THE ENERGY OF NUCLEAR MATTER AND THE MOMENTUM DISTRIBUTION*

J. DĄBROWSKI

Theoretical Division, Soltan Institute for Nuclear Studies
Hoża 69, 00681 Warszawa, Poland

AND

H.S. KÖHLER

Department of Physics, University of Arizona
Tucson, Arizona 85721, USA

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Dedicated to Wiesław Czyż in honour of his 65th birthday

An approximate method of deriving the energy of nuclear matter with an arbitrary momentum distribution from known ground state properties of nuclear matter is discussed. It is shown that for the momentum-dependent Skyrme effective interaction the method is exact.

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1. Introduction

We want to discuss the energy E of nuclear matter (NM) of density $\rho = A/\Omega$ and with an arbitrary nucleon momentum distribution $n(\underline{k}_N)$, where $\underline{k}_N$ denotes the nucleon momentum $\underset{\sim}{k}$ and the z component of its spin and isospin [in most cases $n(\underline{k}_N) = n(\underset{\sim}{k}_N)$]. The energy E is important in describing many phenomena. In particular, in the NM approach to the heavy ion (HI) scattering presented in our previous papers [1–4], the

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combined system of two colliding nuclei was approximated locally by a piece of NM, and the energy E of the system was expressed as

$$\mathcal{E} = \int d\underset{\sim}{r}\mathcal{H}, \tag{1.1}$$

where the energy density

$$\mathcal{H} = f\rho, \tag{1.2}$$

where $f = E/A$ is the energy per nucleon in NM of the local density ρ and with the local momentum distribution n.

The momentum distribution n, discussed in the present paper, is an unperturbed distribution, *i.e.*, the distribution in the non-interacting system, which is modified when the two-body interaction is switched on. The point is that we are concerned here with the energy of the system, calculated as the expectation value of the effective Hamiltonian (with the effective two-body interaction) in the unperturbed state of the system.

In the present paper, we discuss the approximate method of calculating f for a given ρ and n, applied in [1–4]. In our discussion, we assume the effective two-body interaction in NM in the form of the Skyrme force, and show that in this case the approximate method is exact.

The problem of determining the local density ρ and momentum distribution n will not be considered here. For the initial stage of an HI collision, the problem was resolved in [1–4] by introducing the frozen density approximation with a two Fermi sphere momentum distribution. In the latter stage of the collision, the distribution approaches that of one hot Fermi sphere.

The paper is organized as follows. In Section 2 we outline our approximate method of calculating f. In Section 3, we apply the Skyrme force to calculate f, and demonstrate the accuracy of our approximate method. A discussion is given in Section 4.

2. The approximate method of calculating f

To determine the energy E of NM with given ρ and n, we first consider normal NM (*i.e.*, NM in its ground state) of the same density ρ, and with momentum distribution $n_0(\underline{k}_N) = \theta(k_F - k_N)$, where the Fermi momentum $k_F = (3\pi^2\rho/2)^{1/3}$. Its energy is denoted by $E_0 = f_0(\rho)A$. For small changes $\delta n = n - n_0$, we have the following approximate relation between E and E_0 [1] (see also [5]):

$$E = E_0 + \sum_{\underline{k}_N} \left[n(\underline{k}_N) - n_0(\underline{k}_N)\right] \tilde{e}^0_{\underline{k}_N}, \tag{2.1}$$

where $\tilde{e}^0_{\underline{k}_N} = \left[\delta E/\delta n(\underline{k}_N)\right]_{n_0}$ is the s.p. removal energy in normal NM.

If we assume for $\tilde{e}^0$ the effective mass approximation,

$$\tilde{e}^0_{\underline{k}_N} = \frac{\epsilon_{k_N}}{\nu} + \text{ const.}, \tag{2.2}$$

where $\epsilon_{k_N} = \hbar^2 k_N^2/2m$, and $\nu = m^*/m$ is the ratio of the effective to the real nucleon mass, we get from (2.1):

$$f = f_0(\rho) + \frac{\tau - \tau_0}{\nu\rho}, \tag{2.3}$$

where τ and τ_0 are the kinetic energy densities

$$\tau = \frac{\rho}{A}\sum_{\underline{k}_N} n(\underline{k}_N)\epsilon_{k_N}, \qquad \tau_0 = \tfrac{3}{5}\epsilon_{k_F}\rho. \tag{2.4}$$

In [1–4], we assumed for $f_0(\rho)$ the form:

$$f_0(\rho) = \tfrac{3}{5}\epsilon_{k_F} + b\Big(\frac{k_F}{k_{F0}}\Big)^3 + c\Big(\frac{k_F}{k_{F0}}\Big)^4 + d\Big(\frac{k_F}{k_{F0}}\Big)^5, \tag{2.5}$$

where k_{F0} is the Fermi momentum at the equilibrium density ρ_0, and the constants b, c, d are determined by k_{F0}, by the volume energy of NM, $\epsilon_{\text{vol}} = f_0(\rho_0)$, and by the compressibility $K_c = k_{F0}^2\Big(\frac{d^2 f_0}{dk_F^2}\Big)_0$, and for the dependence of ν on ρ the form (suggested in [6]):

$$\nu = \nu(\rho) = \frac{1}{1 + \Big(\frac{1}{\nu(\rho_0)} - 1\Big)\frac{\rho}{\rho_0}}, \tag{2.6}$$

where $\nu(\rho_0)$ was adjusted to the empirical energy dependence of the nucleon optical potential.

Thus our approximate method of calculating f is very simple. With the effective mass $\nu(\rho)$ given in (2.6) [with an empirical value of $\nu(\rho_0)$], we may apply expression (2.3) for f, provided we know $f_0(\rho)$ which we treat as a semiempirical function of form (2.5), adjusted to the empirical values of ρ_0, ϵ_{vol}, and K_c. In this way we bypass the difficult task of solving the NM problem (with an arbitrary momentum distribution n) starting from the NN interaction. This task was undertaken and solved approximately within the Brueckner theory by Feassler and his collaborators [7–10] in the case of the two sphere momentum distribution n. Their "exact" results for f were shown in [1–3] to agree satisfactorily with those obtained by our approximate expression (2.3) even for n markedly different from n_0. In the next section, we present an argument for the accuracy of our approximate method of

calculating f by showing that for the Skyrme effective NN interaction our approximate expression (2.3) for f is exact.

3. Calculation of f with the Skyrme force

We assume the effective interaction v between nucleons 1 and 2 in spin and isospin saturated NM to have in momentum space [$\underset{\sim}{k} = (\underset{\sim}{k}_1 - \underset{\sim}{k}_2)/2$ is the relative momentum of nucleons 1 and 2] the following Skyrme form:

$$\langle \underset{\sim}{k}|v|\underset{\sim}{k}'\rangle = t_0(1+x_0P_\sigma) + \tfrac{1}{2}t_1(1+x_1P_\sigma)(k^2+k'^2) + t_2(1+x_2P_\sigma)\underset{\sim}{k}\underset{\sim}{k}' + \tfrac{1}{6}t_3(1+x_3P_\sigma)\rho^\alpha , \tag{3.1}$$

where the last term is a generalization of the term $\frac{1}{6}t_3\rho$ which results from the three-body part $t_3\delta(\underset{\sim}{r}_1-\underset{\sim}{r}_2)\delta(\underset{\sim}{r}_1-\underset{\sim}{r}_3)$ of the original Skyrme interaction.

For the energy per nucleon of NM, we use the expression:

$$f = A^{-1}E = A^{-1}\sum_{\underline{k}_1} n(\underline{k}_1)\left[\epsilon_{k_1} + \tfrac{1}{2}V_{\underline{k}_1}\right], \tag{3.2}$$

where the s.p. model potential

$$V_{\underline{k}_1} = \sum_{\underline{k}_2} n(\underline{k}_2)(\underline{k}_1\underline{k}_2|v|\underline{k}_1\underline{k}_2 - \underline{k}_2\underline{k}_1). \tag{3.3}$$

In the case of normal NM, we get the s.p. model potential $V^0_{\underline{k}_1}$ by changing n into n_0 in (3.3), and $f_0 = A^{-1}E_0$ by changing n into n_0 and $V_{\underline{k}_1}$ into $V^0_{\underline{k}_1}$ in (3.2).

Inserting (3.1) into (3.3) and (3.2) leads to

$$V_{\underline{k}_1} = \frac{1}{\nu} - 1\epsilon_{k_1} + C, \tag{3.4}$$

$$f = \frac{\tau}{\nu\rho} + \frac{\gamma}{2}, \tag{3.5}$$

where

$$\frac{1}{\nu} - 1 = \frac{m}{8\hbar^2}\left[3t_1 + 5t_2\left(1+\frac{4}{5}x_2\right)\right]\rho, \tag{3.6}$$

$$\gamma = \frac{3}{4}t_0\rho + \frac{1}{8}t_3\rho^{\alpha+1}, \tag{3.7}$$

$$C = \left(\frac{1}{\nu} - 1\right)\frac{\tau}{\rho} + \gamma. \tag{3.8}$$

For the s.p. model energy $e = \epsilon + V$, we get the effective mass form:

$$e_{\underline{k}_1} = \nu^{-1}\epsilon_{k_1} + C\,, \tag{3.9}$$

where according to (3.6) the effective mass ν for a given ρ does not depend on n.

The s.p. removal energy $\tilde{e}_{\underline{k}_1}$ differs from the s.p. model energy $e_{\underline{k}_1}$ by the s.p. rearrangement potential

$$V^R_{\underline{k}_1} = \frac{1}{2}\sum_{\underline{k}_2\underline{k}_3} n(\underline{k}_2)n(\underline{k}_3)(\underline{k}_2\underline{k}_3|\delta v/\delta n(\underline{k}_1)|\underline{k}_2\underline{k}_3 - \underline{k}_3\underline{k}_2)\,. \tag{3.10}$$

Since the Skyrme force depends on n only through $\rho = \Omega^{-1}\sum_{\underline{k}_1} n(\underline{k}_1)$, we have $\delta v/\delta n(\underline{k}_1) = \Omega^{-1}\partial v/\partial\rho = A^{-1}\rho\partial v/\partial\rho$, and we get

$$\tilde{e}_{\underline{k}_1} = e_{\underline{k}_1} + \frac{\alpha}{16}t_3\rho^{\alpha+1} = \nu^{-1}\epsilon_{k_1} + \tilde{C}\,, \tag{3.11}$$

where $\tilde{C} = C + (\alpha/16)t_3\rho^{\alpha+1}$. Thus $\tilde{e}$ has also the effective mass form with the same effective mass as e.

Notice that the linear dependence of $1/\nu - 1$ on ρ, *e.g.* (3.6), leads directly to the dependence of ν on ρ assumed in (2.6).

In the case of normal NM, we get V^0, f_0, e^0, and $\tilde{e}^0$ by changing in (3.4), (3.5), (3.9), and (3.11) τ into τ_0, and C into $C_0 = (1/\nu - 1)\tau_0/\rho + \gamma$. Thus we get

$$f - f_0 = \frac{\tau - \tau_0}{\nu\rho} \tag{3.12}$$

in the case of the Skyrme interaction, which is exactly the result (2.3) of the approximate method of Section 2[1].

Of course, in the case of the Skyrme interaction (3.1), one has for f_0 expression (3.5) with $\tau = \tau_0$, which may be written as

$$f_0(\rho) = \frac{3}{5}\epsilon_{k_F} + \tilde{b}\Big(\frac{k_F}{k_{F0}}\Big)^3 + \tilde{c}\Big(\frac{k_F}{k_{F0}}\Big)^{3(\alpha+1)} + \tilde{d}\Big(\frac{k_F}{k_{F0}}\Big)^5, \tag{3.13}$$

where the constants

$$\tilde{b} = \frac{3}{8}t_0\rho_0\,, \quad \tilde{c} = \frac{t_3\rho_0^{\alpha+1}}{16}\,, \quad \tilde{d} = \frac{3}{80}\Big[3t_1 + 5t_2\Big(1 + \frac{4}{5}x_2\Big)\Big]\rho_0 k_{F0}^2 \tag{3.14}$$

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[1] Eq. (3.12) would hold also if we added to v in (3.1) a density dependent k^2 term, *e.g.*, $\frac{1}{6}t_3'(1+x_3'P_\sigma)\rho^\alpha(k^2+k'^2)/2$. In this case Eqs (3.4)–(3.9) would hold with t_1 replaced by $t_1 + \frac{1}{6}t_3'\rho^\alpha$, and Eq. (3.11) with t_3 replaced by $t_3 + t_3'(m/\hbar^2)\tau/\rho$. Of course, $1/\nu$ would cease to depend linearly on ρ.

may be adjusted to the empirical values of ρ_0, $\epsilon_{\rm vol}$, and K_c. In the special case of $\alpha = \frac{1}{3}$ (the Skyrme interaction SKa and SKb of Ref. [11]), expression (3.13) for f_0 coincides with expression (2.5) applied in Section 2.

There is one difference between the approximate method of Section 2, where ρ_0, $\epsilon_{\rm vol}$, K_c, and $\nu(\rho_0)$ are independent parameters, and our present procedure. Notice that Eqs (3.6) and (3.14) imply that $1/\nu(\rho_0) - 1 = 5\tilde{d}/3\epsilon_{k_{F0}}$. Thus after we adjust $\tilde{d}$ to the empirical values of ρ_0, $\epsilon_{\rm vol}$, and K_c, the value of $\nu(\rho_0)$ is fixed. Namely, we have:

$$\frac{1}{\nu(\rho_0)} = 1 + \frac{-1 - 3\alpha + \frac{5}{3}\frac{K_c + 9(1+\alpha)\epsilon_{\rm vol}}{\epsilon_{k_{F0}}}}{2(2-3\alpha)}. \tag{3.15}$$

4. Discussion

We want to discuss why the results for f of Section 2 — based on approximate relation (2.1) and the effective mass approximation (2.2) — become exact in the case of the Skyrme interaction.

The kinetic energy part of relation (2.1),

$$E^{\rm kin} - E_0^{\rm kin} = \sum_{\underline{k}_1} \Big[n(\underline{k}_1) - n_0(\underline{k}_1) \Big] \epsilon_{k_1}, \tag{4.1}$$

is, of course, trivially satisfied. Let us then discuss the potential energy part of this relation.

In general, the effective interaction $v = v[n]$ is a functional of the momentum distribution, and this dependence of v on n introduces an additional dependence of the s.p. model potential on n:

$$\begin{aligned} V_{\underline{k}_1} &= \sum_{\underline{k}_2} n(\underline{k}_2)(\underline{k}_1\underline{k}_2|v[n]|\underline{k}_1\underline{k}_2 - \underline{k}_2\underline{k}_1) \\ &= V_{\underline{k}_1}[n, n_0] + \sum_{\underline{k}_2} n(\underline{k}_2)(\underline{k}_1\underline{k}_2|v[n] - v[n_0]|\underline{k}_1\underline{k}_2 - \underline{k}_2\underline{k}_1), \end{aligned} \tag{4.2}$$

where we use the notation

$$V_{\underline{k}_1}[n, n_0] = \sum_{\underline{k}_2} n(\underline{k}_2)(\underline{k}_1\underline{k}_2|v[n_0]|\underline{k}_1\underline{k}_2 - \underline{k}_2\underline{k}_1). \tag{4.3}$$

In this notation

$$V_{\underline{k}_1} = V_{\underline{k}_1}[n, n], \qquad V^0_{\underline{k}_1} = V_{\underline{k}_1}[n_0, n_0], \tag{4.4}$$

where n as the second argument indicates the dependence of the effective interaction v on n.

With this notation we have

$$\begin{aligned} E^{\mathrm{pot}} - E_0^{\mathrm{pot}} &= \frac{1}{2}\sum_{\underline{k}_1}\{n(\underline{k}_1)V_{\underline{k}_1} - n_0(\underline{k}_1)V^0_{\underline{k}_1}\} \\ &= \sum_{\underline{k}_1}[n(\underline{k}_1) - n_0(\underline{k}_1)]\left(\tfrac{1}{2}V_{\underline{k}_1}[n, n_0] + \tfrac{1}{2}V^0_{\underline{k}}\right) \\ &\quad + \frac{1}{2}\sum_{\underline{k}_1\underline{k}_2} n(\underline{k}_1)n(\underline{k}_2)(\underline{k}_1\underline{k}_2|v[n] - v[n_0]|\underline{k}_1\underline{k}_2 - \underline{k}_2\underline{k}_1)\,, \end{aligned} \tag{4.5}$$

where we use the identity

$$\frac{1}{2}\sum_{\underline{k}_1}\{n_0(\underline{k}_1)V_{\underline{k}_1}[n, n_0] - n(\underline{k}_1)V^0_{\underline{k}_1}\} = 0\,. \tag{4.6}$$

For small changes $\delta n(\underline{k}_1) = n(\underline{k}_1) - n_0(\underline{k}_1)$, for which we may restrict ourselves to terms linear in δn, Eq.(4.5) takes the approximate form:

$$E^{\mathrm{pot}} - E_0^{\mathrm{pot}} \cong \sum_{\underline{k}_1}\delta n(\underline{k}_1)[V^0_{\underline{k}_1} + V^{\mathrm{R0}}_{\underline{k}_1}]\,, \tag{4.7}$$

where

$$V^{\mathrm{R0}}_{\underline{k}_1} = \frac{1}{2}\sum_{\underline{k}_2\underline{k}_3} n_0(\underline{k}_2)n_0(\underline{k}_3)\left(\underline{k}_2\underline{k}_3\left|\left[\frac{\delta v}{\delta n(\underline{k}_1)}\right]_{n_0}\right|\underline{k}_2\underline{k}_3 - \underline{k}_3\underline{k}_2\right) \tag{4.8}$$

is the s.p. rearrangement potential, Eq. (3.10), in normal NM. The s.p. removal energy $\tilde{e}^0_{\underline{k}_1} = \varepsilon_{k_1} + V^0_{\underline{k}_1} + V^{\mathrm{R0}}_{\underline{k}_1}$, and thus adding Eqs (4.7) and (4.1) we get approximate relation (2.1).

In the case when v has the Skyrme form, Eq. (3.1), it depends on n only through the density ρ, $v[n] = v(\rho[n])$. Since the density ρ is kept constant, $\rho[n] = \rho[n_0]$, the second (rearrangement) part of the r.h.s. of Eq. (4.5) vanishes. For the same reason

$$V_{\underline{k}_1}[n, n_0] = V_{\underline{k}_1}[n, n] = V_{\underline{k}_1} = V^0_{\underline{k}_1} + C - C_0\,, \tag{4.9}$$

where in the last step Eq. (3.4) was used. Since $\sum_{\underline{k}_1}[n(\underline{k}_1) - n_0(\underline{k}_1)](C - C_0) = 0$, we may replace $V_{\underline{k}_1}[n, n_0]$ in (4.5) by $V^0_{\underline{k}_1}$, and — after adding Eq. (4.1) — we get:

$$E - E^0 = \sum_{\underline{k}_1}[n(\underline{k}_1) - n_0(\underline{k}_1)]e^0_{\underline{k}_1}\,. \tag{4.10}$$

Since the second (rearrangement) part in (4.5) vanishes, and it is the source of the s.p. rearrangement potential, we may replace $e^0_{\underline{k}_1}$ in (4.10) by $\tilde{e}^0_{\underline{k}_1}$ (as obviously follows also from (3.11)), and we see that relation (2.1) is satisfied exactly.

In addition to relation (2.1), the effective mass approximation, Eq. (2.2), was used in Section 2. In the case of the Skyrme interaction, this approximation too is satisfied exactly, as is seen from Eqs (3.9) and (3.11).

The approximate method of calculating f of Section 2 becomes exact in the case of the Skyrme interaction v, Eq. (3.1), because the interaction depends on n only through ρ, and its dependence on k is purely quadratic. Because of this quadratic dependence, $V_{\underline{k}_1}$ depends on n only through the constant C, and furthermore the s.p. energy has the effective mass form. The medium dependence of v only through ρ is of course crucial. Although Skyrme interaction has been successfully applied in describing various properties of finite nuclei, and also in the description of HI scattering [12, 13], one cannot expect it to be applicable when n differs markedly from n_0.

In general, it appears impossible to incorporate this n dependence into a reasonably simple form of the effective interaction $v[n]$. The situation is somewhat simpler in the two cases mentioned in Section 1: *(i)* two spheres separated by the relative momentum K_r of the two ions; *(ii)* Fermi distribution for the temperature T. In both cases the distribution n is defined by one parameter (K_r or T), and the problem is reduced to determine $v(K_r)$ or $v(T)$, which still is very difficult. A solution of the problem of $v(K_r)$ has been presented by the Faessler group [14] in the form of numerical tables. The T dependence of the effective interaction v has been recently considered in [15] with the (numerical) result suggesting an increase in v with increasing T.

Of course when the momentum distribution n differs so much from n_0 that it noticeably affects the effective interaction, the approximate method of calculating f, described in Section 2, is expected to break down. Our previous work [1–3] showed nevertheless surprising agreement between our simplified approach and the more detailed but also much more complicated approach in Ref. [7]. The present investigation sheds some light on the reason for the success of our simple approach.

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