SU(1,1)/U(1) COSET GAUGE THEORY

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We construct and investigate an abelian gauge theory on the SU(1,1)/U(1) coset base space. We determine the space of flat connections and solutions of Maxwell equations. The relations to the two-dimensional euclidean black hole are discussed. We study also the interactions of the scalar field with the vacuum of considered model.

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1. Introduction

The coset models appear in physics in several contexts. They can provide, for instant, an elegant way of reducing a high-dimensional theory to a lower-dimensional. If we allow a nontrivial dependence on the extra coordinates and impose simultaneously the condition that some symmetry transformation on these variables is equivalent to a gauge transformation we obtain that although the model feels the existence of the extra dimensions, its dynamics takes place only on the lower-dimensional manifold (see [1] and references therein).

The other context is that of the SL(2,R)/U(1) gauged WZW models, which have led to the conformal theory with a two-dimensional black hole in the target space (see [2, 3] and references therein). This can be understood as an effective theory of gravity coming from the coset valued string theory. Similar effect arises also from examinations of the simplest example of the classical mechanics models on this space [4].

The interesting point is to determine the possible relations between both approaches, *i.e.* to construct a U(1) coset reduction of a gauge field theory on this space. In this paper we shall investigate the simplest example of the classical electrodynamics on the SU(1,1)/U(1) coset model. We shall determine the classical solutions and find the similarities between the considered model and the effective field theory on a Riemannian manifold.

2. Preliminaries

First, let us introduce the notation. We choose the following parametrization of SU(1, 1):

$$\begin{pmatrix} e^{i\varphi}\sqrt{1+vv^*} & v \\ v^* & e^{-i\varphi}\sqrt{1+vv^*} \end{pmatrix} \in \mathrm{SU}(2), \tag{1}$$

where φ is real and v can take values in the complex plane.

In this parametrization the Haar measure is just $d\varphi dv^* dv$ and the basis of the left invariant vector fields can be taken as follows:

$$L_z = \frac{1}{2}i\left(v^*\partial_{v^*} - v\partial_v - i\partial_\varphi\right), \tag{2}$$

$$L_- = \frac{1}{\sqrt{2}}e^{i\varphi}\left(\frac{1}{2}i\frac{v^*}{\sqrt{1+vv^*}}\partial_\varphi + \sqrt{1+vv^*}\partial_v\right), \tag{3}$$

$$L_+ = \frac{1}{\sqrt{2}}e^{-i\varphi}\left(\frac{1}{2}i\frac{v}{\sqrt{1+vv^*}}\partial_\varphi - \sqrt{1+vv^*}\partial_{v^*}\right). \tag{4}$$

Their satisfy the standard commutation relation and conjugation properties of the $su(1,1)$ Lie algebra,

$$[L_+, L_-] = -iL_z \quad [L_-, L_z] = -iL_- \quad [L_+, L_z] = iL_+, \tag{5}$$

$$L_z^* = L_z \qquad L_+^* = -L_-. \tag{6}$$

The Maurer–Cartan forms c^+, c^-, c^z, dual to these vector fields, make the basis of the cotangent bundle of SU(1, 1).

Let us mention here that we choose SU(1,1) for convenience reasons, however all our considerations could be easily reproduced in the case of other isomorphic groups. For instance, to consider a SU(2) theory, we have to substitute $-\bar{v}$ for $\bar{v}$ in all formulas. In such case, however, the values of v must be restricted to the disc $\| v \| \leq 1$.

3. SU(1,1)/U(1) coset electrodynamics

We will consider now the gauge theory constructed on the SU(1, 1) space, which remains invariant under the following transformations,

$$g \to hgh, \quad h \in \mathrm{U}(1), \tag{7}$$

$$h = \begin{pmatrix} e^{i\frac{\psi}{2}} & 0 \\ 0 & e^{-i\frac{\psi}{2}} \end{pmatrix} \in \mathrm{SU}(1,1), \tag{8}$$

which define the SU(1, 1)/U(1) coset condition.

Let us take as the basic object, the algebra of functions, which remain invariant under the transformations (7). The space of coset invariant differential forms is a module over this algebra, generated by forms obtained by the external differentiation of elements of this algebra.

The gauge transformations must belong to the algebra, so they are of the form $\mathcal{U}(v, v^*) = \mathrm{e}^{-iU(v,v^*)}$, where $U(v, v^*)$ is a smooth, real function on the complex plane $\mathbf{C}$. The gauge potential is a one-form,

$$\boldsymbol{A} = i\Phi c^z + \mathrm{e}^{-i\varphi} A c^+ + \mathrm{e}^{i\varphi} A^* c^- , \tag{9}$$

where Φ is a real field and A is a complex valued field on $\mathbf{C}$. The curvature two-form $\boldsymbol{F} = d\boldsymbol{A}$ may be written in terms of A and Φ,

$$F_{+-} = -\frac{1}{\sqrt{2}}\left(\partial_{v^*}(\sqrt{1+vv^*}A^*) + \partial_v(\sqrt{1+vv^*}A)\right) - \Phi , \tag{10}$$

$$F_{+z} = -i\mathrm{e}^{-i\varphi}\left(\frac{1}{\sqrt{2}}\sqrt{1+vv^*}\partial_{v^*}\Phi + \frac{1}{2}(v^*\partial_{v^*} - v\partial_v + 1)A\right) , \tag{11}$$

$$F_{-z} = i\mathrm{e}^{i\varphi}\left(\frac{1}{\sqrt{2}}\sqrt{1+vv^*}\partial_v\Phi - \frac{1}{2}(v^*\partial_{v^*} - v\partial_v - 1)A^*\right) . \tag{12}$$

3.1. Flat Connections

Before we construct the Yang–Mills functional we will analyse the vacua of this model. Let us notice that due to the underlying three-dimensional background of our coset space, we can construct a Chern-Simons action functional, which describes the trivial dynamics of the field $\boldsymbol{F}$, where the flat connections are the only solutions of the equations of motion.

Because the gauge potential $\boldsymbol{A}$ is defined only up to a gauge transformation we can extend the possible class of potentials if we allow for the existence of multi-valued potentials. The only restriction we have to impose, is that the physical variables, like the curvature F, are smooth and the difference between the multiple values is trivial, *i.e.*, it is an exact form. Such potential must have the following form,

$$\boldsymbol{A} = \boldsymbol{A}_R + id(\chi U) , \tag{13}$$

where χ is the angular coordinate around an arbitrary point of the plane, $\boldsymbol{A}_R$ is the regular (single-valued) flat connection and U is an arbitrary smooth real function on $\mathbf{C}$. In order to have the potential well-defined both U and dU must vanish at the origin of the coordinate system. Otherwise, the whole expression would be singular at this point.

Now, the curvature F vanishes, and the multiple values of the potential A differ by an exact form $2n\pi i dU$.

This establishes the one-to-one correspondence between the space of some gauge transformations U, which satisfy the restrictions mentioned earlier, and the space of the multi-valued flat connections. The only task left is to determine the moduli space of regular connections with vanishing curvature.

Before we proceed with the investigation, we will find a convenient gauge. For convenience we will use the spherical coordinates, $v = re^{i\chi}$. Let us notice that by an appropriate transformation the regular part of the field Φ can be chosen to be a function of the radius r. Indeed, any non-constant periodical dependence on the angular variable χ, could be gauged out. However, if we take $\Phi = \Phi(r)$, this cannot be trivial unless $\Phi = 0$. For simplicity, let us take A in a factorized form, $A = e^{i\chi} A_r(r) a_\chi(\chi)$.

If we assume so and impose the vanishing of the curvature, we obtain the following conditions for the field A and Φ. First, from Eq. (10) we get,

$$\sqrt{1+r^2}\,\Phi'(r) + iA_r(r)a'_\chi(\chi) = 0 \tag{14}$$

which we can solve immediately,

$$\Phi(r) = -\Phi_0 \qquad a_\chi(\chi) = a_0 \,. \tag{15}$$

We can assume safely that $a_0 = 1$, as we can incorporate it into A_r. If we insert (15) into the condition for the vanishing of F_{+-}, Eq. (9), we end up with the following equation for the real part of A_r,

$$\mathrm{Re}\left(\frac{1}{\sqrt{2}}\frac{1}{r}\left(r\sqrt{1+r^2}A_r(r)\right)'\right) = \Phi_0 \,. \tag{16}$$

The imaginary part of A_r is irrelevant, as in this case it is always trivial. Since we are interested only in solutions that are not singular, this restricts the possibility to the following,

$$A_r(r) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+r^2}}\Phi_0 r \,. \tag{17}$$

Finally, let us notice that this solution is a direct remnant of the coset background of the theory. It could be easily rewritten as $A = -2i\Phi_0 d\phi$, restricted to the coset space. The free parameter Φ_0, which can take arbitrary real values, labels the one-dimensional moduli space, $\Phi_0 = 0$ is of course the trivial flat connection.

The Yang–Mills action depends on the metric defined on the base space. In our case we would confine our consideration to the simplest case of constant metric, in its canonical form. For simplicity we choose $\eta_{+-} = \eta_{-+} = 1$

with all other components vanishing except for η_{zz}, which we will denote by $1/\varepsilon$. It has a clear physical meaning as is closely connected with the size of the space in the direction orthogonal to the coset space. If we take a closed curve, which is invariant under the coset transformation (7), we obtain that its length is as follows,

$$l = 4\pi\sqrt{\frac{1}{\varepsilon}(1+vv^*)(1+(1-\varepsilon)vv^*)}. \tag{18}$$

This extra dimension is space-like if $0 < \varepsilon < 1$, in which case the length is always real.

3.2. The Yang–Mills Action

The Lagrange Yang–Mills function, which is the squared norm of the curvature in the chosen metric, reads as follows,

$$\mathcal{L} = -\left(2F_{+-}^2 - 4\varepsilon F_{+z}F_{-z}\right), \tag{19}$$

which, if we take into account that $F_{+z} = F_{-z}^*$, may be rewritten as,

$$\mathcal{L} = -2\left(F_{+-}^2 - 2\varepsilon F_{+z}F_{+z}^*\right). \tag{20}$$

Thus, in the case of the euclidean metric, $0 < \varepsilon < 1$, it does not have an absolute minimum. The local minima are set by the Maxwell equations of motion. If we introduce the one-form dual to F,

$$^\star F = F_{+-}c^z + \varepsilon F_{-z}c^- - \varepsilon F_{+z}c^+, \tag{21}$$

the Maxwell equations, $d^\star F = 0$ together with the Bianchi identity $dF = 0$, could be solved immediately. Since we have already established the first cohomology of the coset differential algebra, we can take $^\star F = d\xi + \beta$, where ξ is a real function of v, v^* and β belongs to the first cohomology class. This solves the Maxwell equations, whereas the condition $dF = 0$ gives us $d^\star d\xi = 0$, because $d^\star\beta$ vanishes. The resulting equation is the following,

$$d^\star d\xi = -\left(\varepsilon L_z^2 + (L_+L_- + L_-L_+)\right)\xi = 0. \tag{22}$$

The expression (22) is the SU(1,1) Laplace operator in the chosen form of the metric, which acts on a coset invariant scalar function. It could also be interpreted as a Laplace operator on a two-dimensional curved space, with the volume element modified by a dilaton field. The equation (22) may be derived from the following two-dimensional Lagrangian,

$$\mathcal{L} = \sqrt{g}e^{\Psi}g^{ab}\partial_a\xi\partial_b\xi, \tag{23}$$

where g_{ab} is a two-dimensional metric and Ψ is a dilaton field, which modifies the measure. The dilaton can be determined exactly, whereas the metric is defined only up to a Weyl scaling factor $\Omega(v, v^*)$. In the considered example we have the metric,

$$g_{ab} = -\tfrac{1}{2}\Omega(v, v^*) \begin{pmatrix} -\frac{\varepsilon}{2}v^{*2} & (1-\frac{\varepsilon}{2})vv^* + 1 \\ (1-\frac{\varepsilon}{2})vv^* + 1 & -\frac{\varepsilon}{2}v^2 \end{pmatrix}, \tag{24}$$

modified by the dilaton:

$$\Psi = \tfrac{1}{2}\ln\left((1 + vv^*)(1 + (1-\varepsilon)vv^*)\right). \tag{25}$$

Let us notice that if we take $\varepsilon \to 0$, we obtain the dilatonic field Ψ and the metric g corresponding exactly to the two-dimensional euclidean black hole, after an appropriate choice of the Weyl factor. Therefore, in this limit we can interpret the scalar free field theory in the black hole background as a coset restriction of the U(1) gauge theory on SU(1, 1). The equivalence is not exact, as the Maxwell equations do not detect the wide class of nontrivial flat connections, which exist in this case. The question whether the dilaton field and the metrics in case of $\varepsilon \neq 0$ correspond to some solutions of dilatonic gravity, as well as their physical interpretation, we shall leave for future studies [5].

3.3. Free Field in the Background of U(1) Vacuum

Finally, we shall present an example of the charged scalar field theory in the background of the nontrivial electromagnetic potential. We use the solution (15–17), which we have calculated earlier, and our starting point is the Lagrange function of the complex field B,

$$\mathcal{L} = \eta^{ab}(L_a + iA_a)B^*(L_b - iA_b)B, \tag{26}$$

where the metric η is exactly the one we have taken in the previous section and we set the electric charge to be 1. Now, if the field B is coset invariant, we can reduce the Lagrangian (26) to the following expression,

$$\begin{aligned}\mathcal{L}_{\text{eff}} = \| \partial_\chi B \|^2 \left(\frac{\varepsilon}{4} - \frac{1+r^2}{4r^2}\right) - \frac{1}{4}(1+r^2) \| \partial_r B \|^2 \\ + \left(\varepsilon - \frac{r^2}{1+r^2}\right) \Phi_0^2 \| B \|^2, \end{aligned} \tag{27}$$

where we have dropped the terms that are total derivatives. Let us notice that the presence of the background field contributes only to the mass-like term of the effective Lagrangian. If $0 < \varepsilon < 1$, as assumed, the sign of this term changes as r increases. For symmetric, χ independent solutions, we can discuss their asymptotic behaviour in two regions. In the neighborhood of $r = 0$, B behaves like $B_0(1 - \varepsilon\Phi_0^2 r^2)$ but in the limit $\varepsilon \rightarrow 0$, B has a logarithmic singularity at $r = 0$, $B \sim \ln r$. For large r, the solution, which vanishes at infinity, behaves like $r^{-\alpha}$, where $\alpha = \sqrt{4(1-\varepsilon)\Phi_0^2 + 1} - 1$.

This example shows a rather complicated pattern of the interaction of the scalar field with the nontrivial flat gauge potential. The kinetic terms can be again interpreted as originating in the same metric and dilaton structure, such correspondence, however, cannot be easily maintained for the additional interaction term, which completely changes the evolution of the field. It would be interesting to determine, whether such procedure is possible in general. Another attractive task would be the generalization of the theory to the nonabelian case.

4. Conclusions

We have presented the basic ideas of the SU(1,1)/U(1) coset gauge theory in its simplest possible example, with the abelian gauge group. The theory, which effectively is set on the plane, shows features characteristic to the two-dimensional physics. First, we can allow for the existence of multiple-valued gauge potentials. This leads to the Aharonov–Bohm type effects, as the integral of A along a given curve depends not only on its starting and ending point but on the number of times the curve has encircled the zero of A. The effective equations derived from the Yang–Mills functional give us the free scalar field theory set in the background of dilatonic gravity, with the dilaton field depending on the underlying SU(1,1) metric and the metric determined up to a Weyl scaling.

The coset construction of the theory leads to the presence of nontrivial flat connections. They are not the absolute minima of Yang–Mills functional, as the latter is unbounded from below. However, due the underlying three-dimensional space, we can introduction the Chern–Simons functional, which gives them as the only solutions. These potentials interact with the charged fields on the coset space and although classically they cannot be distinguished from the trivial one, such interaction leads to the known result of the mass-like terms in the effective Lagrangians of the considered field.

Let us also mention here that in addition to the one-dimensional class of regular flat connection, there exists a two-parameter family of singular potentials, for which the curvature is proportional to the δ function.

The SU(1,1)/U(1) coset abelian gauge theory shows many interesting features, which we have only outlined in this paper. It seems that some further investigations of its relation to the two- dimensional dilatonic gravity and the generalization to the non-abelian theories may throw more light on both topics.

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