FIELD THEORY OF PHOTON DUST

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(Received March 27, 1992)

Dedicated to Wiesław Czyż in honour of his 65th birthday

Nonlinear relativistic field theory in $3+1$ dimensions describing the dust of photons is outlined. This theory possesses an infinite hierarchy of conservation laws. It can be viewed as a limiting case of Born–Infeld nonlinear electrodynamics obtained when the critical field tends to zero, *i.e.*, when all fields become overcritical.

PACS numbers: 03.50.-z, 03.50.De, 11.10.Lm

Motto:
This dust will not settle in our time.
And when it does some great roaring machines
will come and whirl it all skyhigh again.
Samuel Beckett, *All That Fall*

In 1983 I have written an essay [1] on Born–Infeld electrodynamics for the birthday volume dedicated to Jan Łopuszański. It gives me great pleasure to publish a continuation of these studies to honor Wiesław Czyż.

1. Introduction

Nonlinear electrodynamics of Born and Infeld [2] is a theory with several exceptional properties and for that reason it may serve as a source of inspiration for theoretical studies. It is a pity that, contrary to the expectations of its discoverers, Born-Infeld (BI) electrodynamics does not describe properly the nonlinearities found in quantum electrodynamics.

In this paper I shall show that a limiting case of BI electrodynamics is a very peculiar field theory. This theory may be viewed as a field theory of photon dust describing the motion of noninteracting massless particles.

It is also a theory whose Lagrangian is just a sum of two constraints. But most of all it is a relativistic nonlinear field theory in $3+1$ dimensions which, like some soliton theories in $1+1$ dimensions has an infinite hierarchy of conservation laws. An infinite hierarchy of conservation laws has usually been considered to be a very interesting feature of field theory. For the Korteweg-de Vries and Sine–Gordon equations the gradual unfolding of these conservation laws produced quite an excitement in the soliton community (*cf.* , for example [4]). The case reported here is probably less remarkable since it does not seem to be related to inverse scattering theory, but on the other hand it is an example of a nonlinear relativistic field theory in $3+1$ dimension. For all these reasons, even though this theory is fictitious, and any resemblance to actual physical theories is purely coincidental, I do believe that this toy model is worth describing in a publication.

2. Nonlinear electrodynamics of Born and Infeld

The nonlinear Maxwell-like equations of BI theory are derived from the following Lagrangian built from the two invariants of the electromagnetic field:

$$L_{\rm BI} = b^2\left(1-\sqrt{1-(\vec{E}^2-\vec{B}^2)/b^2-(\vec{E}\cdot\vec{B})^2/b^4}\right). \tag{1}$$

Field equations derived from this Lagrangian take on the following form when expressed in terms of canonical fields $\vec{B}$ and $\vec{D}$

$$\partial_t\vec{B} = -\nabla\times\frac{\vec{D}-(\vec{D}\times\vec{B})\times\vec{B}/b^2}{\sqrt{1+(\vec{D}^2+\vec{B}^2)/b^2+(\vec{D}\times\vec{B})^2/b^4}}, \tag{2}$$

$$\nabla\cdot\vec{B} = 0, \tag{3}$$

$$\partial_t\vec{D} = \nabla\times\frac{\vec{B}+(\vec{D}\times\vec{B})\times\vec{D}/b^2}{\sqrt{1+(\vec{D}^2+\vec{B}^2)/b^2+(\vec{D}\times\vec{B})^2/b^4}}, \tag{4}$$

$$\nabla\cdot\vec{D} = 0. \tag{5}$$

3. Limiting case of Born–Infeld electrodynamics

It is easy to see that in the limit, when $b\to\infty$, BI electrodynamics tends to linear Maxwell theory, but what happens when $b\to 0$? This question was asked and answered in [1], but I had not discovered there that the limiting theory (called for short UBI, which stands for ultra Born–Infeld) has some unusual properties. Before I exhibit these properties, let me point out that the limit $b \to 0$ is singular: the Lagrangian ceases to exist. I shall return to the question of the Lagrangian later but first let me discuss the limit when $b \to 0$ of the BI field equations expressed in the Hamiltonian formulation. As has been observed in [1], the limit when $b \to 0$ can easily be taken in the formulas (5) and the resulting field equations take on the form

$$\partial_t \vec{B} = \nabla \times (\vec{n} \times \vec{B}), \tag{6}$$

$$\nabla \cdot \vec{B} = 0, \tag{7}$$

$$\partial_t \vec{D} = \nabla \times (\vec{n} \times \vec{D}), \tag{8}$$

$$\nabla \cdot \vec{D} = 0, \tag{9}$$

where $\vec{n}$ is the unit vector in the direction of the Poynting vector,

$$\vec{n} = \frac{\vec{D} \times \vec{B}}{|\vec{D} \times \vec{B}|}. \tag{10}$$

Even though the relations between the $(\vec{B}, \vec{D})$ and $(\vec{H}, \vec{E})$ pairs involve merely a rotation around the vector $\vec{n}$,

$$\vec{E} = -\vec{n} \times \vec{B}, \tag{11}$$

$$\vec{H} = \vec{n} \times \vec{D}, \tag{12}$$

our limiting theory is nevertheless nonlinear.

4. Standard conservation laws

Field theory of electromagnetism described by Eqs (6)–(9) is relativistic, since it is obtained as a limit of BI electrodynamics. The UBI theory is even invariant under the full conformal group since there is no dimensional parameter in the equations. The 16 generators of all conformal transformations can be written in terms of the components of the energy-momentum tensor. For any nonlinear relativistic theory of electromagnetism these components expressed in terms of canonical variables $\vec{B}$ and $\vec{D}$ have the form [3]:

$$T^{00} = \mathcal{H}(\vec{D}, \vec{B}), \tag{13}$$

$$T^{0i} = T^{i0} = (\vec{D} \times \vec{B})^i = \left(\frac{\partial \mathcal{H}}{\partial \vec{D}} \times \frac{\partial \mathcal{H}}{\partial \vec{B}}\right)^i, \tag{14}$$

$$T^{ij} = T^{ji} \tag{15}$$

$$= -\left(\frac{\partial \mathcal{H}}{\partial D}\right)^i D^j - \left(\frac{\partial \mathcal{H}}{\partial \vec{B}}\right)^i B^j + \delta^{ij}\left(\vec{D} \cdot \frac{\partial \mathcal{H}}{\partial \vec{D}} + \vec{B} \cdot \frac{\partial \mathcal{H}}{\partial \vec{B}} - \mathcal{H}\right). \tag{16}$$

For UBI, these relations give:

$$
\begin{aligned}
T^{00} &= |\vec{D} \times \vec{B}| = \mathcal{H}, && (17)\\
T^{0i} &= T^{i0} = (\vec{D} \times \vec{B})^i = \mathcal{H} n^i, && (18)\\
T^{ij} &= T^{ji} = (\vec{n} \times \vec{B})^i D^j - (\vec{n} \times \vec{D})^i B^j + \delta^{ij} |\vec{D} \times \vec{B}| \\
&= \mathcal{H} n^i n^j. && (19)
\end{aligned}
$$

All 16 generators of the conformal group

$$P^0 = \int d^3r\, T^{00}, \tag{20}$$

$$P^i = \int d^3r\, T^{i0}, \tag{21}$$

$$M^{ij} = \int d^3r\, \left(x^i T^{j0} - x^j T^{i0}\right), \tag{22}$$

$$N^i = \int d^3r\, x^i T^{00} - P^i, \tag{23}$$

$$D = \int d^3r\, x^k T^{0k} - tP^0, \tag{24}$$

$$K^0 = \int d^3r\, \vec{r}^{\,2} + t^2 P^0 - 2tD, \tag{25}$$

$$K^i = \int d^3r\, \left(\vec{r}^{\,2} T^{i0} - 2x^i x^k T^{k0}\right) + t^2 P^i + 2tN^i, \tag{26}$$

are conserved for UBI since not only the energy-momentum tensor obeys the continuity equation (this is true for every relativistic nonlinear electrodynamics and it leads to conservation of P^0, P^i, M^{ij}, and N^i) but also the trace of the energy-momentum tensor vanishes.

In addition to conserved quantities (20)–(26), UBI has three additional quantities Λ, Γ_1, and Γ_2 already discussed in [1],

$$\Lambda = \tfrac{1}{2} \int d^3r_1 \int d^3r_2\, \left(\vec{D}_1 r_{12}^{-1} \cdot (\nabla \times \vec{D}_2) + \vec{B}_1 r_{12}^{-1} \cdot (\nabla \times \vec{B}_2)\right), \tag{27}$$

$$\Gamma_1 = \tfrac{1}{2} \int d^3r_1 \int d^3r_2\, \left(\vec{D}_1 r_{12}^{-1} \cdot (\nabla \times \vec{D}_2) - \vec{B}_1 r_{12}^{-1} \cdot (\nabla \times \vec{B}_2)\right), \tag{28}$$

$$\Gamma_2 = \int d^3r_1 \int d^3r_2\, \vec{B}_1 r_{12}^{-1} \cdot (\nabla \times \vec{D}_2), \tag{29}$$

where the subscripts 1 and 2 refer to points $\vec{r}_1$ and $\vec{r}_2$ and r_{12} is the distance between those points. The quantity Λ is also conserved in linear Maxwell theory and it may be interpreted as the total helicity — the generator of duality rotations

$$'\vec{D} = \vec{D}\cos(\alpha) - \vec{B}\sin(\alpha), \tag{30}$$

$$'\vec{B} = \vec{B}\cos(\alpha) + \vec{D}\sin(\alpha). \tag{31}$$

The quantities $\Gamma_{1,2}$ do not have a simple dynamical interpretation, but they can be identified as the generators of the following canonical transformations:

$$'\vec{D} = \vec{D}\cosh(\alpha) + \vec{B}\sinh(\alpha), \tag{32}$$

$$'\vec{B} = \vec{B}\cosh(\alpha) + \vec{D}\sinh(\alpha), \tag{33}$$

and

$$'\vec{D} = \mathrm{e}^{\lambda}\vec{D}, \tag{34}$$

$$'\vec{B} = \mathrm{e}^{-\lambda}\vec{B}. \tag{35}$$

5. Infinite hierarchy of conservation laws

In addition to the conservation laws listed in the preceding section, UBI possesses an infinite hierarchy of nonstandard conservation laws. Their existence can be traced to the fact that UBI describes photon dust. This picturesque phrase is based on the observation that the components of the energy-momentum tensor (19) have the same structure here as for the dust — a fluid without pressure. Indeed we can write (19) in the form:

$$T^{\mu\nu} = \mathcal{H}u^{\mu}u^{\nu}, \tag{36}$$

where

$$(u^{\mu}) = (1, \vec{n}). \tag{37}$$

Since $u^{\mu}u_{\mu} = 0$, the dust particles move with the speed of light. There are two basic equations that characterize the motion of such a dust. The continuity equation for the dust current,

$$\partial_{\mu}(\mathcal{H}u^{\mu}) = 0, \tag{38}$$

and the equations of motion for u^{μ},

$$u^{\nu}\partial_{\nu}u^{\mu} = 0. \tag{39}$$

Both these equations are satisfied on account of field equations (6)–(9). With the use of the equations (38) and (39) we obtain an infinite set of continuity equations of the form:

$$\partial_t(\mathcal{H}n^{i_1}n^{i_2}\cdots n^{i_k}) + \nabla_i(\mathcal{H}n^i n^{i_1}n^{i_2}\cdots n^{i_k}) = 0. \tag{40}$$

These continuity equations lead to the following hierarchy of conserved quantities

$$\int d^3r\ \mathcal{H}n^{i_1}n^{i_2}\cdots n^{i_k} = \text{const.} \tag{41}$$

For $k = 0$ and $k = 1$ the conserved quantities (41) are the energy and the momentum. For $k > 1$ the conserved object does not have a special physical interpretation other than the average of the product of the components of $\vec{n}$ weighted with the energy density.

6. Lagrangian formalism for UBI

Treated as a Hamiltonian system UBI does not show any pathological properties. Its Lagrangian formulation is, however, impossible in the standard form because the Lagrangian defined as

$$L = \vec{E}\cdot\vec{D} - \mathcal{H} \tag{42}$$

is on account of Eq. (11) identically zero. This means that our field theory is a constrained system and indeed from Eq. (11) we find the following two constraints for the Lagrangian field vectors $\vec{B}$ and $\vec{E}$

$$\vec{E}^{\,2} - \vec{B}^{\,2} = 0, \quad \vec{E}\cdot\vec{B} = 0. \tag{43}$$

Let us note that no constraints are imposed on the canonical field vectors $\vec{B}$ and $\vec{D}$ appearing in the Hamiltonian. The presence of constraints leads to Lagrange multipliers that enforce the constraints. Since the Lagrange function here is zero the action reduces just to the sum of two constraint terms

$$W = \int d^4x\big(\lambda_1(\vec{E}^{\,2} - \vec{B}^{\,2}) + \lambda_2\vec{E}\cdot\vec{B}\big). \tag{44}$$

It is matter of a straightforward calculation to show that the equations of motion obtained by varying this action with respect to electromagnetic potentials, after the elimination of the Lagrange multipliers λ_1 and λ_2, coincide with equations (8)-(9).

Solutions for which both invariants vanish are also well known in the linear Maxwell electrodynamics. For such solutions the factorized form (36) of the energy-momentum tensor holds. These special solutions of Maxwell equations and associated with them families of congruences of light rays were the subject of detailed studies that were originated by Robinson [5] and continued by Robinson and Trautman (for a recent review see [6]) in connection with what has been named optical geometry — the geometry of radiative electromagnetic fields. All these special solutions of the Maxwell equations are also solutions of UBI (and for that matter of any relativistic nonlinear electrodynamics). Other solutions are not shared by the two theories.

7. Conclusions

I have presented here a toy model of relativistic field theory in $3+1$ dimensions that possesses some unusual properties, an infinite hierarchy of conservation laws being the most interesting one. Even though this model does not describe directly any realistic physical systems it may serve as an illustration of some properties encountered in field theories. Two most interesting questions were left unanswered. Is this field theory completely integrable and can it be quantized?

I would like to thank Jerzy Kijowski for his profound explanations of the intricacies connected with singular Lagrangians.

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