THE INTEGRABILITY OF THE CLASSICAL X-Y MODEL ON A LINEAR CHAIN*

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The classical $X-Y$ model on a one-dimensional lattice is considered. The $X-Y$ model is investigated as a two-component classical spin. Dynamics of the system is determined by the Hamilton equations for the spin field. By means of the zero curvature representation the integrability in the kinematic sense was proved.

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1. Introduction

We investigate the $X-Y$ model as a two-component classical spin on the one-dimensional chain [1, 2]. The spin vector is a two-dimensional vector of a unit length. The Hamiltonian for the system including interaction up to M-th coordination zone has the following form:

$$H = -\sum_{k=-\infty}^{\infty}\sum_{n=1}^{M} J_n \vec{S}_k \bullet \vec{S}_{k+n} = -\sum_{k=-\infty}^{\infty}\sum_{n=1}^{M} J_n \left(S_k^x S_{k+n}^x + S_k^y S_{k+n}^y\right) , \qquad (1)$$

where J_n is the exchange integral for spins at k-th and $k+n$-th positions. For each lattice point

$$(S_k^x)^2 + \left(S_k^y\right)^2 = 1 . \qquad (2)$$

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Dynamics of the system is determined by the Hamilton equations of motion. For each spin

$$\dot{S}_k^x = \{S_k^x, H\}, \quad \dot{S}_k^y = \{S_k^y, H\}, \tag{3}$$

where { } is the Poisson bracket (PB). We assume the following structure of the PB-s:

$$\{S_i^x, S_j^x\} = \{S_i^y, S_j^y\} = 0, \quad \{S_i^x, S_j^y\} = B_i \delta_{ij} \quad \forall\, i, j. \tag{4}$$

B_i is a real number and in the general case B_i depends on the lattice site. In this approach we do not consider $X - Y$ model as a certain restriction of the Heisenberg model (*e.g.* [3, 4]). There is no particular case of the XYZ model presented in the work [5].

Among physical systems the integrable systems have the special position. In Section 3 we will prove that the $X - Y$ model is integrable in the kinematic sense.

2. Equations of motion

Substituting the Hamiltonian H given by formula (1) into Hamilton equations (3) and taking into account the properties of the PB we derive

$$\dot{S}_k^x = -B_k \sum_{n=1}^{M} J_n \left(S_{k-n}^y + S_{k+n}^y\right),$$
$$\dot{S}_k^y = B_k \sum_{n=1}^{M} J_n \left(S_{k-n}^x + S_{k+n}^x\right). \tag{5}$$

We satisfy (2) constrain by the following ansatz:

$$\vec{S}_k(t) = [\cos f_k(t), \sin f_k(t)], \tag{6}$$

where $f_k(t)$ is a real function depending on time and on the integer variable k labelling lattice points. Substituting (6) into (5) we finally obtain

$$\dot{f}_k = B_k \sum_{n=1}^{M} J_n \Big(\exp\big(i(f_{k-n} - f_k)\big) + \exp\big(i(f_{k+n} - f_k)\big)\Big). \tag{7}$$

Detailed description of the wave like solutions of the (7) equation are presented in papers [1] and [2]. In paper [2] we explained, that the chosen method for testing $X - Y$ model dynamics is in accord with the Dirac method for a constrained system (see [6]).

3. The zero curvature representation in the discrete one-dimensional case

Instead of the discrete nonlinear differential equation one can consider the following pair of linear equations (see [4, 7, 8]):

$$\begin{aligned} F_{k+1} &= L_k(t,\lambda)F_k\,, \\ \dot{F}_k &= V_k(t,\lambda)F_k\,, \end{aligned} \tag{8}$$

where L_k, V_k are reversible matrix operators depending on k, t and complex parameter λ. The agreement condition of these equations (see Sylwester's Theorem) has the form

$$\dot{L}_k(t,\lambda) + L_k(t,\lambda)V_k(t,\lambda) - V_{k+1}(t,\lambda)L_k(t,\lambda) = 0\,. \tag{9}$$

This condition arrises from the following observation:

$$\dot{F}_{k+1} = \dot{L}_kF_k + L_k\dot{F}_k = \dot{L}_kF_k + L_kV_kF_k$$

and

$$\dot{F}_{k+1} = V_{k+1}F_{k+1} = V_{k+1}L_kF_k\,. \tag{10}$$

For the equation of motion (7) we have found the zero curvature representation and thus we have proved, that the $X - Y$ model is integrable in the kinematic sense [4]. We present two independent representations for equation (7).

1.

$$L_k = \begin{pmatrix} \exp(if_k) & \exp(-if_k) \\ 0 & c(\lambda) \end{pmatrix}, \qquad V_k = \begin{pmatrix} a_k & b_k \\ 0 & d(\lambda) \end{pmatrix}. \tag{11}$$

$c(\lambda)$ and $d(\lambda)$ are arbitrary functions of parameter λ (they are independent of time and of k), $c(\lambda) \neq 0$, $d(\lambda) \neq 0$. The components a_k and b_k of the matrix V_k are given by the following recurrent formulas:

$$\begin{aligned} &a_{k+1} - a_k = iP(k)\,, \\ &c(\lambda)b_{k+1}\exp(if_k) - b_k\exp(2if_k) = d(\lambda) - a_k - 2iP(k)\,, \end{aligned} \tag{12}$$

where

$$P(k) \stackrel{\text{def}}{=} B_k \sum_{n=1}^{M} J_n\Big(\exp\big(i(f_{k-n} - f_k)\big) + \exp\big(i(f_{k+n} - f_k)\big)\Big)\,.$$

One can choose for instance $a_0 = \lambda$. Then

$$\text{for} \quad k > 0 \qquad a_k = \lambda + i \sum_{j=0}^{k-1} P(j),$$

$$\text{for} \quad k < 0 \qquad a_k = \lambda - i \sum_{j=-1}^{k} P(j). \tag{13}$$

First we must calculate a_k elements, then b_k (one can choose for instance $b_0 = 0$). The agreement condition (9) has the form

$$\begin{pmatrix} r_k & r_k \\ 0 & 0 \end{pmatrix} = 0, \tag{14}$$

where

$$r_k = \dot{f}_k - B_k \sum_{n=1}^{M} J_n\Big(\exp\big(i(f_{k-n} - f_k)\big) + \exp\big(i(f_{k+n} - f_k)\big)\Big),$$

which is equivalent to the (7).

2.

$$L_k = \begin{pmatrix} \exp(if_k) & e(\lambda) \\ 0 & \exp(-if_k) \end{pmatrix}, \quad V_k = \begin{pmatrix} g_k & h_k \\ 0 & l_k \end{pmatrix}, \tag{15}$$

where $e(\lambda)$ is an arbitrary function of λ parameter. We calculate the elements g_k, h_k and l_k from the following formulas:

$$\begin{aligned} &g_{k+1} - g_k = iP(k), \\ &l_{k+1} - l_k = -iP(k), \\ &h_{k+1}\exp(-if_k) - h_k \exp(if_k) = e(\lambda)(l_k - g_{k+1}). \end{aligned} \tag{16}$$

We can choose for instance $g_0 = l_0 = \lambda$, $h_0 = 0$. The agreement condition has the form

$$\begin{pmatrix} r_k & 0 \\ 0 & r_k \end{pmatrix} = 0, \tag{17}$$

where

$$r_k = \dot{f}_k - B_k \sum_{n=1}^{M} J_n\Big(\exp\big(i(f_{k-n} - f_k)\big) + \exp\big(i(f_{k+n} - f_k)\big)\Big).$$

4. Conclusions

Dynamics of the $X-Y$ model is determined by the Hamilton equations for the spin field. For the equation of motion we have found examples of zero curvature representation. Thus the $X-Y$ model on a one-dimensional lattice is integrable in the kinematic sense.

Basing on the found representations we will be able to try to examine the $X-Y$ model by applying the Inverse Scattering Method to it as it is done *e.g.* for the Toda Lattice (see [4, 7, 9]), and thus to search for new solutions of the equations of motion.

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