PAIR PRODUCTION IN A STRONG ELECTRIC FIELD WITH BACK-REACTION

J.M. Eisenberg, Y. Kluger, B. Svetitsky

School of Physics and Astronomy
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University
69978 Tel Aviv, Israel

(Received December 18, 1991)

Dedicated to Wiesław Czyż in honour of his 65th birthday

We present a summary of the present status of efforts to solve the problem in which pairs are produced in a strong electric field, are accelerated by it, and then react back on it through the counter-field produced by their current. This picture has been used by Białas and Czyż and others as a model for effects that may possibly arise in the study of the quark-gluon plasma. We here give a didactic review of recent developments in this back-reaction problem. We first present a simple version of the theory of pair tunneling from a fixed electric field, and then sketch how this has been applied to the quark-gluon plasma. Then we turn to a field formulation of the problem for charged bosons, which leads to the need to carry out a renormalization program, outlined again in simple terms. Numerical results for this program are presented for one spatial dimension, the corresponding physical behavior of the system is discussed, and the implications for three spatial dimensions are considered. We exhibit a phenomenological transport equation embodying physics that is essentially identical to that of the field formulation, thus helping to tie the model of Białas and Czyż for the quark-gluon plasma to a field-theory formulation. Last, we note the status of extensions to *(i)* the problem with three space dimensions; *(ii)* the fermion case; *(iii)* the formulation in terms of boost-invariant variables (as desirable for the quark-gluon plasma); and *(iv)* transport equations derived in a fundamental and consistent fashion

PACS numbers: 11.15.Kc, 12.20.Ds, 25.70. Np

1. The context of this study

In March 1967 one of us (J.M.E.) first met Wiesław Czyż while riding the daily bus from Tel Aviv to Rehovot, where we were both attending an international conference, and found unusual personal and scientific rapport with him. A few months later, in the aftermath of the June 1967 war, scientific travel between Poland and Israel became impossible, although we continued to meet periodically elsewhere within the scientific community. Thus it was natural, when free exchange again became possible, to invite Wiesław for a visit to Tel Aviv in 1987. His main scientific interest at that moment was his work with Andrzej Białas on oscillations in the quark-gluon plasma as described in a transport-equation formalism [1]. The work we present here is a direct outgrowth of that visit. A brief initial report on it has been published [2], and further work will be appearing soon [3]. The emphasis in this paper is on a didactic summary of the current status of the back-reaction problem. Many technical details are suppressed ruthlessly in the hope of setting forth the main physical issues more clearly. Thus, for purposes going beyond general education, at least Refs [2] through [4] should be consulted.

Fig. 1 provides a very simplistic depiction of a possible scenario for the first stage of the production of a quark-gluon plasma at ultra-relativistic energies. Two highly contracted nuclei are assumed to have collided, generated color charges on each other, and then passed through each other. In their wake they have left a chromoelectric field produced by the color charges on the receding nuclei. In our treatment the chromoelectric field is taken in a radically simplified view: We treat it as an Abelian — and therefore an ordinary electric — field; we further take it to be a classical field [4], and regard it as filling all space homogeneously. These are, of course, highly unrealistic assumptions insofar as the quark-gluon plasma application is concerned, but, as we shall see, even with these grotesque simplifications the back-reaction problem remains remarkably recalcitrant. Thus we exploit all of them in order to make progress with back-reaction, and hope to restore a more realistic framework after that has been done. Out of the (chromo)electric field there now tunnel pairs of partons, quarks and gluons of opposite "charge", that are eventually to comprise the plasma.

The tunneling mechanism in question has been extremely well known for over 60 years now [5], and an exact solution [6] for the pair creation rate in the presence of a fixed, external electric field has been available for some 40 years. In simple terms [7], what is happening in the tunneling process is illustrated in Fig. 2. The pair we are considering is initially latent in the Dirac sea. We imagine a fictitious potential that binds this latent pair at the combined rest-mass energy $2mc^2$. The electric field provides a further potential $-eEx$, and the overall potential then allows the pair to tunnel

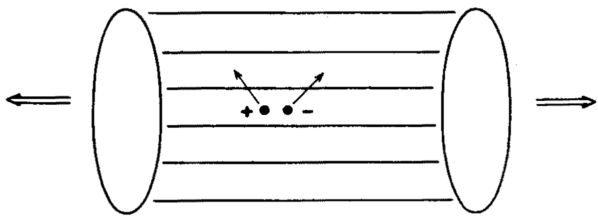

Fig. 1. Schematic view of pair production by a chromoelectric field formed in the wake of two receding ultra-relativistic nuclei. The field is here taken to be an Abelian, and therefore an ordinary electric field, and further assumed to be homogeneous and to fill all space.

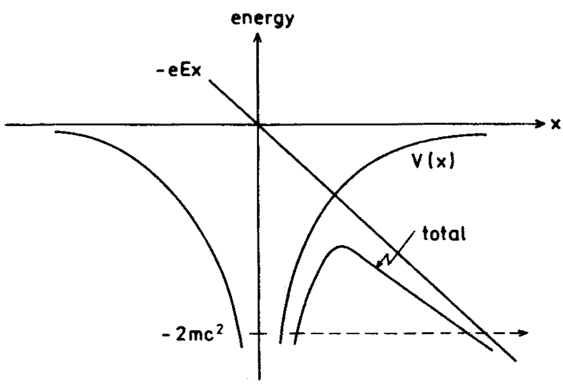

Fig. 2. A simplistic model for the tunneling mechanism by which the presence of a homogeneous electric field allows pair in the Dirac sea to emerge as real particles. The latent pair, at energy $-2mc^2$, corresponding to the combined rest-mass energy, is viewed as bound in a fictitious potential $V(x)$. The presence of the electric field provides a potential $-eEx$, which lowers one side of the barrier and permits tunneling from the total of the two potentials.

through to the outside. The point at which they emerge is $x = 2m/eE$, implying, for small fields, a long tunneling distance. A rough estimate of the rate of pair production is then given by

$$\begin{aligned}\text{rate} &\sim \exp\left(-\int |p|dx\right)\\ &\sim \exp\left(-2\int_0^{2m/eE} \tfrac{1}{2}(2m\cdot 2m)^{1/2}dx\right)\\ &\sim \exp\left(-\frac{4m^2}{eE}\right) \to \exp\left(-\frac{\pi m^2}{eE}\right),\end{aligned} \tag{1}$$

where the last expression is that to emerge from a more precise treatment. As is to be expected, the tunneling process is quantal by its nature, and no perturbative expansion about $E = 0$ is possible for it.

Now the application [1,8,9] of this pair tunneling rate to the picture for quark-gluon plasma production of Fig. 1 has been made through the use of a Boltzmann equation in which the pair rate serves as a source term on the right-hand side,

$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{(\vec{p}^{\,2} + m^2)^{1/2}} \cdot \frac{\partial f}{\partial \vec{x}} + e\vec{E}(t) \cdot \frac{\partial f}{\partial \vec{p}} = \text{pair rate}$$
$$= \cdots \exp\left(-\frac{\pi m^2}{eE(t)}\right), \qquad (2)$$

where, of course, $f(\vec{x}, \vec{p}, t)$ is the density of particles at position $\vec{x}$ with momentum $\vec{p}$ and at time t. We have again indicated only very loosely the structure of the equation; more precise forms for our present context will eventually be given below. Further, the applications to the quark--gluon plasma are generally made in terms of boost-invariant variables and coordinates. We shall return briefly to this point towards the end of this paper, but otherwise we shall here work with conventional space and time variables in order to study the back-reaction issue. Indeed, the question of a correct description of back-reaction deserves an acceptable answer even in ordinary electrodynamics, without the further motivation provided by thoughts of eventual application to the quark-gluon plasma. (We note that a very closely analogous back-reaction problem arose years ago in inflationary cosmology [4,10]; there the time-dependent metric plays the role of the time--dependent electric field here.)

In allowing the electric field in Eq. (2) to be time-dependent, we have anticipated the inevitable appearance of back-reaction: Once the charged pairs make their appearance, they will be accelerated by the electric field, producing a current, which in turn produces an electric field. This field will oppose the direction of the original field, and eventually field and plasma oscillations will be set up; indeed it was just those oscillations that were of interest to Białas and Czyż [1]. This effect enters our theoretical description through a simple application of Maxwell's laws for this case,

$$\dot{\vec{E}}(t) = -\vec{j}(t) = -2e \int \frac{d\vec{p}}{(2\pi)^3} \frac{\vec{p}}{(\vec{p}^{\,2} + m^2)^{1/2}} f(\vec{x}, \vec{p}, t) + \text{polarization term}, \qquad (3)$$

where, for the case of a homogeneous system filling all of space, only a constant magnetic field can arise, which we ignore. The two equations (2) and (3) now form a coupled set which must be solved to incorporate back-reaction and exhibit oscillatory behavior.

It is clear, however, that there is an inconsistency built into the construction of this set of coupled equations: The expression for the rate of pair production used on the right-hand side of Eq. (2) is derived [5-7] for the case of a field *fixed* (by an external agent) in time, while back-reaction necessarily involves a changing field. Furthermore, one might easily suppose that pairs produced directly from the time variation of the field will be generated at a faster rate than those that emerge from tunneling [11]. In fact, Eq. (2) has not really been "derived" from any basic field equations through the use of a Wigner representation , say, but instead has been put together on the grounds of a healthy physical intuition for the problem. This immediately raises the two questions, *(i)* How would back-reaction emerge in a description of the same physical system based on field theory, and *(ii)* can one derive a transport equation resembling Eq. (2) from the field-theory formulation? In this work, we provide an answer for the first of these questions, and are able to find a transport equation very much along the lines of Eq. (2) above, whose solutions bear a striking resemblance — at the quantitative level — to those of the field theory. The question of a direct derivation of the transport equations from the field equations is the subject of ongoing study.

2. Formulation of back-reaction in field theory

In line with our didactic purpose here, we shall continue to formulate the back-reaction problem in its simplest form; a far more complete discussion is given in Ref. [4]. We take a system of charged bosons of mass m satisfying the Klein–Gordon equation

$$-(\partial - ieA)^{\mu}(\partial - ieA)_{\mu}\phi + m^2\phi = 0, \tag{4}$$

where for the homogeneous field filling all space we take a vector potential (in a particular gauge)

$$A_{\mu} = \left(0; \vec{A}(t)\right), \tag{5}$$

which satisfies the Maxwell equation

$$\ddot{\vec{A}}(t) = \vec{\jmath} = -ie[\phi^{\dagger}\vec{\nabla}\phi - (\vec{\nabla}\phi^{\dagger})\phi] - 2e^2\vec{A}\phi^{\dagger}\phi. \tag{6}$$

The form of the boson field after second quantization is

$$\phi(\vec{x},t) = \int \frac{d\vec{k}}{(2\pi)^3}\left(f_{\vec{k}}(t)a_{\vec{k}}\exp(i\vec{k}\cdot\vec{x}) + f^*_{-\vec{k}}(t)b^{\dagger}_{\vec{k}}\exp(-i\vec{k}\cdot\vec{x})\right), \tag{7}$$

where $a_{\vec{k}}$ and $b^{\dagger}_{-\vec{k}}$ are the particle annihilation and antiparticle creation operators, respectively. The forms $f_{\vec{k}}$ are the mode amplitudes for bosons with momentum $\vec{k}$, which, upon substituting Eq. (7) into Eq. (4), are seen to satisfy

$$\ddot{f}_{\vec{k}}(t) + \omega^2_{\vec{k}}(t) f_{\vec{k}}(t) = 0, \tag{8}$$

where

$$\omega^2_{\vec{k}}(t) = (\vec{k} - e\vec{A}(t))^2 + m^2, \tag{9}$$

just as one would anticipate. The boson commutation relations imply a constraint on the mode amplitudes,

$$[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = [b_{\vec{k}}, b^{\dagger}_{\vec{k}'}] = (2\pi)^3 \delta(\vec{k} - \vec{k}') \Rightarrow f_{\vec{k}} \dot{f}^*_{\vec{k}} - f^*_{\vec{k}} \dot{f}_{\vec{k}} = i. \tag{10}$$

Last, using Eq. (7) the Maxwell equation becomes

$$\ddot{\vec{A}}(t) = \langle \vec{\jmath}(t) \rangle = e \int \frac{d\vec{k}}{(2\pi)^3} (\vec{k} - e\vec{A}(t)) 2 |f_{\vec{k}}(t)|^2, \tag{11}$$

where the brackets on $\vec{\jmath}(t)$ are necessary in order to yield a classical electric field, according to the restriction we have chosen. They will imply, for our present purposes, an expectation value in the initial vacuum, although more general formulations are, of course, possible [4]. There immediately arises the question as to whether this integral for the current converges.

3. Renormalization

As we shall see in a moment, the convergence of the integral in Eq. (11) is by no means guaranteed, and a renormalization procedure is required for the back-reaction problem [4]. It is a rather intricate case since the dynamics of the system are intrinsically interwoven with the renormalization. Put another way, there is no simple answer to the question of the convergence of the integral for the current, since that convergence is governed by the behavior of $f_{\vec{k}}(t)$ at large $\vec{k}$, which in turn is to be known only after the solution of the coupled equations describing the system dynamics. And, of course, no such solution is possible so long as there is no finite result for the current. The general considerations pertinent to the problem of renormalizing — an issue having a long history — are presented very lucidly in Ref. [4], where previous literature on the subject is also noted. The only viable approach appears to be to study the high-k behavior of $f_{\vec{k}}(t)$ using a WKBJ-like ansatz, which, as one might expect, rather readily lends itself to the investigation of the high-energy limit, and an assumption of adiabaticity. We therefore make the general ansatz

$$f_{\vec{k}}(t) = \frac{1}{[2\Omega_{\vec{k}}(t)]^{1/2}} \exp\Big(-i \int^t \Omega_{\vec{k}}(t') dt' \Big), \tag{12}$$

where it is easily seen that, as for the usual WKBJ treatment in quantum mechanics, $\Omega_{\vec{k}}(t)$ satisfies

$$\Omega^2_{\vec{k}}(t) + \frac{\ddot{\Omega}_{\vec{k}}}{2\Omega_{\vec{k}}} - \frac{3\dot{\Omega}^2_{\vec{k}}}{4\Omega^2_{\vec{k}}} = \omega^2_{\vec{k}}(t). \tag{13}$$

The Maxwell equation (11) then becomes

$$\ddot{\vec{A}}(t) = \langle \vec{\jmath}(t) \rangle = e \int \frac{d\vec{k}}{(2\pi)^3} \big(\vec{k} - e\vec{A}(t)\big) \frac{1}{\Omega_{\vec{k}}(t)}. \tag{14}$$

In order to proceed with an investigation of the behavior of $f_{\vec{k}}(t)$ at high $\vec{k}$, we have no choice but to suppose that $\Omega_{\vec{k}}(t)$ varies adiabatically in this limit. This is in some sense consistent with one's expectations for a renormalization program since a violent variation in the time dependence of the physical quantities at large momenta would certainly appear to rule out any hope of renormalizing them meaningfully. In the present case, we shall, at the end, find a kind of *a posteriori* justification for this procedure in that the physics that emerges from it is consistent with a quite different phenomenological formulation of the problem. Assuming adiabaticity, we replace the time derivatives of $\Omega_{\vec{k}}(t)$ with those of $\omega_{\vec{k}}(t)$, and suppose $\dot{\omega}_{\vec{k}}(t)/\omega^2_{\vec{k}}(t) \ll 1$, $\ddot{\omega}_{\vec{k}}(t)/\omega^3_{\vec{k}}(t) \ll 1$; we then expand in these small ratios. For the quantity entering in the Maxwell equation, we have

$$\frac{1}{\Omega_{\vec{k}}(t)} = \frac{1}{\omega_{\vec{k}}(t)} \Big(1 - \frac{3\dot{\omega}^2_{\vec{k}}}{8\omega^4_{\vec{k}}} + \frac{\ddot{\omega}_{\vec{k}}}{4\omega^3_{\vec{k}}} + \cdots\Big). \tag{15}$$

To the order in $1/k$ that we need, terms in the first and second time derivatives suffice, and, again to the necessary order, these derivatives are

$$\dot{\omega}_{\vec{k}} = \frac{-e\dot{\vec{A}} \cdot (\vec{k} - e\vec{A})}{\omega_{\vec{k}}} \quad \text{and} \quad \ddot{\omega}_{\vec{k}} = \frac{-e\ddot{\vec{A}} \cdot (\vec{k} - e\vec{A})}{\omega_{\vec{k}}} + \mathcal{O}(1/k). \tag{16}$$

It is then clear from the antisymmetry of the integral in (14) in the variable $(k - eA)$ that the first term on the right-hand side of Eq. (15) — the "1" in the parantheses — makes no contribution. The second term gives a finite integral. But the third term diverges logarithmically. We eliminate this divergence by adding and subtracting the logarithmically divergent integral

$$e \int \frac{d\vec{k}}{(2\pi)^3} \frac{k_3^2 e\ddot{\vec{A}}(t)}{4\omega^5_{\vec{k}}(0)} = e \int \frac{d\vec{k}}{(2\pi)^3} \frac{k^2 e\ddot{\vec{A}}(t)}{12\omega^5_{\vec{k}}(0)}.$$

This added integral can then be regrouped with that of Eq. (14) to produce a finite result, and the subtraction of the identical integral is absorbed into the definition of the renormalized charge, $e_{\rm R}^2 = Ze^2$, where the infinite renormalization constant is

$$Z = \left(1 + e^2 \int \frac{d\vec{k}}{(2\pi)^3} \frac{k^2}{12\omega_{\vec{k}}^5(0)}\right)^{-1}.$$

The electromagnetic field is correspondingly renormalized by $\vec{A}_{\rm R} = \vec{A}/Z^{1/2}$, so that the combination eA which appears throughout is unchanged, and we therefore need not bother to label all the quantities e and A with a subscript R. The renormalized Maxwell equation now reads

$$\ddot{\vec{A}}(t) = e \int \frac{d\vec{k}}{(2\pi)^3} \left(\frac{\vec{k} - e\vec{A}(t)}{\Omega_{\vec{k}}(t)} + \frac{k^2 e\ddot{\vec{A}}(t)}{12\omega_{\vec{k}}^5(0)} \right), \tag{17}$$

where it is to be understood that all the electromagnetic quantities appearing refer to their renormalized values, and the integral on the right-hand side is finite.

Still a problem remains, however: There is no guarantee that, for a given choice of initial values, the resulting solutions for $f_{\vec{k}}(t)$, or, equivalently, for $\Omega_{\vec{k}}(t)$, will remain adiabatic at all future times, a property upon which the entire renormalization scheme rests through the use of Eq. (15). Cooper and Mottola [4] therefore proposed to require that at all times these quantities fulfill

$$\frac{1}{\Omega_{\vec{k}}(t)} = \frac{1}{\omega_{\vec{k}}(t)} - \frac{e(\vec{k} - e\vec{A}) \cdot \ddot{\vec{A}}}{4\omega_{\vec{k}}^5(t)} + r(\vec{k}, t), \tag{18}$$

where $r(\vec{k}, t)$ must fall off faster than $1/k^4$, thus guaranteeing adiabaticity at all times. Then

$$\ddot{\vec{A}}(t) = e \int \frac{d\vec{k}}{(2\pi)^3} (\vec{k} - e\vec{A}(t)) r(\vec{k}, t). \tag{19}$$

This integral is, of course, finite, and the coupled equations (13) and (19) can now, in principle, be solved, at each stage in time imposing Eq. (18). In fact, it would appear, rather peculiarly, that Eq. (19) is no longer required except at $t = 0$, since Eq. (18) will allow the determination of $\ddot{\vec{A}}$ at each time by isolating — as we must do anyway to carry through this procedure — the part of $1/\Omega_{\vec{k}}(t)$ that falls off with $1/k^4$ at large momentum. It is as if the main use of the Maxwell equation were to establish the need for renormalization, after which that stiff requirement alone suffices to generate solutions. In reality it emerges that this strange situation is not the one encountered physically.

4. Results for the case of one spatial dimension

Both because of the intricacy of the renormalization procedure and the numerical difficulties of the full three-dimensional problem, it has proved important first to study [2] the back-reaction problem in one spatial dimension $(1+1)$ in order to gain some insight in preparation for attacking the problem with three spatial dimensions $(3+1)$. In this case, insofar as the adiabatic assumption for $\Omega_k(t)$ is valid, no renormalization is needed. Let us start by looking at the particle distribution in the integrand in Eq. (14) as a function of momentum k after some time has passed in this $1+1$ case. The result is shown in Fig. 3, where we see on the left-hand side a highly oscillatory result for it. At the technical numerical level this oscillatory behavior means that a very fine grid in k is required from the start of the problem at $t=0$. For this reason reliable numerical results are tedious to come by. Furthermore, it rules out a renormalization program based on isolating terms that decay as a reciprocal power in k, which becomes impossible in the face of the oscillations. We return to this point briefly below.

Figure 4 shows $\tilde{E}(t)$ and its derivative $\tilde{\jmath}(t)$ as functions of time for an initial value $\tilde{E}(0)=4$ and $e^2/m^2=0.1$. The quantities are scaled for $1+1$ such that they become dimensionless, *i.e.*, $\tilde{E}\equiv eE/m^2$, $\tilde{\jmath}\equiv ej/m^3$, and $\tau\equiv mt$, and adiabatic initial values have been taken for $\Omega_k(0)=\omega_k(0)$ and $\dot{\Omega}_k(0)=\dot{\omega}_k(0)$. All these quantities show plasma oscillations with slightly increasing frequency as time goes by, corresponding to the additional production of pairs in the electric field, mainly at its peak values. The current $j(t)$ shows a quite flat plateau where its first (and, in some cases, second) oscillatory peak is expected. This occurs because the initial number of pairs produced is not very great, and the subsequent acceleration of the particles brings them to the speed of light, leading to a saturation of the current. This is in fact a rather useful property since it gives us a way to measure the number of particles present at the early times: The current is given by $j=2nev$, where n is the density of particles (or antiparticles) of charge e, and $\pm v$ is their velocity. At the plateau $|v|=1$, and n may be read off from the plateau height. Since the precise definition of particle number is unclear in a field formulation so long as the electric field is nonzero, this gives us a physical way to extract the operative number. At later times we may use the connection between the relativistic plasma frequency and the particle number for this purpose.

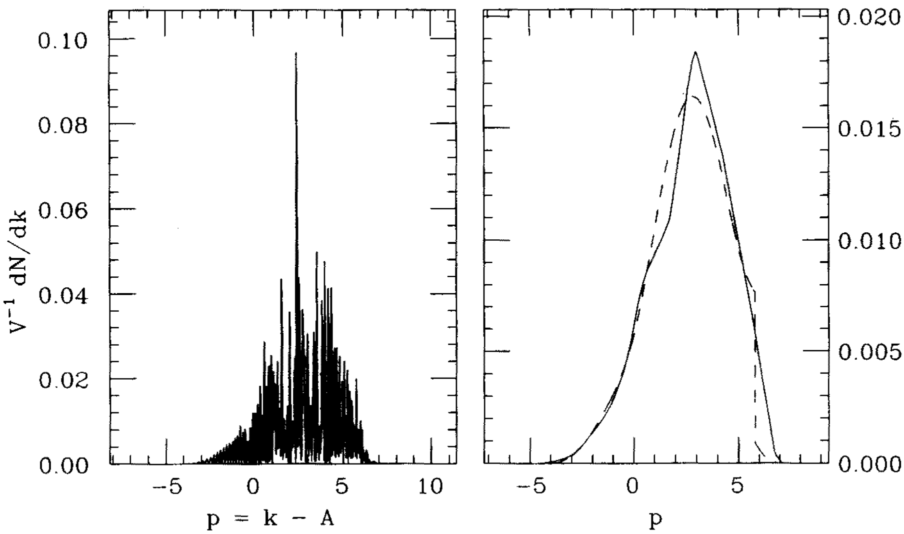

Fig. 3. The number of pairs produced per unit volume (or length, since we have only one spatial dimension here) and momentum (in units of the particle mass m) interval dk at $p = k - eA$ for $e^2/m^2 = 1$ and $\widetilde{E}(0) = eE(0)/m^2 = 1$ at time $\tau = mt = 130$, all in scaled, dimensionless units as indicated. In the right-hand figure, the solid line shows the result of smoothing the exact numerical solution on the left by combining 75 bins into one, and the dashed line is the solution of Eq. (20) for $f(p,t)$ under these same conditions.

5. A classical Boltzmann model yielding equivalent results

All of these physical points may be sharpened considerably by considering a Boltzmann model [2] close to that of Eqs (2) and (3), but now with benefit of the field solutions of Eqs (13) and (14) to test it. The model in question has the transport equation

$$\frac{\partial f}{\partial t} + eE(t)\frac{\partial f}{\partial p} = |eE(t)| \log\left(1 + \exp\left(-\frac{\pi m^2}{|eE(t)|}\right)\right)\delta(p), \tag{20}$$

where the right-hand side is the pair-production rate of the tunneling mechanism in one spatial dimension, with a distribution in momentum space suggested by microscopic arguments on pair tunneling (see especially the first paper noted in Ref. [7]). This can be inserted into the Maxwell equation (3), and the resulting single equation in $A(t)$ and its derivatives is easily solved. A comparison of this solution, shown by the dashed line, with that obtained from the field equations is given in Fig. 4, and is seen to reproduce the initial oscillatory and plateau behavior quite remarkably. Later, the oscillations drift out of phase, presumably because the classical Boltzmann

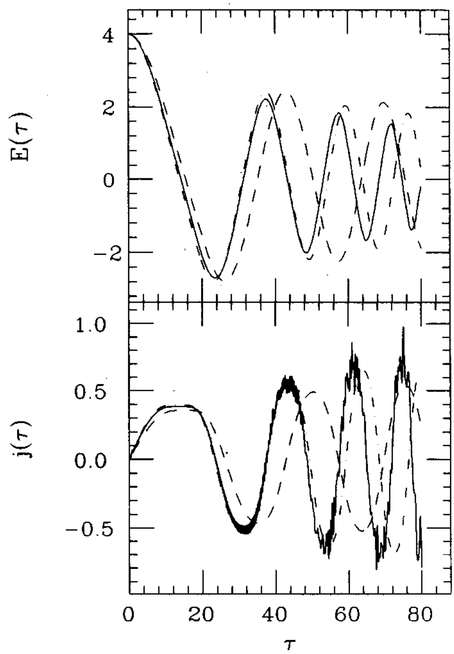

Fig. 4. Results for $\widetilde{E}(\tau)$ and $\tilde{\jmath}(\tau)$ for $\widetilde{E}(0) = 4$ and $e^2/m^2 = 0.1$ in the same units defined in the caption to the previous figure. The solid line shows the solutions to the coupled field equations; the dashed line is for the Boltzmann equation (20); and the dot-dash line — which for short times is almost indistinguishable from the solid line — is the Boltzmann equation modified by a boson enhancement factor.

equation has no mechanism for direct production of pairs through time variations in $E(t)$, as is allowed in the field case. The agreement is made even more striking — shown in Fig. 4 by the dot-dash line — if an enhancement factor for induced boson emission $(2f+1)$ is inserted to multiply the right-hand side of Eq. (20). Of course, the highly oscillatory numerical results for $1/V\, dN/dk$ bear no resemblance to the smooth distribution $f(p,t)$, but if we smooth the former, say by grouping every 100 momentum points into one bin, we obtain the curve shown in Fig. 3 on the right-hand side, which again shows a remarkable resemblance to the result for $f(p,t)$ given on the right-hand side of Fig. 3 by the dashed line. Thus for purposes of physical interpretation the Boltzmann problem can in a major degree replace the field formulation, very much along the lines of the applications [1,8,9] to the quark-gluon plasma.

6. Summary and outlook

We believe that the present calculation provides, for the first time, reliable numerical results for the formulation of the back-reaction problem as one of coupled fields in 1 + 1 dimensions. It allows the elucidation of the main physical effects of plateaux in the current at early times, and plasma oscillations throughout. It also implies that a modification of the renormalization scheme based on Eq. (19) is required. Last it provides a mapping to an equivalent classical and phenomenological Boltzmann formulation for the problem, which may be of great value for considering the physics of systems where back-reaction is important, but quantum details may not be essential.

Much has yet to be done [3] before one can feel satisfied with our mastery of even this simplified problem involving a homogeneous system filling all of space:

1. Progress has been made in handling the 3+1 problem of three spatial dimensions, which, of course, requires renormalization. A consistency loop,

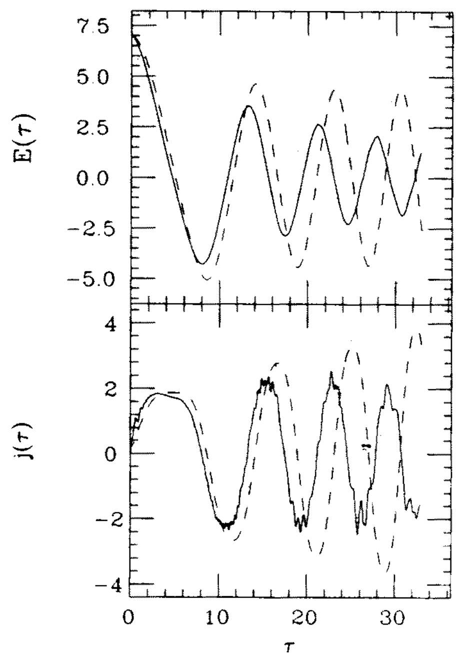

Fig. 5. Results in three spatial dimensions (3 + 1) for $\widetilde{E}(\tau)$ and $\tilde{\jmath}(\tau)$ for $\widetilde{E}(0) = 7$ and $e^2/m^2 = 4$ in the same units defined in the caption of Fig. 3. The solid line shows the solutions to the coupled field equations and the dashed line is for the Boltzmann equation (20).

based on Eq. (19) and guaranteeing that Eq. (18) is indeed satisfied for all t, has been inserted into the coupled equations. The already tedious numerical work of the $1+1$ problem becomes, however, even more difficult in this case. Results are shown in Fig. 5 for $\widetilde{E}(0) = 7$ and $e^2/m^2 = 4$.

2. The same field methods may be applied to the fermion case, and a similar Boltzmann equation may be constructed, this time with fermion suppression in place of boson enhancement. The agreement between the two methods is again very striking.

3. The formulation of the problem in terms of boost-invariant variables is required for applications to the quark-gluon plasma [1,8,9], and this has now been carried out [3].

4. The perplexing question remains as to how to derive a transport equation directly from the field formulation by use of a Wigner representation. In fact this issue becomes rather more mysterious in the light of our results here, since it is well known [12,13] that transport equations derived in that fashion — and exhibiting pair production — are generally *homogeneous* in the transport function, which makes the appearance of a source term on the right-hand side of Eq. (20) quite surprising.

This work was partially supported by the German–Israel Foundation, by a Wolfson Research Award (administered by the Israel Academy of Sciences and Humanities), and by the Ne'eman Chair in Theoretical Nuclear Physics at Tel Aviv University. It was written while J.M.E. was visiting the Institute of Theoretical Physics at the University of Frankfurt, and he wishes to thank the institution for its warm hospitality and the Alexander von Humboldt Stiftung for its support.

REFERENCES

[1] A. Białas, W. Czyż, *Acta Phys. Pol.* **B17**, 635 (1986).
[2] Y.Kluger, J.M. Eisenberg, B. Svetitsky, F. Cooper, E. Mottola, *Phys. Rev. Lett.* **67**, 2427 (1991).
[3] Y. Kluger, B. Svetitsky, F. Cooper, E. Mottola, in preparation; Y. Kluger, J.M. Eisenberg, B. Svetitsky, F. Cooper, E. Mottola, *Phys. Rev.* **D**, to be published.
[4] For a more complete formulation of the problem in this regard, as well as all that pertains to the renormalization program below, see F. Cooper, E. Mottola, *Phys. Rev.* **D40**, 456 (1989).
[5] F. Sauter, *Z. Phys.* **69**, 742 (1931).
[6] J. Schwinger, *Phys. Rev.* **82**, 664 (1951).
[7] This simple view is presented, for example, in A. Casher, H. Neuberger, S. Nussinov, *Phys. Rev.* **D20**, 179 (1979), where the pair-production mechanism is used to develop the flux-tube model, and in C. Itzykson, J.-B. Zuber,

Quantum field theory, McGraw-Hill, New York, 1985, where a rigorous treatment of pair tunneling in QED is also given. See also W.Greiner, B. Müller, J. Rafelski, *Quantum electrodynamics of strong fields*, Springer, Berlin, 1985, for an extensive treatment of this last problem.
[8] G. Gatoff, A.K. Kerman, T. Matsui, *Phys. Rev.* **D36**, 114 (1987).
[9] A. Białas, W.Czyż, *Ann. Phys. (N.Y.)* **187**, 97 (1988); A. Białas, W. Czyż, A. Dyrek, W. Florkowski, *Nucl. Phys.* **B296**, 611 (1988).
[10] See, for example, the review of Ya.B. Zel'dovich, in *Magic Without Magic: John Archibald Wheeler*, Freeman, San Francisco, 1972, p.277.
[11] C. Martin, D. Vautherin, J. Cugnon, *Phys. Lett.* **B227**, 301 (1989).
[12] J.M. Eisenberg, G.Kälbermann, *Phys. Rev.* **D37**, 1197 (1988).
[13] I. Białynicki-Birula, P. Górnicki, J. Rafelski, *Phys. Rev.* **D44**, 1825 (1991).