A NEW REALIZATION OF THE Sp(2, R) ALGEBRA

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We present a new realization of the Sp(2, R) algebra and show its connections with a quantum anharmonic oscillator. This new realization admits only one unitary irreducible representation, comprising states with both even and odd parities. Coherent states for this realization exhibit squeezing, though weaker than the standard squeezed vacuum states.

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1. Introduction

The Sp(2, R) algebra contains three elements satisfying the following commutation rules:

$$[K_+, K_-] = -2K_0\,, \qquad [K_0, K_\pm] = \pm K_\pm\,. \tag{1}$$

It plays a crucial role in many areas of physics. It has been used to study giant monopole resonances (breathing modes) in nuclei or in the context of commutation rules–preserving canonical transformations of multiboson systems [1]. It has been also pointed out [2,3] that so-called squeezed states of quantum optics are in fact coherent states for this algebra in the sense of Perelomov [4]. However, all these applications have been connected with one specific realization of the operators (1), given by

$$K_+ = \tfrac{1}{2}\mathbf{a}^\dagger\mathbf{a}^\dagger\,,\ K_- = \tfrac{1}{2}\mathbf{a}\mathbf{a}\,,\ K_0 = \tfrac{1}{2}\mathbf{a}^\dagger\mathbf{a} + \tfrac{1}{4}\,, \tag{2}$$

where $\mathbf{a}^\dagger$, $\mathbf{a}$ are creation and annihilation operators for the harmonic oscillator. It is interesting to note that the Casimir operator of the Sp(2, R) algebra [4]

$$C = K_0^2 - \tfrac{1}{2}(K_+K_- + K_-K_+) = k(k-1) \tag{3}$$

admits two values of k in the realization (2), which both are positive: $k = 1/4$ and $k = 3/4$. Consequently, there exist two unitary irreducible representations of the algebra (2), each defined by one of the values of k, and it is easy to show that the number states of the harmonic oscillator $|0\rangle$, $|1\rangle$, are the lowest-weight states [4] for the representations $k = 1/4$ and $k = 3/4$, respectively. Since $K_\pm$ change the number of excitations by 2, all even parity states belong to the $k = 1/4$ representation, while all odd parity states to the $k = 3/4$ one [5].

In the present paper, we construct a new realization of the algebra (1) and examine its coherent states. This paper is organized as follows: In Section 2 we consider a quantum anharmonic oscillator and show its connection with the new realization of the Sp$(2, R)$ algebra, which we also construct. In Section 3 we construct coherent states for the new realization of the algebra, and in Section 4 we discuss squeezing properties of these states. A short discussion is given in Section 5.

2. Anharmonic oscillator

Consider a quartic one-dimensional oscillator

$$H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}x^2 + \frac{2m^2\omega^2\chi}{3}x^4 , \tag{4}$$

where the coefficient at x^4 has been chosen in a form simplifying further calculations, and both ω, $\chi \geq 0$. Upon introduction of standard creation and annihilation operators $\mathrm{a}^\dagger$, a:

$$x = \frac{1}{\sqrt{2m\omega}}(\mathrm{a}^\dagger + \mathrm{a}) , \qquad p = i\sqrt{\frac{m\omega}{2}}(\mathrm{a}^\dagger - \mathrm{a}) , \qquad (\hbar = 1) \tag{5}$$

the Hamiltonian (4) changes into

$$\begin{aligned} H &= (\omega + \chi)(\mathrm{a}^\dagger\mathrm{a} + \tfrac{1}{2}) + \chi(\mathrm{a}^{\dagger 2} + \mathrm{a}^2) \\ &\quad + \chi\left[(\mathrm{a}^\dagger\mathrm{a})^2 + \tfrac{1}{6}(\mathrm{a}^{\dagger 4} + \mathrm{a}^4) + \tfrac{2}{3}(\mathrm{a}^{\dagger 3}\mathrm{a} + \mathrm{a}^\dagger\mathrm{a}^3)\right] . \end{aligned} \tag{6}$$

If we now remove terms that do not conserve the number of excitations (rotating-wave approximation), we obtain (up to an additive constant)

$$H = (\omega + \chi)\mathrm{a}^\dagger\mathrm{a} + \chi(\mathrm{a}^\dagger\mathrm{a})^2 . \tag{7}$$

Such approximation is valid if $(\chi/\omega) \ll 1$.

A Hamiltonian similar to (7) has been expressed in terms of Sp(2, *R*) generators (2) and its interaction with squeezed electromagnetic field has been studied by Gerry [3]. Here we present another decomposition of (7). Since any function of an operator commutes with this operator, (7) may be also written as

$$H = \chi\sqrt{(\mathbf{a}^\dagger\mathbf{a}+\lambda)}\,\mathbf{a}^\dagger\mathbf{a}\sqrt{(\mathbf{a}^\dagger\mathbf{a}+\lambda)}\,, \tag{8}$$

where

$$\lambda = \frac{\omega+\chi}{\chi}\,. \tag{9}$$

Note that the square roots in (8) are Hermitian and well defined. Note also that $\lambda \geq 1$ for all $\chi \geq 0$, and for physical reasons (small nonlinearity) we expect rather large values of λ.

Now we introduce new operators

$$\begin{aligned} K_+ &= \sqrt{(\mathbf{a}^\dagger\mathbf{a}+\lambda)}\,\mathbf{a}^\dagger\,,\\ K_- &= \mathbf{a}\sqrt{(\mathbf{a}^\dagger\mathbf{a}+\lambda)}\,,\\ K_0 &= \mathbf{a}^\dagger\mathbf{a}+\tfrac{1}{2}(1+\lambda)\,. \end{aligned} \tag{10}$$

It is easy to check that the operators (10) satisfy the commutation rules (1) and thus they constitute a new realization of the Sp(2, *R*) algebra, which may be called a Holstein-Primakoff [6] realization. To see that the realization (10) is fundamentally different from (2), let us compute the Casimir operator (3). Using (10) and the standard commutation rules of $\mathbf{a}$, $\mathbf{a}^\dagger$, we get

$$C = \tfrac{1}{4}(\lambda^2-1) = k(k-1)\,, \tag{11}$$

from which we obtain

$$k = \tfrac{1}{2}(1-\lambda) \leq 0 \quad \text{or} \quad k = \tfrac{1}{2}(1+\lambda) \geq 1\,. \tag{12}$$

Since only positive values of k are connected with unitary irreducible representations of Sp(2, *R*) [4], we can see that for the realization (10) only one such representation exists. For the ground state $|0\rangle$ of the harmonic oscillator we have

$$K_0\,|0\rangle = \tfrac{1}{2}(1+\lambda)\,|0\rangle\,, \quad \text{and} \quad K_-\,|0\rangle = 0\,, \tag{13}$$

and the state $|0\rangle$ is the lowest-weight state for this representation. Because the new operators $K_\pm$ change the number of excitations by 1, we can see that the new representation comprises both even and odd parity states of the harmonic oscillator.

The Hamiltonian (7) can be now expressed as

$$H = \chi K_+ K_- . \tag{14}$$

Note that although the generators (10) formally diverge as $\chi \to 0$, the Hamiltonian (14) remains finite.

3. Coherent states

Coherent states for higher algebras can be defined in two ways: as eigenstates of the generalized annihilation operator (K_- in our case), or, as proposed by Perelomov [4], through the action of a generalized Glauber operator on the lowest-weight state of a representation:

$$D(z)\,|0\rangle = \exp\left(zK_+ - \bar{z}K_-\right)|0\rangle \; . \tag{15}$$

These two approaches give the same results for the harmonic oscillator only.

Coherent states for the Sp$(2, R)$ algebra defined as eigenstates of K_- have been examined by Barut and Girardello [7] for a "general" case, that is to say, without specifying the form of the generators (1). However, as Perelomov's approach proved to be very useful in many areas of physics, we concentrate here on the states (15). These states, with $K_\pm$ defined by (2), have been discussed by Gerry in his numerous papers (see *e.g.* [3,5]). Since formal properties of the states (15) result from algebraic properties of (1), and not from a specific form of the generators, we leave out the calculations and only quote results for the generators (10).

A parameterization $z = (\theta/2)\exp(-i\phi)$ is conventionally used. Another complex number ξ is usually introduced, $\xi = \tanh(\theta/2)\exp(-i\phi)$. Thus the states (15), though formally covering the entire complex plane z (the operator $D(z)$ is unitary), are confined to the unit Siegel disk [4] if described in terms of ξ.

From the commutation rules (1) and the Baker-Hausdorff formula we get

$$D(z) = \exp(\xi K_+)\exp(\ln(1 - |\xi|^2)K_0)\exp(-\bar{\xi}K_-) . \tag{16}$$

Thus the coherent states

$$|\xi\rangle = D(z)\,|0\rangle = (1 - |\xi|^2)^k \exp(\xi K_+)\,|0\rangle \; , \tag{17}$$

where k labels the representation. Using (10), after some algebra we finally get

$$|\xi\rangle = (1-|\xi|^2)^{\frac{1}{2}(1+\lambda)} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(n+1+\lambda)}{n!\Gamma(1+\lambda)}}\, \xi^n \,|n\rangle \,, \tag{18}$$

where $|n\rangle$ are number states of the harmonic oscillator. We can see that the states (18) are built up of both even and odd parity states, contrary to the standard squeezed vacuum states [2,3].

The states (15) are also over-complete, *i.e.* they are nonorthogonal but constitute a resolution of identity. Details, which do not depend on a specific realization of the generators, may be found in [5].

4. Squeezing properties of the coherent states

Consider two Hermitian operators

$$X_1 = \tfrac{1}{2}\left(\mathrm{a}+\mathrm{a}^\dagger\right), \qquad X_2 = \tfrac{1}{2i}\left(\mathrm{a}-\mathrm{a}^\dagger\right). \tag{19}$$

They do not commute, and as it is known from textbooks, their variances

$$V_i = \langle {X_i}^2\rangle - \langle X_i\rangle^2 \tag{20}$$

must satisfy the inequality

$$V_1 V_2 \geq \tfrac{1}{16}\,. \tag{21}$$

The variances (20) are calculated in a given quantum state. For the vacuum $|0\rangle$ we get $V_{1,\mathrm{vac}} = V_{2,\mathrm{vac}} = \frac{1}{4}$. If either of the variances is less than its vacuum value (while the other one is enhanced in order to satisfy (21)), the corresponding quantum state is called a squeezed state. It is now well known [2,3] that coherent states for the realization (2) of the Sp(2, R) algebra are squeezed, unless a specific choice of phase ϕ is made; they are even nicknamed squeezed vacuum states.

For the states (18) we obtain

$$V_1 = \tfrac{1}{4} + \tfrac{1}{2}\bar{n} - \tfrac{1}{2}B + \tfrac{1}{2}(A-B)\cos 2\phi\,, \tag{22a}$$

$$V_2 = \tfrac{1}{4} + \tfrac{1}{2}\bar{n} - \tfrac{1}{2}B - \tfrac{1}{2}(A-B)\cos 2\phi\,, \tag{22b}$$

where

$$\bar{n} = \left\langle \mathrm{a}^\dagger \mathrm{a}\right\rangle = (1+\lambda)\frac{|\xi|^2}{1-|\xi|^2}\,, \tag{23}$$

and

$$A = |\xi|^2 (1 - |\xi|^2)^{1+\lambda} \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\lambda)}{n!\Gamma(1+\lambda)} \sqrt{(n+1+\lambda)(n+2+\lambda)}\, |\xi|^2 , \tag{24a}$$

$$B = |\xi|^2 \left[(1 - |\xi|^2)^{1+\lambda} \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\lambda)}{n!\Gamma(1+\lambda)} \sqrt{(n+1+\lambda)}\, |\xi|^2 \right]^2 . \tag{24b}$$

As one can see, the degree of squeezing is fully determined by the average number of photons in the state $\bar{n}$, and by the phase ϕ. In particular, for $\phi = \pi/4$, no squeezing occurs. For simplicity, we put $\phi = 0$, so the variance V_2 takes values smaller than $V_{2,\mathrm{vac}}$, and squeezing is maximal.

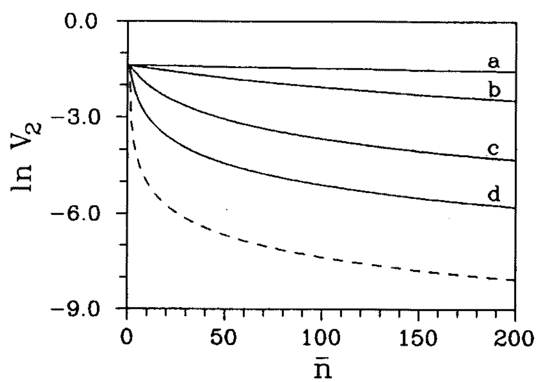

Fig. 1. $\ln V_2$ as a function of $\bar{n} = \langle a^\dagger a \rangle$ for various values of λ and $\phi = 0$. Line a corresponds to $\lambda = 1000$, b to $\lambda = 100$, c to $\lambda = 10$, and d to $\lambda = 1$. The dashed line corresponds to a squeezed vacuum state (*cf.* (25)).

Fig. 1 shows numerical results for a logarithm of V_2 as a function of $\bar{n}$ for various values of λ. In addition, V_2 for squeezed vacuum states, as calculated by Gerry [3],

$$V_{2,\mathrm{sq.vac}} = \tfrac{1}{2}(\bar{n} + \tfrac{1}{2}) - \tfrac{1}{2}\sqrt{\bar{n}(\bar{n}+1)}\,, \tag{25}$$

is shown. Apparently the greater the average number of photons, the greater the squeezing, although it is always smaller than squeezing for a squeezed vacuum state with the same average number of photons. The states (18) also remain, to a good numerical approximation, minimum uncertainty states (*i.e.* $V_1 V_2 = 1/16$). However, for large values of λ, corresponding to physically realistic nonlinearities in (7), squeezing is marginal.

5. Discussion

We have constructed a Holstein-Primakoff realization of the Sp$(2, R)$ algebra, and shown how it naturally arises from considering the anharmonic oscillator (7). This new realization is fundamentally different from the "standard" realization (2). Coherent states for the new realization exhibit some interesting properties, including squeezing, which, however, is weaker than squeezing displayed by coherent states of the "standard" realization.

It is interesting to note that similar Holstein-Primakoff realizations of the Sp$(2, R)$ algebra have been already introduced in an auxiliary Fock space for a multidimensional oscillator [8] and in a Hilbert space of analytic functions [9]. Here we have presented a Holstein-Primakoff realization in a "real" physical space for the first time. However, the problem of a possible generation of quantum states (18) remains open.

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