THE QUANTIZED COULOMB FIELD AND IRREDUCIBLE UNITARY REPRESENTATIONS OF THE PROPER, ORTOCHRONOUS LORENTZ GROUP

A. STARUSZKIEWICZ

Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Cracow, Poland

(Received December 18, 1991)

Dedicated to Wiesław Czyż in honour of his 65th birthday

The quantized Coulomb field is decomposed into irreducible unitary representations of the proper, ortochronous Lorentz group. It is shown that for $e^2/\hbar c > \pi$ the Coulomb field contains representations from the main series. For $0 < e^2/\hbar c < \pi$ there is additionally a representation from the supplementary series corresponding to the special value of the Casimir operator $-\frac{1}{2}M_{\mu\nu}M^{\mu\nu} = \frac{e^2}{\pi\hbar c}\left(2 - \frac{e^2}{\pi\hbar c}\right)$.

PACS numbers: 12.20.Ds, 11.10.Jj

1. Introduction

In Ref. [1] we constructed a closed dynamical system which is linear and contains the electric charge as one of its degrees of freedom. Quantization of this system gives quantization of the electric charge in units equal to the constant e. This gives us a unique opportunity to investigate the quantized Coulomb field in a rigorous way. Such an investigation is presented in this paper. The quantized Coulomb field is decomposed into irreducible unitary representations of the proper, ortochronous Lorentz group. It is shown that for $e^2/\hbar c > \pi$ the Coulomb field contains representations from the main series. For $0 < e^2/\hbar c < \pi$ there is additionally a representation from the supplementary series corresponding to the special value of the Casimir operator

$$-\frac{1}{2}M_{\mu\nu}M^{\mu\nu} = \frac{e^2}{\pi\hbar c}\left(2 - \frac{e^2}{\pi\hbar c}\right).$$

This confirms the observation, first given in [1], that the value $e^2/\hbar c = \pi$ is distinguished and separates two regimes with markedly different kinematical properties.

2. Formulation of the problem

The quantized Coulomb field is a state vector $|u\rangle$, associated with a given four-velocity u and having the following properties.

1° $|u\rangle$ is an eigenstate of the total charge $Q : Q|u\rangle = e|u\rangle$.

2° $|u\rangle$ is spherically symmetric in the rest frame of $u : \epsilon^{\alpha\beta\mu\nu}u_\beta M_{\mu\nu}|u\rangle = 0$, where $M_{\mu\nu}$ are generators of the Lorentz group.

3° $|u\rangle$ does not contain the transversal photons: $N(u)|u\rangle = 0$, where $N(u)$ is the operator of the number of transversal photons in the rest frame of u. If u is the four-velocity of the reference frame in which the partial waves expansion in Ref. [1] is carried out, then $N(u) = (4\pi e^2)^{-1}\sum_{l=1}^{\infty}\sum_{m=-l}^{l} c^+_{lm}c_{lm}$, c_{lm} being the annihilation operators for transversal photons.

These three conditions determine the state vector $|u\rangle$ up to a phase factor.

Consider now a state vector $|f\rangle$ of the form $|f\rangle = \int du f(u)|u\rangle$, where

$$du = \frac{du^1 du^2 du^3}{u^0}$$

is the invariant measure on the set of four-velocities *i.e.* on the Lobachevsky space of unit, future oriented time-like vectors. To avoid irrelevant mathematical problems $f(u)$ is assumed to be of compact support.

$$\langle f|f\rangle = \int du\overline{f(u)}\int dv f(v)\langle u|v\rangle .$$

It was shown in [1] that ($\hbar = 1 = c$)

$$\langle u|v\rangle = \exp\left\{ -\frac{e^2}{\pi}(\lambda\coth\lambda - 1)\right\},$$

where λ is the hyperbolic angle between u and v: $uv = g_{\mu\nu}u^\mu v^\nu = \cosh\lambda$. Thus $\langle u|v\rangle$ is an invariant kernel in the Lobachevsky space; this suggests immediately that properties of the norm $\langle f|f\rangle$ will be seen best when both $f(u)$ and $\langle u|v\rangle$ are Fourier transformed.

The theory of the Fourier transform in the Lobachevsky space was given by Gelfand, Graev and Vilenkin [2]. The theory is summarized in two formulae (formulae (20) and (21) on page 477 in Ref. [2]; we write ν instead of $\rho/2$, as it is customary in more recent literature [3]) which give the Fourier transform and its inverse in the Lobachevsky space:

$$\check{f}(k;\nu) = \int du f(u)(ku)^{i\nu-1},$$

$$f(u) = \frac{1}{(2\pi)^3}\int_0^\infty d\nu\nu^2 \int d^2k \check{f}(k;\nu)(ku)^{-i\nu-1}.$$

Here k is a future oriented null vector, $(ku)^{-i\nu-1}$ is a plane wave in the Lobachevsky space, d^2k is the invariant measure on the set of null directions, applicable when the integrand is a homogeneous of degree -2 function of k, which is the case in the last formula.

We have

$$\int dv f(v)\langle u|v\rangle = \int dv \frac{1}{(2\pi)^3}\int_0^\infty d\nu\nu^2 \int d^2k \check{f}(k;\nu)(kv)^{-i\nu-1}\langle u|v\rangle.$$

If $e^2/\pi > 1$ we can change the order of integrations:

$$\int dv f(v)\langle u|v\rangle = \frac{1}{(2\pi)^3}\int_0^\infty d\nu\nu^2 \int d^2k \check{f}(k;\nu)\int dv (kv)^{-i\nu-1}\langle u|v\rangle,$$

the last integral being convergent for $e^2/\pi > 1$. Since $\langle u|v\rangle$ is an invariant kernel, there is a function $K(\nu; e^2/\pi)$ such that

$$\int dv (kv)^{-i\nu-1}\langle u|v\rangle = (ku)^{-i\nu-1}\cdot K\left(\nu;\frac{e^2}{\pi}\right).$$

Hence

$$\langle f|f\rangle = \frac{1}{(2\pi)^3}\int_0^\infty d\nu\nu^2 K\left(\nu;\frac{e^2}{\pi}\right)\int d^2k \mid \check{f}(k;\nu)\mid^2,$$

which means that the irreducible component $\check{f}(k;\nu)$ enters the state vector $|f\rangle$ with the weight $K(\nu; e^2/\pi)$.

We can formulate now the main goal of this paper: to find the equivalent of the last formula in the physically interesting case $0 < e^2/\pi < 1$.

3. Calculation of the integral $K(\nu; e^2/\pi)$

It is easy to show that

$$K\left(\nu; \frac{e^2}{\pi}\right) = \frac{4\pi}{\nu} \int_0^\infty d\lambda \sinh\lambda \sin(\nu\lambda) e^{-\frac{e^2}{\pi}(\lambda \coth\lambda - 1)} .$$

Put $e^2/\pi = z$ and consider the integral

$$f(\nu; z) = \int_0^\infty d\lambda \sinh\lambda \sin(\nu\lambda) e^{-z\lambda\coth\lambda} .$$

This integral exists for $z > 1$ and defines an analytic function of two real variables ν and z. Differentiating and integrating by parts one can show that

$$f(\nu; z) = -\frac{z^2}{\nu^2 + (z-1)^2}\left(1 + 2\frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \nu^2}\right) g(\nu; z) ,$$

$$g(\nu; z) = \int_0^\infty d\lambda \frac{\sin(\nu\lambda)}{\sinh\lambda} e^{-z\lambda\coth\lambda} .$$

The last integral exists for $z > -1$ and satisfies the partial differential equation

$$z\left(\frac{\partial^2 g}{\partial \nu^2} + \frac{\partial^2 g}{\partial z^2}\right) + \nu\frac{\partial g}{\partial \nu} + (z+1)\frac{\partial g}{\partial z} + g = 0 . \qquad (1)$$

Moreover

$$g(\nu; 0) = \int_0^\infty d\lambda \frac{\sin(\nu\lambda)}{\sinh\lambda} = \frac{\pi}{2}\tanh\left(\frac{\pi}{2}\nu\right) = \sum_{n=0}^{\infty}\left\{\frac{1}{\nu + i(2n+1)} + \text{c.c.}\right\} .$$

Thus $g(\nu; z)$ is the solution of Eq. (1), analytic around $z = 0$ and having the initial value $g(\nu; 0)$ given above. Eq. (1) can be separated in the variables $\nu + c$, $z/(\nu + c)$, c being a constant. One finds thus that the function

$$\frac{[\nu + i(2n+1-z)]^n}{[\nu + i(2n+1+z)]^{n+1}}$$

is the solution of Eq. (1), analytic around $z = 0$ and having the initial value $[\nu + i(2n+1)]^{-1}$. Using the superposition principle one has finally

$$g(\nu; z) = \sum_{n=0}^{\infty} \left\{ \frac{[\nu + i(2n+1-z)]^n}{[\nu + i(2n+1+z)]^{n+1}} + \text{c.c.} \right\}.$$

Hence

$$f(\nu; z) = -z^2 \sum_{n=-\infty}^{+\infty} \frac{[\nu + i(2n+1-z)]^{n-1}}{[\nu + i(2n+1+z)]^{n+2}}.$$

Let us summarize our results. For $z > 1$

$$\int_0^{\infty} d\lambda \sinh\lambda \sin(\nu\lambda) e^{-z\lambda\coth\lambda} = -z^2 \sum_{n=-\infty}^{+\infty} \frac{[\nu + i(2n+1-z)]^{n-1}}{[\nu + i(2n+1+z)]^{n+2}}.$$

The right hand side of this identity is absolutely convergent for all values of ν and z except for the poles. Thus it gives the analytic extension of the integral on the left hand side, valid for all values of ν and z except for the poles.

4. Expression for the scalar product valid for $0 < e^2/\pi < 1$

If $f(u)$ is of compact support, as it is assumed, the scalar product

$$\langle f | f \rangle = \int du \overline{f(u)} \int dv f(v) \langle u | v \rangle$$

is an analytic function of $z = e^2/\pi$ which for $z > 1$ can be written in the form

$$\langle f | f \rangle = \frac{1}{(2\pi)^3} \int_0^{\infty} d\nu \nu^2 K(\nu; z) \int d^2k \mid \check{f}(k; \nu) \mid^2,$$

$\check{f}(k; \nu)$ being the Gelfand, Graev and Vilenkin transform of $f(u)$:

$$\check{f}(k; \nu) = \int du f(u) (ku)^{i\nu - 1}.$$

We have seen in the previous section that the function $K(\nu; z)$ has the analytic extension valid for all ν and z. It is thus tempting to extend the validity of the last expression for $\langle f | f \rangle$ replacing the integral $K(\nu; z)$ by its analytic extension. This is basically correct but one has to note what follows.

For $z > 1$ the norm $\langle f|f\rangle$ can be written as

$$\langle f|f\rangle = \frac{1}{(2\pi)^3}\int_0^\infty d\nu\nu^2 K(\nu;z)\int d^2k\int du\overline{f(u)}(ku)^{-i\nu-1}\int dv f(v)(kv)^{i\nu-1}$$
$$= \int du\overline{f(u)}\int dv f(v)\cdot\frac{4\pi}{(2\pi)^3}\int_0^\infty d\nu\nu^2 K(\nu;z)\frac{\sin(\nu\lambda)}{\nu\sinh\lambda},$$

λ being the hyperbolic angle between u and v. For $z > 1$ we can use the integral representation for $K(\nu;z)$, which reproduces the original form of the norm:

$$\langle f|f\rangle = \int du\overline{f(u)}\int dv f(v)\langle u|v\rangle .$$

Replace now $K(\nu;z)$ by its analytic extension and assume that $0 < z < 1$. The integral

$$\frac{4\pi}{(2\pi)^3}\int_0^\infty d\nu\nu^2 K(\nu;z)\frac{\sin(\nu\lambda)}{\nu\sinh\lambda}$$

can be treated as a contour integral in the complex ν-plane. It is seen then that when z changes from $z > 1$ to $0 < z < 1$, certain poles of $K(\nu;z)$ cross the contour. The contribution from these poles has to be subtracted, since $\langle f|f\rangle$ is a given analytic function of z, the same for all z. Thus the correct prescription for extension of the norm $\langle f|f\rangle$ to the segment $0 < z < 1$ is this: replace the integral $K(\nu;z)$ by its analytic extension and simultaneously remove contribution from those poles of $K(\nu;z)$ which cross the contour when z changes from $z > 1$ to $0 < z < 1$.

It is easy to calculate that the scalar product $\langle f|f\rangle$ for $0 < z < 1$, worked out in accordance with the above prescription, has the form

$$\langle f|f\rangle = \frac{1}{(2\pi)^3}\int_0^\infty d\nu\nu^2 K(\nu;z)\int d^2k \mid \check{f}(k;\nu)\mid^2$$
$$+\frac{(1-z)^2(2e)^z}{16\pi^2}\int\int\frac{d^2k d^2l}{(kl)^z}\overline{\check{f}\left(k;i(1-z)\right)}\check{f}\left(l;i(1-z)\right).$$

(e is the basis of natural logarithms.) In this way we have realized the goal indicated at the end of the first section: to find the expression for the norm $\langle f|f\rangle$ valid for $0 < z < 1$.

The second term in the norm $\langle f|f\rangle$ is the scalar product characteristic for the supplementary series of unitary representations ([1], [2]); it corresponds to the special value of the Casimir operator [2]

$$-\frac{1}{2}M_{\mu\nu}M^{\mu\nu} = z(2-z), \qquad z = \frac{e^2}{\pi\hbar c}.$$

5. Conclusions

Our calculation confirms the observation, first given in Ref. [1], that the value $e^2 = \pi$ is distinguished and separates two regimes with markedly different kinematical properties. It is also interesting to note that for small e^2 almost entire norm of the Coulomb field is concentrated upon the second (discrete) part of the norm: for $|f\rangle = \int du\delta(u;u_0)|u\rangle = |u_0\rangle$ the first (continuous) part of the norm equals $1 - e^z(-z+1)$ while the second (discrete) part of the norm equals $e^z(1-z)$, which is very close to 1 for small $z = e^2/\pi$.

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