Section 4 and using the reference frames defined in [7].

The case, when various spin states contribute to a given decay channel, has been much discussed in the physics of strong interactions, where often resonances interfere with each other, or with a nonresonant background. *Cf.* [4] for examples. It reappeared in the physics of weak interactions, where the virtual W-boson is usually far off-shell and its spin zero component sometimes cannot be neglected (*cf.* [6]). With interference [4,5]:

$$W(\theta,\phi) = \sum_{JM}\sum_{kl} F_{kl}(J)T^J_M(k,l)Y^{J*}_M(\theta,\phi) \quad (36)$$

and for a joint decay distribution of a pair of particles

$$W(\theta_1,\phi_1;\theta_2,\phi_2) = \sum_{J_1 M_1 J_2 M_2} F_{ij}(J_1)F_{kl}(J_2)T^{J_1 J_2}_{M_1 M_2}(i,j;k,l)Y^{J_1 *}_{M_1}(\theta_1,\phi_1)Y^{J_2 *}_{M_2}(\theta_2,\phi_2). \quad (37)$$

Here the arguments i,j,k,l label the interfering objects. The statistical tensors for decays of single particles

$$T^J_M(k,l) = \sum_{m_k m_l} (-1)^{s_k+m_k-J}\langle s_k,-m_k;s_l,m_l|JM\rangle\rho_{m_k m_l} \quad (38)$$

and for joint decays of particle pairs

$$T^{J_1 J_2}_{M_1 M_2}(i,j;k,l) = \sum_{m_i m_j m_k m_l} (-1)^{s_k+s_i+m_k+m_i-J_1-J_2}$$
$$\langle s_k,-m_k;s_l,m_l|J_1 M_1\rangle\langle s_i,-m_i;s_j,m_j|J_2 M_2\rangle\rho^{m_k m_l}_{m_i m_j} \quad (39)$$

contain the information about the spin states of the decaying objects *i.e.* about the production dynamics. The information about the decay dynamics affects only the coefficients F_{ij}, which for the simplest case of two-body decays are [4]

$$F_{kl}(J) = N\sqrt{\frac{4\pi}{2s_l+1}}$$
$$\sum_{\lambda_1\lambda_2} M_k(\lambda_1,\lambda_2)M_l^*(\lambda_1,\lambda_2)(-1)^{s_k-s_l}\langle s_k,\mu;J,0|s_l,\mu\rangle. \quad (40)$$

The normalization condition (3) yields now

$$\sqrt{4\pi}\sum_k T^0_0(k,k)F_{k,k}(0) = 1, \quad (41)$$

but other normalization conventions may be more convenient.

For instance, substituting the decay constants from Section 5, we find for the decays of a W-boson into a massive antilepton and a neutrino

$$F_{00}(0) = \frac{1}{\sqrt{4\pi}}\frac{3\epsilon}{1+\epsilon}, \quad (42)$$

$$F_{10}(1) = -F_{01}(1) = -\sqrt{\frac{3}{4\pi}}\frac{\epsilon}{1+\epsilon}, \quad (43)$$

$$F_{11}(0) = \sqrt{\frac{3}{4\pi}}, \quad (44)$$

$$F_{11}(1) = \sqrt{\frac{3}{8\pi}}\frac{1}{1+\epsilon}, \quad (45)$$

$$F_{11}(2) = \sqrt{\frac{3}{40\pi}}\frac{1-2\epsilon}{1+\epsilon}. \quad (46)$$

Here the normalization is chosen so that condition (3) is satisfied for $\epsilon = 0$. Substituting these formulae and the formulae obtained in Section 3 into (34) one finds, using suitable reference frames, the formula given for the joint decay distribution in the process $B \to D^*W \to D\pi\bar{l}\nu$ in Ref. [6]. A generalization of formula (40) to more-than-two-body decays can be found in Ref. [5].

When for the more-than-two-body decays the integration over the third Euler angle ψ is not performed, one obtains instead of formula (35) the formula [5]:

$$W(\theta, \phi, \psi) = \sum_{JMN} T^J_M(k,l) F^J_N(k,l) D^J_{MN}(\phi, \theta, \psi). \tag{47}$$

Again all the decay dynamics is contained in the coefficients $F^J_N(k,l)$. Since, however, these formulae have found little application, we shall not discuss them any further referring the reader to Ref. [5].

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