CALCULATION OF THE PERTURBATION SERIES TO 9-TH ORDER IN A HIGH ENERGY LIMIT OF THE GAUGE FIELD THEORY*

J. Kraśkiewicz

Department of Theoretical Physics
M. Curie-Skłodowska University
Nowotki 8, 20-031 Lublin, Poland

AND

M. Pindor

Institute for Theoretical Physics
Faculty of Physics, Warsaw University
Hoża 69, 00-681 Warsaw, Poland

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Perturbation expansion for the imaginary part of the elastic fermion-fermion amplitude in the vacuum channel for the high energy limit (LLA) of the gauge field theory is calculated to the 9-th order in α. Technical and numerical details of the calculations are discussed.

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1. Introduction

High energy limit of a spontaneously broken gauge field theory with SU(N) gauge group has been studied by Lipatov and co-workers [1], and then by Cheng and Lo [2] and others. Cheng and Lo have calculated four perturbation terms (up to fifth order in α) for the imaginary part of the elastic scattering amplitude for fermion-fermion process in this limit and

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guessed the general procedure for calculating higher terms. In the forthcoming paper [4] the precise method (equivalent, as it appeared, to Cheng and Lo formula) for calculating perturbation expansion is proposed and results of calculations performed until ninth order (in α) are presented and used. It is the aim of the present paper to give some details of these calculations in a hope that they may be interesting for other people working in the field.

The plan of the paper is as follows: in Section 2 we express coefficients of the perturbation series through multiple integrals over two-dimensional Euclidean momenta. We shortly summarize here a line of reasoning from Ref. [3]. Next we show how some of these integrals look like — the full list of integrals, up to 9-th order, is given in the Appendix. In Section 3 we describe manipulations that significantly reduce number of integrations from the initial 2*(order of the term-1) and discuss problems with numerical evaluation of the integrals. In Section 4 we discuss actual numerical computations, algorithm used, and list values of some of the integrals. Finally we discuss prospects for calculating even more perturbation terms.

2. Expressions

It can be concluded from [1], as it is shown in more detail in [2], that the function $F_z^T(q^2)$, a Mellin transform of the imaginary part of the fermion-fermion elastic scattering amplitude:

$$\mathrm{Im}\, M^T(s,t) = \left(\frac{Ng^2}{2(2\pi)^3}\right) Bs \int\limits_{\delta-i\infty}^{\delta+i\infty} \frac{d\omega}{2\pi i} \left(\frac{s}{s_0}\right)^{\omega} F_\omega^T(\vec{q}), \qquad \vec{q}^{\,2} = -t, \; B = \mathrm{const} \tag{2.1}$$

is given, in a high energy limit, by the formula:

$$F_\omega^T(q^2) = -\frac{1}{A_T(q^2)} + \frac{\frac{Ng^2}{2(2\pi)^3}}{\omega} \int d^2k \frac{f_\omega(\vec{q};\vec{k})}{(k^2+1)\left((\vec{q}-\vec{k})^2+1\right)}, \tag{2.2}$$

where f_ω satisfies some integral equation.

We have shown in [3] that one can write an integral equation directly for F_ω^T and it has the solution:

$$F_z^T(q) = -\frac{1}{A_T(q^2)} + \left\langle \psi \middle| \frac{1}{z - K_q^T} \middle| \varphi \right\rangle \tag{2.3}$$

(the first term in this formula is actually unnecessary in the vacuum channel) $\mathcal{K}_q^T$ is a linear operator acting in $\mathcal{L}^2(\mathbb{R}^2)$, $|\varphi\rangle$ and $|\psi\rangle$ are some vectors in $\mathbb{R}^2$; $z = \left(\frac{Ng^2}{2(2\pi)^3}\right)/\omega$.

In [3] an effort was made to calculate (2.3) numerically. In this paper we show how one can calculate many orders of the perturbation expansion of $M^{T=0}$. One way of taking advantage of this medium order expansion will be presented in [4].

We are interested in σ_{TOT} at high energy limit, therefore we consider $q = 0$ (forward scattering) and $T = 0$ (vacuum channel). Neglecting sub- and superscript at $\mathcal{K}$ and using $\alpha = Ng^2/(2\pi)^2/2$ as an expansion parameter, we arrive at the general formula for n-th perturbation term for the imaginary part of the forward vacuum channel amplitude in the LLA approximation for spontaneously broken gauge field theory with SU(N) group:

$$b_n(s) = C\alpha^{n+1}\frac{\langle\varphi|\mathcal{K}^{n-1}|\varphi\rangle}{(2\pi)^{n-1}\pi}\frac{\log\left(\frac{s}{s_0}\right)}{(n-1)!}\,. \tag{2.4}$$

$\mathcal{K}$ is an integral operator with a kernel:

$$K(\vec{p},\vec{q}) = -\frac{A}{(k^2+1)(p^2+1)} + \frac{4}{(\vec{p}-\vec{k})^2+1} - 2\alpha(k^2)\delta(\vec{p}-\vec{k})\,, \tag{2.5}$$

where: $A = 2(N^2+1)/N^2$; C is a constant immaterial here and

$$\begin{aligned}\alpha(k^2) &= (k^2+1)\int\frac{d^2q}{(q^2+1)((\vec{q}-\vec{k})^2+1)}\\ &= 4\pi\frac{k^2+1}{k\sqrt{k^2+4}}\log\left(\frac{k+\sqrt{k^2+4}}{2}\right).\end{aligned} \tag{2.6}$$

$\mathcal{K}$ acts in $\mathcal{L}^2(\mathbb{R}^2)$ and $|\varphi\rangle$ is a vector from this space:

$$\langle\vec{r}|\varphi\rangle = \frac{1}{r^2+1}\,. \tag{2.7}$$

Now, when we calculate powers of $\mathcal{K}$, the second and the third term in (2.5) give rise to non trivial integrals of higher and higher multiplicity (they correspond to what Cheng and Lo call "nonseparable diagrams" — we think that their diagramatical technique is of no great use because not all diagrams, one can draw, actually appear in the expansion), while the first term and its operator product with the remaining two, lead to products of integrals that has already appeared in lower orders ("separable diagrams" of Cheng and Lo). This way in each order we have, in principle, 2^{n-1} new integrals. However, many of them are equivalent, so that the number of new integrals appearing each time we calculate the next order is actually smaller. Using symbols $C_1,\ldots,C_{43}$ for the integrals we have ($p_i = \langle\varphi|\mathcal{K}^i|\varphi\rangle/(2\pi)^i/\pi$):

$$
\begin{aligned}
p_1 &= -\frac{A}{2} + 2C_1\,,\\
p_2 &= \frac{A^2}{4} - 2AC_1 + 4C_2\,,\\
p_3 &= -\frac{A^3}{8} - \frac{3}{2}A^2C_1 - 2AC_1^2 - 4AC_2 + 8(C_3 - C_4)\,,\\
p_4 &= \frac{A^4}{16} - A^3C_1 + 3A^2C_1^2 + 3A^2C_2 - 8AC_1C_2\\
&\quad - 8A(C_3 - C_4) + 4C_5 - 4C_6 + C_7\,,\\
p_5 &= -\frac{A^5}{32} + \frac{5}{8}A^4C_1 - 3A^3C_1^2 + 2A^2C_1^3\\
&\quad - 2A^3C_2 - 8AC_2^2 + 12A^2C_1C_2 + 6A^2(C_3 - C_4)\\
&\quad - 16AC_1(C_3 - C_4) - A(4C_5 - 4C_6 + C_7) + 8C_8\\
&\quad - 8C_9 - 4C_{10} + 4C_{11} + 2C_{12} - C_{13}\,,\\
p_6 &= \frac{A^6}{64} - \frac{3}{8}A^5C_1 + \frac{5}{2}A^4C_1^2 + \frac{5}{4}A^4C_2 - 4A^3C_1^3\\
&\quad - 12A^3C_1C_2 - 4A^3(C_3 - C_4) + 12A^2C_1^2C_2 + 12A^2C_2^2\\
&\quad + 24A^2C_1(C_3 - C_4) + \frac{3}{4}A^2(4C_5 - 4C_6 + C_7)\\
&\quad - 32AC_2(C_3 - C_4) - 2AC_1(4C_5 - 4C_6 + C_7)\\
&\quad - A(8C_8 - 8C_9 - 4C_{10} + 4C_{11} + 2C_{12} - C_{13})\\
&\quad + 16C_{14} - 16C_{15} - 16C_{16} + 8C_{17} + 8C_{18}\\
&\quad + 4C_{19} - 4C_{20} + 4C_{21} - 4C_{22} + C_{23}\,,\\
p_7 &= -\frac{A^7}{128} + \frac{7}{32}A^6C_1 - \frac{15}{8}A^5C_1 - \frac{3}{4}A^5C_2 + 5A^4C_1\\
&\quad + \frac{5}{2}A^4(C_3 - C_4) + 10A^4C_1C_2 - 2A^3C_1^4 - 12A^3C_2^2 - 24A^3C_1^2C_2\\
&\quad - 24A^3C_1(C_3 - C_4) - \frac{A^3}{2}(4C_5 - 4C_6 + C_7) + 24A^2C_1(C_3 - C_4)\\
&\quad + \frac{3}{4}A^2(8C_8 - 8C_9 - 4C_{10} + 4C_{11} + 2C_{12} - C_{13})\\
&\quad + 48A^2C_2(C_3 - C_4) + 24A^2C_1C_2^2 + 3A^2C_1(4C_5 - 4C_6 + C_7)\\
&\quad - 32A(C_3 - C_4)^2 - 4AC_2(4C_5 - 4C_6 + C_7)\\
&\quad - 2AC_1(8C_8 - 8C_9 - 4C_{10} + 4C_{11} + 2C_{12} - C_{13})\\
&\quad - A(16C_{14} - 16C_{15} - 16C_{16} + 8C_{17} + 8C_{18} + 4C_{19}\\
&\quad - 4C_{20} + 4C_{21} - 4C_{22} + C_{23}) + 32C_{24} - 32C_{25}
\end{aligned}
$$

$$- 32C_{26} - 16C_{27} + 16C_{28} + 16C_{29} + 16C_{30} + 16C_{31} + 8C_{32} + 8C_{33} - 8C_{34} - 4C_{35} - 8C_{36} - 8C_{37} - 8C_{38} + 4C_{39} + 2C_{40} + 4C_{41} - 4C_{42} - C_{43} . \tag{2.8}$$

Using notation $Q_{i,k,\ldots,l} = (q_i + q_k + \cdots + q_l)^2 + 1$ we can write integrals C_n as:

$$C_1 = \frac{1}{2\pi^2} \iint \frac{d^2q_1 d^2q_2}{Q_1 Q_{1,2} Q_2}, \tag{2.9}$$

$$C_2 = \frac{1}{4\pi^3} \iiint \frac{d^2q_1 d^2q_2 d^2q_3}{Q_1 Q_{1,2} Q_{2,3} Q_3}, \tag{2.10}$$

$$C_3 = \frac{1}{4\pi^4} \iiiint \frac{d^2q_1 d^2q_2 d^2q_3 d^2q_4}{Q_1 Q_{1,2} Q_{2,3} Q_{3,4} Q_4}, \tag{2.11}$$

$$C_4 = \frac{1}{8\pi^4} \iiiint \frac{Q_3 d^2q_1 d^2q_2 d^2q_3 d^2q_4}{Q_1 Q_{1,3} Q_2 Q_{2,3} Q_4 Q_{3,4}} . \tag{2.12}$$

The remaining integrals can be found in the Appendix.

3. Integrals

The two extreme cases in the n-th order are:

a)

$$\frac{1}{\pi^n} \int \cdots \int \frac{d^2q_1 d^2q_2 d^2q_3 \cdots d^2q_n}{Q_1 Q_{1,2} Q_{2,3} \cdots Q_{n-1,n} Q_n}, \tag{3.1}$$

b)

$$\frac{1}{\pi^n} \int \cdots \int \frac{Q_n^{n-3} d^2q_1 d^2q_2 \cdots d^2q_n}{Q_1 Q_{1,n} Q_2 Q_{2,n} \cdots Q_{n-1} Q_{n-1,n}} . \tag{3.2}$$

Now it is very useful to define a function:

$$\varphi_n(q_n) = \frac{1}{(2\pi)^{n-1}} \int \cdots \int \frac{d^2q_1 d^2q_2 \cdots d^2q_{n-1}}{Q_1 Q_{1,2} Q_{2,3} \cdots Q_{n-1,n}} . \tag{3.3}$$

The first of the integrals above is just $2^n \varphi_{n+1}(0)$ while the second one is:

$$2^n \int (q^n + 1)^{n-3} \varphi_2^{n-1}(q) q dq \tag{3.4}$$

because:

$$\int \frac{d^2q_i}{Q_i Q_{i,n}} = (2\pi)\varphi_2(q_n) . \tag{3.5}$$

It can also be observed that $\alpha(k^2) = 2\pi(k^2+1)\varphi_2(k)$.

It can now be shown that φ_n can be expressed as a single integral only. In fact, through a simple change of variables, we can show that:

$$\varphi_n(q_n) = \frac{1}{(2\pi)^{n-1}} \int \cdots \int \frac{d^2q_1 \dots d^2q_{n-1}}{Q_1 Q_2 \cdots Q_{n-1} Q_{1,2,\dots,n-1,n}}$$
$$= \frac{1}{(2\pi)^{n-1}} \int \cdots \int \frac{d^2q_1 \dots d^2q_{n-1} d^2q}{Q_1 Q_2 \cdots Q_{n-1} Q_q} \delta(\vec{q}_1 + \vec{q}_2 + \cdots + \vec{q}_{n-1} + \vec{q}_n + \vec{q}),$$

where $Q_q = (q^2+1)$. Using the Fourier representation for δ we have:

$$\varphi_n(q_n) = \frac{1}{(2\pi)^{n+1}} \int \cdots \int \frac{d^2q_1 \dots d^2q_{n-1} d^2q}{Q_1 Q_2 \cdots Q_{n-1} Q_p} d^2x$$
$$\exp\left(\vec{x}(\vec{q}_1 + \vec{q}_2 + \cdots + \vec{q}_n + \vec{q})\right)$$
$$= \frac{1}{(2\pi)^{n+1}} \int d^2x \exp(\vec{x}\vec{q}_n) \Big(\int \frac{\exp(\vec{x}\vec{q})}{(q^2+1)} d^2q \Big)^n .$$

But

$$\int_0^{2\pi} \exp(ixp\cos\vartheta) d\vartheta = 2\pi J_0(xp) \quad \text{and} \quad \int \frac{\exp(\vec{x}\vec{q})}{(q^2+1)} d^2q = 2\pi K_0(x).$$

Therefore finally:

$$\varphi_n(q) = \int K_0^n(x) J_0(qx) x dx, \tag{3.6}$$

where J_0 and K_0 are Bessel and modified Bessel functions.

As we have seen above, for $n = 2$ the integral can be expressed as an elementary function. For $n = 3$ there exists also another formula, much more convenient for numerical integration, as it has a positive function as an integrand:

$$\varphi_3(q) = \int_0^\infty \frac{\varphi_2(p) p dp}{\sqrt{(p^2+q^2+1)^2 - 4p^2q^2}}. \tag{3.7}$$

For higher n's we have found nothing better than (3.6). Anyway the formula (3.6) for φ_n, expresses it through a single integral and, with some minor modifications described below, appeared useful for further calculations. Using the formulae for φ_n we can express all C_n's, except for $n = 31, 32$ and 33, as, at most, double integrals. The remaining three can be expressed as triple integrals, but with completely positive integrands.

We give some examples:

$$C_{26} = 2^8 \int_0^\infty (p^2+1)\varphi_2(p)\varphi_3(p)\varphi_5(p)p\,dp\,, \tag{3.8}$$

$$C_{36} = 2^8 \int_0^\infty \left(\int_0^\infty \frac{(q^2+1)\varphi_2^2(q)q\,dq}{s(p,q)} \right) (p^2+1)^2 \varphi_2^2(p)\varphi_3(p)p\,dp\,, \tag{3.9}$$

where

$$s(p,q) = \sqrt{(p^2+q^2+1)^2 - 4p^2q^2}\,. \tag{3.10}$$

The triple integrals are:

$$C_{31} = 2^8 \int_0^\infty \left(\int_0^\infty \frac{(p^2+1)\varphi_2^2(p)p\,dp}{s(p,v)} \int_0^\infty \frac{(q^2+1)\varphi_2(q)\varphi_3(q)q\,dq}{s(q,v)} \right) v\,dv, \tag{3.11}$$

$$C_{32} = 2^8 \int_0^\infty \left(\int_0^\infty \frac{(q^2+1)\varphi_2(q)\varphi_3(q)q\,dq}{s(p,q)} \right) (p^2+1)\varphi_2(p)\varphi_3(p)p\,dp\,, \tag{3.12}$$

$$C_{33} = 2^8 \int_0^\infty\!\!\int_0^\infty \left(\int_0^\infty \frac{(u^2+1)\varphi_2^2(u)u\,du}{s(u,p)} \int_0^\infty \frac{(v^2+1)\varphi_2^2(v)v\,dv}{s(v,q)} \right) \frac{pq\,dp\,dq}{s(p,q)}\,. \tag{3.13}$$

We remind that φ_3 is defined through a single integral see (3.7).

4. Numerical calculations

Although we have expressed all C_n's through, at most, triple integrals, finding their numerical values is not as easy as it seems. The reason is that we need them with higher and higher precision. This fact can easily be understood when we look at an example from 8-th order: C_{24} is 114,874.48 and it enters p_7 with a coefficient 32, but the final value of p_7 is 4796. Therefore we aimed at achieving relative accuracy of C_n's to be at least $5 \cdot 10^{-7}$.

To obtain this we used automatic integration routine INTX of Bartnik, Górski and Pindor [5] — iteratively for double and triple integrals (*i.e.* we used two or three copies of the routine with different names: INTY and INTZ). Unfortunately, for some of the integrals this was not enough, because, as it can be seen from (3.6), calculation of $\varphi_n(q)$ can be very time consuming for large q (we have then many oscillations of the integrand even for small x where they are not yet damped by $K_0(x)$). We have found a formula for φ_n, that is much faster convergent for large q, by integrating $xK_0^n(x)J_0(qx)$ twice by parts. This way we obtained:

$$\varphi_n(q) = \frac{n}{q^3}\int_0^\infty K_0^{n-2}\left(\frac{z}{q}\right)\left\{\frac{z}{q}\left[K_0^2\left(\frac{z}{q}\right) + (n-1)K_1^2\left(\frac{z}{q}\right)\right] + 2K_0\left(\frac{z}{q}\right)K_1\left(\frac{z}{q}\right)\right\}\left(\frac{2}{z}J_1(z) - J_0(z)\right)dz\,. \qquad (4.1)$$

Although the routine INTX proved earlier to be very stable and reliable [5] we have taken special provisions to make sure that the values we obtained are trustworthy. We calculated, namely, C_n's for different relative accuracies required and checked that in fact differences of results layed within errors estimated by the routine. When it was possible (with respect to the computer time required) we calculated some C_n's with even higher precision, and again checked that, in fact, the relative difference was smaller than 10^{-7}. Table I illustrates this procedure.

TABLE I

Integral	acc. required	value	acc. estimated
C_{27}	10^{-5}	169,403.310	$0.9 \cdot 10^{-5}$
	10^{-6}	169,403.600	$0.4 \cdot 10^{-6}$
	10^{-7}	169,403.676	$0.4 \cdot 10^{-7}$
	10^{-8}	169,403.678	$0.9 \cdot 10^{-8}$
C_{32}	10^{-4}	238,760.170	$0.7 \cdot 10^{-4}$
	10^{-5}	238,760.380	$0.9 \cdot 10^{-5}$
	10^{-6}	238,760.260	$0.4 \cdot 10^{-6}$
	10^{-7}	238,760.250	$0.7 \cdot 10^{-7}$
C_{39}	10^{-5}	477,520.280	$0.1 \cdot 10^{-5}$
	10^{-6}	477,520.030	$0.4 \cdot 10^{-6}$
	10^{-7}	477,520.010	$0.6 \cdot 10^{-7}$
	10^{-8}	477,520.012	$0.4 \cdot 10^{-8}$

5. Conclusions

We have shown that using some tricks it was possible to reduce multiplicity of integrals appearing in the "degenerate field theory" (replacing the standard field theory in the high energy limit) so much that we were able to calculate perturbation terms up to ninth order. The question arises how far can one go this way. Obstacles are of two types:

1° multiplicity of integrals will grow obviously above three for an order of perturbation high enough;

2° values of integrals grow much faster than values of subsequent perturbation terms, therefore higher and higher precision of numerical integration will be needed.

Therefore we think that without some new concept on how to reduce multiplicity of the integrals even more, or how to represent φ_n's by integrals with completely positive integrands, chances to calculate more perturbation terms without an unreasonable computation effort, seem small.

Appendix

$$C_5 = \frac{1}{\pi^5}\int\cdots\int\frac{d^2q_1\ldots d^2q_5}{Q_1Q_{1,2}Q_{2,3}Q_{3,4}Q_{4,5}Q_5}$$

$$C_6 = \frac{1}{\pi^5}\int\cdots\int\frac{Q_3d^2q_1\ldots d^2q_5}{Q_1Q_{1,3}Q_2Q_{2,3}Q_5Q_{5,4}Q_{5,3}}$$

$$C_7 = \frac{1}{\pi^5}\int\cdots\int\frac{Q_4^2d^2q_1\ldots d^2q_5}{Q_1Q_{1,4}Q_2Q_{2,4}Q_3Q_{3,4}Q_5Q_{5,4}}$$

$$C_8 = \frac{1}{\pi^6}\int\cdots\int\frac{d^2q_1\ldots d^2q_6}{Q_1Q_{1,2}Q_{2,3}Q_{3,4}Q_{4,5}Q_{5,6}Q_6}$$

$$C_9 = \frac{1}{\pi^6}\int\cdots\int\frac{Q_3d^2q_1\ldots d^2q_6}{Q_1Q_{1,3}Q_2Q_{2,3}Q_4Q_{4,5}Q_{5,6}Q_{6,3}}$$

$$C_{10} = \frac{1}{\pi^6}\int\cdots\int\frac{Q_3d^2q_1\ldots d^2q_6}{Q_1Q_{1,2}Q_{2,3}Q_4Q_{3,4}Q_5Q_{5,6}Q_{6,3}}$$

$$C_{11} = \frac{1}{\pi^6}\int\cdots\int\frac{Q_6^2d^2q_1\ldots d^2q_6}{Q_1Q_{1,6}Q_2Q_{2,6}Q_3Q_{3,6}Q_4Q_{4,5}Q_{5,6}}$$

$$C_{12} = \frac{1}{\pi^6}\int\cdots\int\frac{Q_3Q_5d^2q_1\ldots d^2q_6}{Q_1Q_{1,3}Q_2Q_{2,3}Q_{3,5}Q_4Q_{4,5}Q_6Q_{5,6}}$$

$$C_{13} = \frac{1}{\pi^6}\int\cdots\int\frac{Q_6^3d^2q_1\ldots d^2q_6}{Q_1Q_{1,6}Q_2Q_{2,6}Q_3Q_{3,6}Q_4Q_{4,6}Q_5Q_{5,6}}$$

$$C_{14} = \frac{1}{\pi^7}\int\cdots\int\frac{d^2q_1\ldots d^2q_7}{Q_1Q_{1,2}Q_{2,3}\cdots Q_{6,7}Q_7}$$

$$C_{15} = \frac{1}{\pi^7} \int \cdots \int \frac{Q_7 d^2 q_1 \ldots d^2 q_7}{Q_1 Q_{1,7} Q_2 Q_{2,7} Q_3 Q_{3,4} Q_{4,5} Q_{5,6} Q_{6,7}}$$

$$C_{16} = \frac{1}{\pi^7} \int \cdots \int \frac{Q_7 d^2 q_1 \ldots d^2 q_7}{Q_1 Q_{1,2} Q_{2,7} Q_3 Q_{3,4} Q_{4,5} Q_{5,7} Q_6 Q_{6,7}}$$

$$C_{17} = \frac{1}{\pi^7} \int \cdots \int \frac{Q_7^2 d^2 q_1 \ldots d^2 q_7}{Q_1 Q_{1,7} Q_2 Q_{2,7} Q_3 Q_{3,7} Q_4 Q_{4,5} Q_{5,6} Q_{6,7}}$$

$$C_{18} = \frac{1}{\pi^7} \int \cdots \int \frac{Q_3 Q_7 d^2 q_1 \ldots d^2 q_7}{Q_1 Q_{1,3} Q_2 Q_{2,3} Q_{3,7} Q_4 Q_{4,7} Q_5 Q_{5,6} Q_{6,7}}$$

$$C_{19} = \frac{1}{\pi^7} \int \cdots \int \frac{Q_7^2 d^2 q_1 \ldots d^2 q_7}{Q_1 Q_{1,2} Q_{2,7} Q_3 Q_{3,4} Q_{4,7} Q_5 Q_{5,7} Q_6 Q_{6,7}}$$

$$C_{20} = \frac{1}{\pi^7} \int \cdots \int \frac{Q_7^3 d^2 q_1 \ldots d^2 q_7}{Q_1 Q_{1,7} Q_2 Q_{2,7} Q_3 Q_{3,7} Q_4 Q_{4,7} Q_5 Q_{5,6} Q_{6,7}}$$

$$C_{21} = \frac{1}{\pi^7} \int \cdots \int \frac{Q_3 Q_5 d^2 q_1 \ldots d^2 q_7}{Q_1 Q_{1,3} Q_3 Q_{2,3} Q_{3,4} Q_{4,5} Q_6 Q_{6,5} Q_7 Q_{7,5}}$$

$$C_{22} = \frac{1}{\pi^7} \int \cdots \int \frac{Q_4^2 Q_7 d^2 q_1 \ldots d^2 q_7}{Q_1 Q_{1,4} Q_2 Q_{2,4} Q_3 Q_{3,4} Q_{4,7} Q_5 Q_{5,7} Q_6 Q_{6,7}}$$

$$C_{23} = \frac{1}{\pi^7} \int \cdots \int \frac{Q_7^4 d^2 q_1 \ldots d^2 q_7}{Q_1 Q_{1,7} Q_2 Q_{2,7} \cdots Q_6 Q_{6,7}}$$

$$C_{24} = \frac{1}{\pi^8} \int \cdots \int \frac{d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,2} Q_{2,3} \cdots Q_{7,8} Q_8}$$

$$C_{25} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_8 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,8} Q_2 Q_{2,8} Q_3 Q_{3,4} Q_{4,5} \cdots Q_{7,8}}$$

$$C_{26} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_8 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,8} Q_2 Q_{2,3} Q_{3,8} Q_4 Q_{4,5} Q_{5,6} Q_{6,7} Q_{7,8}}$$

$$C_{27} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_8 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,8} Q_2 Q_{2,3} Q_{3,4} Q_{4,8} Q_5 Q_{5,6} Q_{6,7} Q_{7,8}}$$

$$C_{28} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_8^2 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,8} Q_2 Q_{2,8} Q_3 Q_{3,8} Q_4 Q_{4,5} Q_{5,6} Q_{6,7} Q_{7,8}}$$

$$C_{29} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_8^2 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,8} Q_2 Q_{2,8} Q_3 Q_{3,4} Q_{4,8} Q_5 Q_{5,6} Q_{6,7} Q_{7,8}}$$

$$C_{30} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_3 Q_8 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,3} Q_2 Q_{2,3} Q_{3,8} Q_4 Q_{4,8} Q_5 Q_{5,6} Q_{6,7} Q_{7,8}}$$

$$C_{31} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_3 Q_5 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,3} Q_2 Q_{2,3} Q_{3,4} Q_{4,5} Q_6 Q_{6,5} Q_7 Q_{7,8} Q_{8,5}}$$

$$C_{32} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_4 Q_8 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,4} Q_2 Q_{2,3} Q_{3,4} Q_{4,8} Q_5 Q_{5,8} Q_6 Q_{6,7} Q_{7,8}}$$

$$C_{33} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_3 Q_8 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,3} Q_2 Q_{2,3} Q_{3,4} Q_{4,5} Q_{5,8} Q_6 Q_{6,8} Q_7 Q_{7,8}}$$

$$C_{34} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_8^3 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,8} Q_2 Q_{2,8} Q_3 Q_{3,8} Q_4 Q_{4,8} Q_5 Q_{5,6} Q_{6,7} Q_{7,8}}$$

$$C_{35} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_8^3 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,8} Q_2 Q_{2,8} Q_3 Q_{3,8} Q_4 Q_{4,5} Q_{5,8} Q_6 Q_{6,7} Q_{7,8}}$$

$$C_{36} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_5^2 Q_8 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,5} Q_2 Q_{2,5} Q_3 Q_{3,4} Q_{4,5} Q_{5,8} Q_6 Q_{6,8} Q_7 Q_{7,8}}$$

$$C_{37} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_4 Q_8^2 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,4} Q_2 Q_{2,3} Q_{3,4} Q_{4,8} Q_5 Q_{5,8} Q_6 Q_{6,8} Q_7 Q_{7,8}}$$

$$C_{38} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_4^2 Q_8 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,4} Q_2 Q_{2,4} Q_3 Q_{3,4} Q_{4,5} Q_{5,8} Q_6 Q_{6,8} Q_7 Q_{7,8}}$$

$$C_{40} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_4^2 Q_8^2 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,4} Q_2 Q_{2,4} Q_3 Q_{3,4} Q_{4,8} Q_5 Q_{5,8} Q_6 Q_{6,8} Q_7 Q_{7,8}}$$

$$C_{41} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_5^3 Q_8 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,5} Q_2 Q_{2,5} Q_3 Q_{3,5} Q_4 Q_{4,5} Q_{5,8} Q_6 Q_{6,8} Q_7 Q_{7,8}}$$

$$C_{42} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_3 Q_5 Q_8 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,3} Q_2 Q_{2,3} Q_{3,5} Q_4 Q_{4,5} Q_{5,8} Q_6 Q_{6,8} Q_7 Q_{7,8}}$$

$$C_{43} = \frac{1}{\pi^8} \int \cdots \int \frac{Q_8^5 d^2 q_1 \ldots d^2 q_8}{Q_1 Q_{1,8} Q_2 Q_{2,8} \cdots Q_7 Q_{7,8}}$$

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