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TWISTOR–DIAGRAM REPRESENTATION OF MASS–SCATTERING INTEGRAL EXPRESSIONS FOR DIRAC FIELDS

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An explicit colored-graph description of the processes of mass-scattering of Dirac fields in real Minkowski space is used to show how the corresponding twistorial mass-scattering formulae can be entirely derived from a set of simple rules for the graphs. A correspondence between graphs and twistor diagrams is then suggested.

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1. Introduction

In an earlier work [1], null initial data (NID) techniques [2, 3] were used to describe completely the null dynamics of classical Dirac fields in real Minkowski space $\mathbb{RM}$. The above referred work was based essentially upon the fact that the Van der Waerden form of the source-free Dirac equation [3] enables one to look upon the Dirac fields as the elements of an invariant exact set of interacting spinor fields. In this NID framework, the mass $m = \sqrt{2}\hbar\mu$ of the fields ($2\pi\hbar$ being the usual Planck's constant), plays effectively the role of a coupling constant. Likewise, the Dirac fields are split up into an infinite number of elementary contributions satisfying Dirac-like equations on the interior V_0^+ of the future cone of an origin 0 of $\mathbb{RM}$. These elementary fields propagate for a while along null geodesics of $\mathbb{RM}$, and scatter off each other at points lying in V_0^+. Such interior points appear to be appropriate vertices of certain forward null zigzags [1, 3] that start at 0 and terminate at a fixed point of V_0^+. Moreover, they are indeed the points at which the NID for the mass-scattering processes are specified. Accordingly, the entire fields are explicitly recovered by four series of terms, each of which being expressed by a finite integral taken over a space of null graphs that possess a suitable number of edges. The choice of the future null cone C_0^+ of 0 as the NID hypersurface for all the elementary fields led us to a formally simple set of scattering formulae. Each scattering integral was then naturally associated with a (simple) connected colored graph. In addition, this choice allowed us to carry out an explicit twistorial transcription of the field integrals in a straightforward way [4]. One of the most striking features of the resulting twistorial mass-scattering integrals is the fact that they involve scaling invariant (SI) holomorphic structures defined on products of subsets of Riemann spheres that represent appropriate vertices of null graphs.

The present paper is mainly aimed at representing the Dirac fields in terms of twistor diagrams. For the sake of completeness, the graphical description of the scattering processes as given by Cardoso [1] is reviewed (Section 2). We reinstate the twistorial scattering formulae by employing a set of rules for the colored graphs (Section 3). Each twistorial scattering integral turns out to be represented by a twistor diagram. A remarkable feature of our diagrams is that some of the singularity lines join vertices of the same type. Indeed, the associated twistor inner products used here are somewhat different from the standard ones [5, 6]. This appears to be directly related to the fact that all the twistors involved in any scattering integral expression possess the same valence [4]. A correspondence between the scattering graphs and twistor diagrams then arises, yielding a manifestly diagrammatic representation of the entire fields (Section 4). Throughout this work, the unprimed and primed fields will be called left-handed and right-handed fields, respectively, but no attempt shall be made herein to regard them as quantum fields.

There are two reasons for undertaking the present work. First, there is the fact that it seems to be worthwhile to set up simple graphical rules whereby the mass-scattering formulae may be entirely derived. Second, there is hope that an explicit representation of the scattering integrals in terms of twistor diagrams can eventually provide new insights into the theory of twistors.

2. Mass-scattering graphs

A mass-scattering null zigzag (MSNZZ) is [1] a simple graph ζ_{N+2} ($N \geq 0$) whose vertex-set

$$V(\zeta_{N+2}) = \{\overset{0}{x}{}^{AA'}, \overset{1}{x}{}^{AA'}, \overset{2}{x}{}^{AA'}, \ldots, \overset{N+2}{x}{}^{AA'}\}, \tag{2.1}$$

consists of $N+3$ vertices such that $\overset{n+1}{x}{}^{AA'}$ is future-null-separated from $\overset{n}{x}{}^{AA'}$, $n = 0, 1, 2, \ldots, N+1$. The starting-vertex $\overset{0}{x}{}^{AA'}$ is effectively identified with 0 and, therefore, $\overset{1}{x}{}^{AA'} \in C_0^+$. All the $(N+1)$ vertices

$\overset{2}{x}{}^{AA'}, \overset{3}{x}{}^{AA'}, \ldots, \overset{N+2}{x}{}^{AA'}$ belong to V_0^+. The vertex $\overset{N+1}{x}{}^{AA'}$ is indeed the point at which the scattering giving rise to an outgoing field of order N happens, the end-vertex $\overset{N+2}{x}{}^{AA'}$ appearing as the (fixed) point at which the explicit evaluation of the elementary field is to be carried out. We shall see shortly that the vertices belonging to $V(\zeta_{N+2}) - \{\overset{0}{x}{}^{AA'}, \overset{N+2}{x}\}$ are the only points that appear to be relevant to the construction of the whole integrand of the scattering integral for the field. It is convenient to endow ζ_{N+2} with a forward spin-basis set (FSBS)

$$\Big\{\{\overset{0}{o}{}^A, \overset{1}{o}{}^A\}, \{\underset{0}{\bar{o}}{}^{A'}, \underset{1}{\bar{o}}{}^{A'}\}, \{\overset{1}{o}{}^A, \overset{2}{o}{}^A\}, \{\underset{1}{\bar{o}}{}^{A'}, \underset{2}{\bar{o}}{}^{A'}\}, \ldots, \\ \{\overset{N+1}{o}{}^A, \overset{N+2}{o}{}^A\}, \{\underset{N+1}{\bar{o}}{}^{A'}, \underset{N+2}{\bar{o}}{}^{A'}\}\Big\}. \tag{2.2}$$

By definition this set consists of $2N+4$ pairs of conjugate spin bases, with the (real) null vectors $\overset{n}{o}{}^A \underset{n}{\bar{o}}{}^{A'}$ and $\overset{n+1}{o}{}^A \underset{n+1}{\bar{o}}{}^{A'}$ pointing in future null directions through $\overset{n}{x}{}^{AA'}$. The elements of the pair $\{\overset{n+1}{o}{}^A, \underset{n+1}{\bar{o}}{}^{A'}\}$ are indeed taken to be covariantly constant along the (null geodesic) generator γ_{n+1} of the future null cone C_n^+ of $\overset{n}{x}{}^{AA'}$ that passes through $\overset{n+1}{x}{}^{AA'}$. In addition, the conjugate spin-inner products at $\overset{n}{x}{}^{AA'}$

$$\overset{n}{z} = \overset{n}{o}{}^A \overset{n+1}{o}{}_A, \quad \underset{n}{\bar{z}} = \underset{n}{\bar{o}}{}^{A'} \underset{n+1}{\bar{o}}{}_{A'} \tag{2.3}$$

are held fixed, the ones set up at 0 being useful only when we consider explicit twistor structures for odd-order fields (see Section 3). The edge-set of ζ_{N+2} is defined by

$$E(\zeta_{N+2}) = \{\overset{1}{r}, \overset{2}{r}, \overset{3}{r}, \ldots, \overset{N+2}{r}\}, \tag{2.4}$$

which consists of $N+2$ edges, with $\overset{n+1}{r}$ being a suitable (positive) affine parameter on γ_{n+1} which actually connects the vertices $\overset{n}{x}{}^{AA'}$ and $\overset{n+1}{x}{}^{AA'}$.

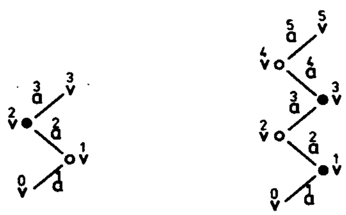
To construct the scattering formulae in a concise way, we need first to define what is called [1] a mass-scattering graph (MSG). It is a (simple) connected oriented colored graph σ_{N+2} [7, 8] whose (vertex-set) edge-set is suitably associated with $(V(\zeta_{N+2}))$ $E(\zeta_{N+2})$. In effect, we have the correspondences

$$\overset{m}{v} \xrightarrow{\vartheta} \overset{m}{x}{}^{AA'}, \overset{m}{v} \in V(\sigma_{N+2}), \quad m = 0, 1, 2, \ldots, N+2, \tag{2.5}$$

and

$$\overset{n+1}{a} \overset{e}{\longrightarrow} \overset{n+1}{r}, \overset{n+1}{a} \in E(\sigma_{N+2}), \tag{2.6}$$

where n runs over the same values as before. The vertices $\overset{n}{v}$ and $\overset{n+1}{v}$ are connected by $\overset{n+1}{a}$, the latter vertex being "forwardly" separated from the former (see (2.1)). This "forwardness" is actually what defines the orientation of σ_{N+2}. We restrict ourselves to considering here only zigzag-like MSG's. Both the starting-vertex $(\overset{0}{v})$ and the end-vertex $(\overset{N+2}{v})$ of σ_{N+2} carry no color. Each of the $N+1$ colored vertices carries either the color white or the color black. For $N > 0$ any one of the N internal edges always joins vertices carrying different colors, the allowable "forward" configurations thus being <white–black> and <black–white>. Any (white) black vertex can be thought of as representing particularly the scattering of a (left-handed) right-handed elementary field. It will be seen that the MSG's carry all the information about the actual scattering processes. The examples of MSG's are shown in Fig. 1.



(a) (b)

Fig. 1. Two MSG's. The internal edges connect vertices bearing different colors. Each white (black) vertex represents the mass scattering of a left- (right-) handed elementary contribution. For either field of order N the graph carries $N+1$ edges. The number of white and black vertices in each case appears to be intimately related to the order and handedness of the relevant outgoing field: (a) the outgoing right-handed field of order 1; (b) the outgoing left-handed field of order 3.

It must be observed that, as far as the construction of any MSG is concerned, what really appears to be of importance is the colored-vertex configuration along with the number of edges. From now on we shall for simplicity delete the v's and a's.

We now consider the N^{th}-scattering process. It takes the elementary contributions $\{\overset{N}{\psi}_A(x), \underset{N}{\chi}^{A'}(x)\}$ as the outgoing fields and, in each case, is graphically described by a σ_{N+2}. To any order, the denominator of the scattering datum entering into either field integral carries the product of N affine parameters with $N+1$ spin-inner products. The parameters involved correspond via (2.6) to the internal edges of the relevant $E(\sigma_{N+2})$, and are therefore set as

$$AP_N = \left\{ \overset{2}{r}, \overset{3}{r}, \ldots, \overset{N+1}{r} \right\}. \tag{2.7}$$

Each spin-inner product carriers elements of the FSBS for ζ_{N+2}. It is always defined at a vertex of $V(\zeta_{N+2})$ which corresponds to a colored vertex of $V(\sigma_{N+2})$. Furthermore, it appears to be barred or unbarred according to whether the associated colored vertex is black or white. For odd-order contributions, both MSG's carry the same number of white and black vertices which actually equals $(N+1)/2$. In the case of even-order fields, the number of white (resp. black) vertices is equal to either $(N/2)+1$ or $N/2$ (resp. $N/2$ or $(N/2)+1$), depending upon whether we consider the left-handed or the right-handed outgoing field. It follows that, for the left-handed field, the vertex $\overset{1}{v}$ carries the color white or black according to whether N is even or odd. It should be clear at this stage that, for $\underset{N}{\chi}^{A'}(x)$, these configurations turn out to be reversed. In fact, the inner products $\overset{1}{z}$ and $\underset{1}{\bar{z}}$ are suitably carried by the $\hat{\pi}$-NID on C_0^+ which actually enter into the numerators of the scattering data (see Ref. [1]). Such null data involve the (real) conformal-invariant directional derivative operators that are carried by the integrands of the Kirchoff–D'Adhemar–Penrose (KAP) expressions for spinning massless free fields [3, 9]. Indeed these null-datum quantities generate the scattering data for all the elementary contributions. We thus have colored-vertex structures

$$\text{Even-order left-handed} \quad \overset{1}{\circ}\underset{2}{\bullet}\overset{3}{\circ}\underset{4}{\bullet}\cdots\underset{N}{\bullet}\ \overset{N+1}{\circ} \tag{2.8a}$$

$$\text{Odd-order left-handed} \quad \underset{1}{\bullet}\overset{2}{\circ}\underset{3}{\bullet}\overset{4}{\circ}\cdots\underset{N}{\bullet}\ \overset{N+1}{\circ} \tag{2.8b}$$

$$\text{Even-order right-handed} \quad \underset{1}{\bullet}\overset{2}{\circ}\underset{3}{\bullet}\overset{4}{\circ}\cdots\overset{N}{\circ}\ \underset{N+1}{\bullet} \tag{2.9a}$$

$$\text{Odd-order right-handed} \quad \overset{1}{\circ}\underset{2}{\bullet}\overset{3}{\circ}\underset{4}{\bullet}\cdots\overset{N}{\circ}\ \underset{N+1}{\bullet}\,. \tag{2.9b}$$

We must emphasize that the structures carrying the same number of vertices can be built up from one another by applying a simple white-black interchange rule.

The numerators of the integrands of the scattering integrals involve appropriately also the "outgoing" spinors $\left\{ \overset{N+2}{o}_A, \underset{N+2}{\bar{o}}^{A'} \right\}$ together with an SI $3N+2$-differential form $\overset{123\ldots N+1}{\underset{\sim}{K}}$ defined on the abstract (compact) space $\overset{123\ldots N+1}{\mathbb{K}}$ of MSNZZ's whose edge-sets possess $N+2$ edges [1]. In the case of either field, this generalized scattering form actually appears multiplied by the mass-factor $(\mu/2\pi)^N$, and effectively carries the contributions emanating from those RM-vertices which correspond to the colored vertices of the relevant MSG. It now becomes evident that all the colored vertices contribute also suitable differential forms to the integrand of the field integral expressions, either MSG thus appearing to be associated with a scattering integral. Each (colored) vertex $\overset{j}{v}$, with j running over the values $1, 2, 3, \ldots, N$ contributes, in effect, an SI three-volume form which is set up at the corresponding element $\overset{j}{x}{}^{AA'}$ of the vertex-set of some appropriate MSNZZ. This form is given as the wedge-product between $d\overset{j}{r}\,/\,\overset{j}{r}$ and $\overset{j}{\underset{\sim}{S}}$, the latter factor defining an SI element of two-surface area provided by the (space-like) intersection of appropriate null cones of suitable RM-vertices (see (3.10) below). Its integration always bears the topology $S^1 \times S^2$. To either integral the pertinent vertex $\overset{N+1}{v}$ contributes, in turn, a KAP-like two-form set up at $\overset{N+1}{x}{}^{AA'}$ whose integration effectively takes up the adequate "outgoing" spinor.

We should observe that the above prescription still applies to the case wherein $N = 0$. The corresponding process can be referred to as the "zero$^{\text{th}}$-order process" which actually involves the massless free-field contributions. In this case, both "scattering data" appear to be nothing else but the $\hat{\pi}$-NID on C_0^+, the relevant differential form accordingly being the KAP-form on $\overset{1}{\mathbb{K}}$.

In accordance with the above rules, we have the SI scattering formulae

$$\overset{2K}{\psi}_A(x) = \frac{c_K}{2\pi} \int_{\overset{123\ldots 2K+1}{\mathbb{K}}} \overset{2K+2}{o}{}_A \frac{\underset{1}{\hat{\pi}}{}^{1/2-}\,\overset{0}{\psi}_{\mathrm{L}}(\overset{1}{o}{}^{M};\overset{1}{x})\;\overset{123\ldots 2K+1}{\underset{\sim}{K}}}{\left(\prod_{n=1}^{K}\overset{2n+1}{z}\right)\left(\prod_{m=1}^{K}\underset{2m}{\bar{z}}\right)\left(\prod_{p=1}^{2K}\overset{p+1}{r}\right)}, \tag{2.10}$$

$$\underset{2K}{\chi}{}^{A'}(x) = \frac{c_K}{2\pi} \int_{\overset{123\ldots 2K+1}{\mathbb{K}}} \underset{2K+2}{\bar{o}}{}^{A'} \frac{\underset{1}{\hat{\pi}}{}^{1/2+}\,\underset{0}{\chi}{}_{\mathrm{R}}(\underset{1}{\bar{o}}{}^{M'};\overset{1}{x})\;\overset{123\ldots 2K+1}{\underset{\sim}{K}}}{\left(\prod_{m=1}^{K}\overset{2m}{z}\right)\left(\prod_{n=1}^{K}\underset{2n+1}{\bar{z}}\right)\left(\prod_{p=1}^{2K}\overset{p+1}{r}\right)}, \tag{2.11}$$

for the even-order fields (with $c_K = (\mu/2\pi)^{2K}$), and

$$\overset{2K+1}{\psi}{}_A(x) = \frac{C_K}{2\pi}\int_{\substack{123\ldots 2K+2\\ \mathbb{K}}} \overset{2K+3}{o}{}_A \frac{\underset{1}{\hat{\pi}}{}_{1/2+}\,\underset{0}{\chi}{}_{\mathrm{R}}(\underset{1}{\bar{o}}{}^{M'};\overset{1}{x})\,\underset{\sim}{\overset{123\ldots 2K+2}{K}}}{\left(\prod_{m=1}^{K+1}\overset{2m}{z}\right)\left(\prod_{n=1}^{K}\underset{2n+1}{\bar{z}}\right)\left(\prod_{q=1}^{2K+1}\overset{q+1}{r}\right)}, \tag{2.12}$$

$$\underset{2K+1}{\chi}{}^{A'}(x) = \frac{C_K}{2\pi}\int_{\substack{123\ldots 2K+2\\ \mathbb{K}}} \underset{2K+3}{\bar{o}}{}^{A'} \frac{\underset{1}{\hat{\pi}}{}_{1/2-}\,\overset{0}{\psi}{}_{\mathrm{L}}(\overset{1}{o}{}^{M};\overset{1}{x})\,\underset{\sim}{\overset{123\ldots 2K+2}{K}}}{\left(\prod_{n=1}^{K}\overset{2n+1}{z}\right)\left(\prod_{m=1}^{K+1}\underset{2m}{\bar{z}}\right)\left(\prod_{q=1}^{2K+1}\overset{q+1}{r}\right)}, \tag{2.13}$$

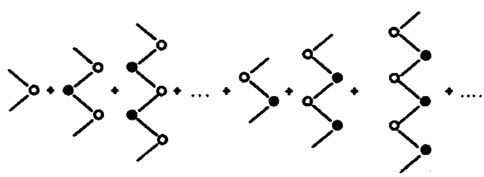


Fig. 2. Graphical recovery of the left-handed element of the Dirac pair.

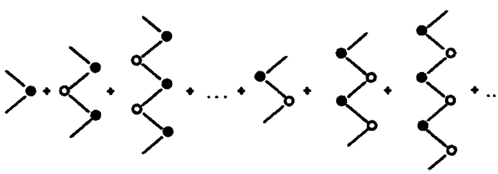


Fig. 3. Graphical recovery of the right-handed element of the Dirac pair.

for the odd-order fields (with $C_K = (\mu/2\pi)^{2K+1}$). It must be observed that the KAP-character of these integrals can be brought about by defining (SI) field densities on future null cones of adequate vertices of appropriate $\mathbb{RM}$-graphs (see Ref. [1]). Roughly speaking, the crucial point here is that a field of order N is evaluated at the (end-) vertex $\overset{N+2}{x}{}^{AA'}$ of some suitable $V(\zeta_{N+2})$ from a null-datum quantity specified at the vertex $\overset{N+1}{x}{}^{AA'} \in V(\zeta_{N+2})$ which is effectively null-separated from $\overset{N+2}{x}{}^{AA'}$. In relation to this situation it is convenient to recall also that the vertex appearing as explicit argument on the left-hand sides of the above integral expressions has to be identified with end-vertices of MSNZZ's carrying appropriate vertex-edge structures. The graphical recovery of the entire Dirac pair is shown in Figs 2 and 3.

3. Twistorial formulae

Towards building up the twistorial formulae for MSNZZ's we define the set of conjugate null twistors for some ζ_{N+2}

$$T(\zeta_{N+2}) = WT(\zeta_{N+2}) \cup ZT(\zeta_{N+2}), \tag{3.1}$$

where

$$WT(\zeta_{N+2}) = \left\{ \overset{0}{W}_{\alpha}, \overset{1}{W}_{\alpha}, \ldots \overset{N+2}{W_{\alpha}} \right\}, \tag{3.2a}$$

and

$$ZT(\zeta_{N+2}) = \left\{ \underset{0}{Z}^{\beta}, \underset{1}{Z}^{\beta}, \ldots, \underset{N+2}{Z}^{\beta} \right\}, \tag{3.2b}$$

with

$$\overset{m}{W}_{\alpha} = (\overset{m}{o}_{A}, -i \overset{m}{x}{}^{AA'} \overset{m}{o}_{A}), \quad \underset{m}{Z}^{\beta} = (i \overset{m}{x}{}^{AA'} \underset{m}{\bar{o}}_{A'}, \underset{m}{\bar{o}}_{A'}) = \bar{W}^{\beta}, \tag{3.3}$$

the elements of the pair $\{\overset{m}{o}_{A}, \underset{m}{\bar{o}}_{A'}\}$ being involved in the definition of the FSBS for ζ_{N+2}, and $\overset{m}{x}{}^{AA'} \in V(\zeta_{N+2})$. These twistors satisfy the (conjugate) incidence relations at $\overset{n}{x}{}^{AA'}$

$$\underset{n+1}{Z}^{\mu} \overset{n}{W}_{\mu} = 0 = \underset{n}{Z}^{\mu} \overset{n+1}{W_{\mu}}, \tag{3.4}$$

where n runs over the same values as in the definition (2.1). It is clear that the twistors (3.3) are both associated with the null geodesic γ_m whenever m takes on the values $1, 2, 3, \ldots, N+2$. The twistors $\overset{0}{W}_{\alpha}$, $\underset{0}{Z}^{\beta}$ actually play an auxiliary role, and enter into the field integrals for odd-order contributions [4] (see also (3.19) and (3.20) below). We now define a correspondence between the associated edge-set $E(\sigma_{N+2})$ and each of the sets $WT(\zeta_{N+2}) - \{\overset{0}{W}_{\alpha}\}$, $ZT(\zeta_{N+2}) - \{\underset{0}{Z}^{\beta}\}$, such that (see (2.5) and (2.6))

$$\overset{n+1}{a} \xrightarrow{\tau_x} \underset{n+1}{X}^{\bullet}, \tag{3.5}$$

where $\underset{n+1}{X}^{\bullet}$ stands for either $\overset{n+1}{W_{\alpha}}$ or $\underset{n+1}{Z}^{\beta}$. In Section 4 it will be shown that this relationship, in particular, enables one to associate MSG's with twistor diagrams. We will see that the twistorial expressions arising here are closely related to the vertex-edge structure of σ_{N+2}. In general, it will be necessary to assume that σ_{N+2} represents a higher-order field.

There are two conjugate twistorial expressions for any element $\overset{h}{r}$ of (2.7). Each of their numerators carries one of the ordinary inner products that involve the twistors "surrounding" the edge. Hence the label of the edge is equal to the arithmetic mean of the labels of the twistors occurring in either numerator. In fact, this result is what gives rise to a pair of conjugate expressions for $\overset{h}{r}$, and appears to be related to the structure of the colored configurations involved. The denominator of either expression carries two of the factors which are associated with the spin-inner products at the relevant connected vertices $\overset{h-1}{x}{}^{AA'}$ and $\overset{h}{x}{}^{AA'}$. These spin-inner products are always defined at different vertices, one involving W_α-twistors and the other, Z^β-twistors. It follows that both denominators actually involve the twistors $\overset{h}{W}_\alpha$ and $\underset{h}{Z}{}^\beta$. In either case, one way of removing the conjugation ambiguity is to introduce the two inner products that carry the twistors appearing in the relevant numerator. This procedure implies, in effect, that the structure of each denominator is automatically built up so as to make $\overset{h}{r}$ bear the appropriate spin-weight. We thus have the expressions

$$\overset{h}{r} = i\,\frac{\underset{h+1}{Z^\mu}\,\overset{h-1}{W_\mu}}{\left(I^{\alpha\beta}\,\overset{h-1}{W_\alpha}\,\overset{h}{W_\beta}\right)\left(I_{\lambda\tau}\,\underset{h}{Z}{}^\lambda\,\underset{h+1}{Z}{}^\tau\right)}, \tag{3.6}$$

and

$$\overset{h}{r} = (-i)\,\frac{\underset{h-1}{Z^\mu}\,\overset{h+1}{W_\mu}}{\left(I^{\alpha\beta}\,\overset{h}{W_\alpha}\,\overset{h+1}{W_\beta}\right)\left(I_{\lambda\tau}\,\underset{h-1}{Z}{}^\lambda\,\underset{h}{Z}{}^\tau\right)}, \tag{3.7}$$

where the i-factors have been appropriately introduced to guarantee the reality oh $\overset{h}{r}$. Here, as elsewhere, $I^{\alpha\beta}$ and $I_{\lambda\tau}$ denote the usual infinity twistors (see, for instance, Ref. [6]).

For the product of two adjacent edges $\overset{k}{r}$, $\overset{k+1}{r}$ joining vertices of $V(\zeta_{N+2})$ associated with colored vertices of $V(\sigma_{N+2})$ we can once again build up two conjugate expressions. The numerator of either of these structures involves one of the (totally skew-symmetric) ε-twistor pieces which carry the twistors of the same valence that "surround" and "contain" the edges. Clearly, there are two allowable triples of $\mathbb{RM}$-vertices which are connected by the edges under consideration, the associated (internal) "forward" colored configurations accordingly being <white-black-white> and <black-white-black>.

Each denominator carries the twistor form of the conjugate spin-inner products at the appropriate middle vertex $\overset{k}{x}{}^{AA'}$ together with the twistor form of two suitable spin-inner products defined at the other vertices of the relevant $\mathbb{R}\mathbb{M}$-configuration. In both cases the twistors involved in these inner products bear the same valence as that borne by the twistors occurring in the numerators, and are indeed always defined at different vertices. We thus have

$$\overset{k}{r}\overset{k+1}{r} = (-1)\frac{\varepsilon^{\alpha\beta\gamma\delta}\overset{k-1}{W}_{\alpha}\overset{k}{W}_{\beta}\overset{k+1}{W}_{\gamma}\overset{k+2}{W}_{\delta}}{\left(I^{\mu\nu}\overset{k-1}{W}_{\mu}\overset{k}{W}_{\nu}\right)\left(I^{\alpha\beta}\overset{k}{W}_{\alpha}\overset{k+1}{W}_{\beta}\right)\left(I_{\rho\sigma}\underset{k}{Z}^{\rho}\underset{k+1}{Z}^{\sigma}\right)\left(I^{\lambda\tau}\overset{k+1}{W}_{\lambda}\overset{k+2}{W}_{\tau}\right)}, \tag{3.8}$$

$$\overset{k}{r}\overset{k+1}{r} = (-1)\frac{\varepsilon_{\alpha\beta\gamma\delta}\underset{k-1}{Z}^{\alpha}\underset{k}{Z}^{\beta}\underset{k+1}{Z}^{\gamma}\underset{k+2}{Z}^{\delta}}{\left(I_{\mu\nu}\underset{k-1}{Z}^{\mu}\underset{k}{Z}^{\nu}\right)\left(I^{\alpha\beta}\overset{k}{W}_{\alpha}\overset{k+1}{W}_{\beta}\right)\left(I_{\rho\sigma}\underset{k}{Z}^{\rho}\underset{k+1}{Z}^{\sigma}\right)\left(I_{\lambda\tau}\underset{k+1}{Z}^{\lambda}\underset{k+2}{Z}^{\tau}\right)}, \tag{3.9}$$

where the factor (-1) is related to the reality of the involved edges, and the ε-twistors are in fact the usual alternating twistors for the standard frame [6]. Actually the form of the inner products entering into the denominators of (3.8) and (3.9) appears to be intimately related to the color of the vertices carried by the three-vertex graphical configurations. In the case of a higher even-order (right-handed) left-handed field, in effect, any (white) black vertex plays the role of a middle vertex, while any (black) white vertex contributes a factor involving (Z^{β}-twistors) W_{α}-twistors. For a higher odd-order field the above handedness-vertex prescription still works, but the (uncolored) starting-vertex of the associated MSG now contributes an appropriate inner product to the relevant expression for $\overset{1}{r}\,\overset{2}{r}$. Indeed, the reason why the (external) edge $\overset{1}{r}$ comes into play here is that in the latter case, each $\hat{\pi}$-NI datum on C_0^+ has to be adequately modified when the explicit twistorial transcription of the corresponding scattering integral is carried out [4]. What results from this modification is that, for either handedness, the relevant vertex $\overset{1}{x}{}^{AA'}$ appears also as a "middle" vertex at every stage of the evaluation of the $\mathbb{R}\mathbb{M}$-integral in question whence the number of edges entering into the denominator of the pertinent scattering datum turns out to be even. In particular, the inner product at 0 involves W_{α}-

or Z^{β}-twistors, depending upon whether the outgoing field is left-handed or right-handed. Evidently, these considerations are also pertinent to the colored-vertex structures involved in (3.6) and (3.7). In respect to the above situation, it should be noticed that the relations (3.6)–(3.9) still hold for $h = 1 = k$. Particularly, we see that the conjugation of (3.6) and (3.7) can now be eliminated once and for all.

We next consider the SI two-form of surface area $\overset{j}{\underset{\sim}{S}}$ defined at

$$\overset{j}{x}{}^{AA'} \in C^{-}_{j+1} \cap C^{+}_{j-1}\,, \tag{3.10}$$

where C^{-}_{j+1} is the backward null cone of $\overset{j+1}{x}{}^{AA'} \in V(\zeta_{N+2})$, and C^{+}_{j-1} is the forward null cone of $\overset{j-1}{x}{}^{AA'} \in V(\zeta_{N+2})$, the label j running now over the values $1, 2, 3, \ldots, N+1$. Notice that the only relevant vertices of $V(\zeta_{N+2})$ here are the ones associated with the colored vertices of $V(\sigma_{N+2})$, independently of whether the element of the pair $\{\overset{N}{\psi}_A(x), \underset{N}{\chi}{}^{A'}(x)\}$, with which we are eventually dealing, is an even- or an odd-order field. In fact, $\overset{N+1}{\underset{\sim}{S}}$ enters appropriately into the KAP-form at $\overset{N+1}{x}{}^{AA'}$. Once again, two conjugate expressions arise. The denominator of either expression involves the twistor form of the two inner products at $\overset{j}{x}{}^{AA'}$ whence no conjugation ambiguity emerges at this stage. In each case, the numerator consists of the wedge-product of two one-forms involving the twistor incident at $\overset{j}{x}{}^{AA'}$ in such a way that the resulting product is non-vanishing. The differentiated twistor possess the same valence in either case and, therefore, carry different edge labels. Whenever the expression is read from left to right these twistors appear in the "forward" order, contributing either a $(-i)$-factor or an i-factor to the numerator according to whether they are W_{α}- or Z^{β}-twistors. We are thus led to the SI conjugate forms

$$\overset{j}{\underset{\sim}{S}} = (-i)\frac{\underset{j+1}{Z^{\mu}}\, d\overset{j}{W}_{\mu} \wedge \underset{j}{Z}{}^{\mu}\, d\overset{j+1}{W}_{\nu}}{\left(I^{\alpha\beta}\overset{j}{W}_{\alpha}\overset{j+1}{W}_{\beta}\right)\left(I_{\rho\sigma}\underset{j}{Z}{}^{\rho}\underset{j+1}{Z}{}^{\sigma}\right)}\,, \tag{3.11}$$

and

$$\overset{j}{\underset{\sim}{S}} = i\frac{\overset{j+1}{W}_{\mu}\, d\underset{j}{Z}{}^{\mu} \wedge \overset{j}{W}_{\mu}\, d\underset{j+1}{Z}{}^{\nu}}{\left(I^{\alpha\beta}\overset{j}{W}_{\alpha}\overset{j+1}{W}_{\beta}\right)\left(I_{\rho\sigma}\underset{j}{Z}{}^{\rho}\underset{j+1}{Z}{}^{\sigma}\right)}\,. \tag{3.12}$$

Each of the SI holomorphic expressions for the $(3N+2)$-volume-form on $\overset{123\ldots N+1}{\mathbb{K}}$ involves twistors of the same valence. This result rests upon the fact that, for one handedness, the spin inner product carrying spinors bearing the other handedness "evaporate" when the intermediate steps of the actual twistorial transcription of the Minkowskian pattern are effectively worked out [4]. To build up the expression that involves W_α-twistors, say, we apply the following rules. The relevant (external) edges $\overset{1}{a}$ and $\overset{N+2}{a}$ contribute the projective one-forms $I^{\mu\nu}\overset{1}{W}_\mu d\overset{1}{W}_\nu$ and $I^{\alpha\beta}\overset{N+2}{W}_\alpha d\overset{N+2}{W}_\beta$ to the numerator of the expression, respectively. Indeed the type of these contributions is related to the fact that both the starting-vertex and end-vertex of any element of $\overset{123\ldots N+1}{\mathbb{K}}$ are held fixed when each scattering integral is actually performed. Any internal edge $\overset{h}{a}$ effectively contributes a three-form $\Delta\overset{h}{W} = (1/3!)\varepsilon^{\alpha\beta\gamma\delta}\overset{h}{W}_\alpha d\overset{h}{W}_\beta \wedge d\overset{h}{W}_\gamma \wedge d\overset{h}{W}_\delta$ to the numerator, while each colored vertex contributes an i-factor to the numerator and a squared inner-product factor $(I^{\mu\nu}\overset{j}{W}_\mu \overset{j+1}{W}_\nu)^2$ to the denominator regardless of whether its color is white or black. Upon being read from the left to the right, the twistors appear in the "forward" order. The rules for building up expression that involves Z^β-twistors are similar to those given above, but each colored vertex now particularly contributes a $(-i)$-factor to the numerator. We thus have the conjugate forms

$$\overset{123\ldots N+1}{\underset{\sim}{K}} = (i)^{N+1}\frac{I^{\alpha\beta}\overset{1}{W}_\alpha d\overset{1}{W}_\beta \wedge \left(\overset{N+1}{\underset{h=2}{\Lambda}}\Delta\overset{h}{W}\right)\wedge I^{\mu\nu}\overset{N+2}{W}_\mu d\overset{N+2}{W}_\nu}{\prod\limits_{j=1}^{N+1}\left(I^{\rho\sigma}\overset{j}{W}_\rho\overset{j+1}{W}_\sigma\right)^2}, \tag{3.13}$$

and

$$\overset{123\ldots N+1}{\underset{\sim}{K}} = (-i)^{N+1}\frac{I_{\alpha\beta}\underset{1}{Z}^\alpha d\underset{1}{Z}^\beta \wedge \left(\overset{N+1}{\underset{h=2}{\Lambda}}\Delta\underset{h}{Z}\right)\wedge I_{\mu\nu}\underset{N+2}{Z}^\mu d\underset{N+2}{Z}^\nu}{\prod\limits_{j=1}^{N+1}\left(I_{\rho\sigma}\underset{j}{Z}^\rho\underset{j+1}{Z}^\sigma\right)^2}. \tag{3.14}$$

Of course, the twistors involved in the above expressions belong all to twistor sets of suitable MSNZZ's. It should be manifested that these forms are associated with two MGS's whose vertex configurations can be constructed from one another by applying the white-black interchange rule. Indeed, it must be emphatically observed that the inner-product simplification to which we referred before must be regarded in both cases as being essentially due to the involved colored-graph structures. In fact, the forms (3.13) and (3.14) are appropriate for the cases wherein σ_{N+2} represents $\overset{N}{\psi}_A(x)$ and $\overset{N}{\chi}{}^{A'}(x)$, respectively. The graphs describing the "zero$^{\text{th}}$-order process" evidently carry no internal edge at all whence, in this case, (3.13) and (3.14) are reduced so as to involve in their numerators only the infinity-twistor one-forms.

We are now in a position to carry out the construction of the twistorial scattering integral formulae. As was previously seen (Section 2), the number of affine parameters entering explicitly into either of the expressions for the NID for the N^{th}-scattering process is equal to the order N of the involved outgoing fields. It follows that, in the even-order case, we have $N/2$ pairs of edges connecting colored vertices of each of associated MSG's. It has also been mentioned that, in the case of either odd-order field, the relevant edge $\overset{1}{r}$ comes about when we modify the appropriate $\hat{\pi}$-NI datum on C_0^+, thereby giving rise to a scattering datum involving explicitly an even number of edges in its denominator. Therefore, in the latter case, we have $(N+1)/2$ pairs of edges occurring in either $E(\sigma_{N+2})$, but only N of these edges actually connect colored vertices. The numerator of the integrand of the scattering integral for any even-order left-handed elementary field carries the holomorphic (homogeneous) twistor-datum one-form [4]

$$\left(I^{\mu\nu}\overset{1}{W}_\mu\overset{2}{W}_\nu\right)^2\overset{0}{\underset{\sim}{\Psi}}_{\mathrm{L}}\left(\overset{1}{W}_\alpha,\overset{2}{W}_\alpha\right)$$
$$= (-1)\left(I^{\mu\nu}\overset{1}{W}_\mu\overset{2}{W}_\nu\right)^2\frac{\partial}{\partial\overset{2}{W}_\rho}\overset{0}{\underset{\sim}{\Psi}}_{\mathrm{L}}\left(\overset{1}{W}_\alpha,\overset{2}{W}_\alpha\right)d\overset{1}{W}_\rho, \quad (3.15)$$

which represents the coupling of the appropriate $\hat{\pi}$-NI datum on C_0^+ with $I^{\alpha\beta}\overset{1}{W}_\alpha d\overset{1}{W}_\beta$. The remaining parts of the structure of the numerator come from (3.13) multiplied by the mass-factor $(\mu/2\pi)^N$ with $N = 2K$, $K \in \mathbb{N}\cup\{0\}$, $\mathbb{N}$ being the set of natural numbers. Each of the colored vertices of the associated MSG contributes an inner-product factor of the form $(I^{\mu\nu}\overset{j}{W}_\mu\overset{j+1}{W}_\nu)^r$ to the denominator, which is defined at the corresponding vertex of some suitable MSNZZ. Here r assumes either the value 1 or 2, the value 2 holding only for the factor defined at the vertex at which the scattering giving rise to the outgoing field happens. Hence the twistorial scattering integral for the massless free elementary field ($K = 0$) carries explicitly no such inner product at all. Any one of the K pairs of affine parameters entering into the scattering datum for the field thus contributes a factor of the type (3.8) to the denominator of the integrand. The prescription for the odd-order left-handed fields is essentially the same as the previous one. In this case we have $N = 2K + 1$, and each of the relevant denominators then carries $K + 1$ alternating-twistor pieces. The holomorphic twistor-datum one-form coming now into play represents the coupling of the modified (right-handed) $\hat{\pi}$-NI datum on C_0^+ with $I^{\alpha\beta} \overset{1}{W}_\alpha d \overset{1}{W}_\beta$. It is given explicitly by

$$\left(I^{\mu\nu} \overset{0}{W}_\mu \overset{1}{W}_\nu\right)\left(I^{\alpha\beta} \overset{1}{W}_\alpha \overset{2}{W}_\beta\right)^2 \underset{\sim}{X}_{\mathrm{R}}\left(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha\right)$$
$$= (-1)\left(I^{\mu\nu} \overset{0}{W}_\mu \overset{1}{W}_\nu\right)\left(I^{\alpha\beta} \overset{1}{W}_\alpha \overset{2}{W}_\beta\right)^2 \frac{\partial}{\partial \overset{2}{W}_\rho} X_{\mathrm{R}}\left(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha\right) d \overset{1}{W}_\rho, \tag{3.16}$$

where the two-variable twistor function involved bears automatically a suitable symmetry-homogeneity property.

As was observed earlier, the scattering graph for each right-handed field can be obtained from the one for the corresponding left-handed field of the same order by interchanging white and black vertices. As far as the integrals are concerned, this amounts particularly to replacing W_α- by Z^β- twistors. We thus have the SI formulae

$$\overset{2K}{\psi}_A(x) = \frac{c_K}{2\pi i} \oint_{\substack{123\ldots 2K+2 \\ \Gamma_{\mathrm{L}}}} \overset{2K+2}{o}{}_A$$
$$\times \frac{\left(I^{\lambda\tau} \overset{1}{W}_\lambda \overset{2}{W}_\tau\right)^2 \overset{0}{\underset{\sim}{\Psi}}_{\mathrm{L}}\left(\overset{1}{W}_\alpha, \overset{2}{W}_\alpha\right) \wedge \left(\overset{2K+1}{\underset{h=2}{\Lambda}} \Delta \overset{h}{W}\right) \wedge I^{\mu\nu} \overset{2K+2}{W}_\mu d \overset{2K+2}{W}_\nu}{\left(I^{\alpha\beta} \overset{2K+1}{W}_\alpha \overset{2K+2}{W}_\beta\right)^2 \left(\prod_{n=1}^{2K} I^{\rho\sigma} \overset{n}{W}_\rho \overset{n+1}{W}_\sigma\right)}$$
$$\times \frac{1}{\left(\prod_{\text{even } p=0}^{2K-2} \varepsilon^{\mu\nu\lambda\tau} \overset{p+1}{W}_\mu \overset{p+2}{W}_\nu \overset{p+3}{W}_\lambda \overset{p+4}{W}_\tau\right)}, \tag{3.17}$$

$$\underset{2K}{\chi}{}^{A'}(x) = \frac{c_K}{2\pi i} \oint_{\substack{123\ldots 2K+2 \\ \Gamma_{\mathrm{R}}}} \underset{2K+2}{\bar{o}}{}^{A'}$$
$$\times \frac{\left(I_{\lambda\tau} \underset{1}{Z}{}^\lambda \underset{2}{Z}{}^\tau\right)^2 \underset{\sim}{\overset{}{X}}{}_{\underset{0}{\mathrm{R}}}\left(\underset{1}{Z}{}^\beta, \underset{2}{Z}{}^\beta\right) \wedge \left(\overset{2K+1}{\underset{h=2}{\Lambda}} \Delta \underset{h}{Z}\right) \wedge I_{\mu\nu} \underset{2K+2}{Z}{}^\mu d \underset{2K+2}{Z}{}^\nu}{\left(I_{\alpha\beta} \underset{2K+1}{Z}{}^\alpha \underset{2K+2}{Z}{}^\beta\right)^2 \left(\prod_{n=1}^{2K} I_{\rho\sigma} \underset{n}{Z}{}^\rho \underset{n+1}{Z}{}^\sigma\right)}$$

$$\times \frac{1}{\left(\prod_{\text{even } p=0}^{2K-2} \varepsilon_{\mu\nu\lambda\tau} \underset{p+1}{Z^{\mu}} \underset{p+2}{Z^{\nu}} \underset{p+3}{Z^{\lambda}} \underset{p+4}{Z^{\tau}}\right)}, \tag{3.18}$$

and

$$\overset{2K+1}{\psi}{}_A(x) = \frac{C_K}{2\pi i} \oint_{\substack{123\ldots 2K+3 \\ \Gamma_{\mathrm{L}}}} \overset{2K+3}{o}{}_A$$

$$\times \frac{\underset{\sim}{X}_{\mathrm{R}}\left(\overset{1}{W}_\alpha \overset{2}{W}_\alpha\right) \wedge \left(\bigwedge_{h=2}^{2K+2} \Delta \overset{h}{W}\right) \wedge I^{\mu\nu} \overset{2K+3}{W}{}_\mu d \overset{2K+3}{W}{}_\nu}{\left(I^{\alpha\beta} \overset{2K+2}{W}{}_\alpha \overset{2K+3}{W}{}_\beta\right)^2 \left(\prod_{n=1}^{2K+1} I^{\rho\sigma} \overset{n}{W}_\rho \overset{n+1}{W}{}_\sigma\right)}$$

$$\times \frac{\left(I^{\lambda\tau} \overset{0}{W}_\lambda \overset{1}{W}_\tau\right)\left(I^{\rho\sigma} \overset{1}{W}_\rho \overset{2}{W}_\sigma\right)^2}{\left(\prod_{\text{even } q=0}^{2K} \varepsilon^{\mu\nu\lambda\tau} \overset{q}{W}_\mu \overset{q+1}{W}_\nu \overset{q+2}{W}_\lambda \overset{q+3}{W}_\tau\right)}, \tag{3.19}$$

$$\underset{2K+1}{\chi}{}^{A'}(x) = \frac{C_K}{2\pi i} \oint_{\substack{123\ldots 2K+3 \\ \Gamma_{\mathrm{R}}}} \underset{2K+3}{\bar{o}}{}^{A'}$$

$$\times \frac{\underset{\sim}{\Psi}_{\mathrm{L}}\left(\underset{1}{Z}^{\beta}, \underset{2}{Z}^{\beta}\right) \wedge \left(\bigwedge_{h=2}^{2K+2} \Delta \underset{h}{Z}\right) \wedge I_{\mu\nu} \underset{2K+3}{Z}{}^{\mu} d \underset{2K+3}{Z}{}^{\nu}}{\left(I_{\alpha\beta} \underset{2K+2}{Z}{}^{\alpha} \underset{2K+3}{Z}{}^{\beta}\right)^2 \left(\prod_{n=1}^{2K+1} I_{\rho\sigma} \underset{n}{Z}{}^{\rho} \underset{n+1}{Z}{}^{\sigma}\right)}$$

$$\times \frac{\left(I_{\lambda\tau} \underset{0}{Z}{}^{\lambda} \underset{1}{Z}{}^{\tau}\right)\left(I_{\rho\sigma} \underset{1}{Z}{}^{\rho} \underset{2}{Z}{}^{\sigma}\right)^2}{\left(\prod_{\text{even } q=0}^{2K} \varepsilon_{\mu\nu\lambda\tau} \underset{q}{Z^{\mu}} \underset{q+1}{Z^{\nu}} \underset{q+2}{Z^{\lambda}} \underset{q+3}{Z^{\tau}}\right)}. \tag{3.20}$$

At this stage, we have just been concerned with representing MGS's in terms of twistorial scattering integrals. Hence, we shall not consider explicitly here the details of the definition of the contours over which these integrals are taken (see Ref. [4]). However, we should bear in mind that such contours must be effectively tied in with the diagrammatic equalities that will be constructed in Section 4.

4. Diagrammatic equalities

In this section, we give the prescriptions whereby each MSG that enters into the graphical expansions exhibited in Figs 2 and 3 may be expressed as a twistor diagram. A correspondence between MSG's and twistor diagrams is then automatically suggested. We shall make particular use of the twistor-diagram conventions provided in Refs [5, 6]. The twistor expressions for the spin-inner products appearing in the ℝ𝕄-field integrals, play here the role of twistor inner products. This yields diagrammatic structures for massive contributions in which some of the singularity lines connect twistor vertices of the same type.

The basic idea is to replace each ε-piece involved in the denominators of the integrands of the massive integrals by an independent twistor integral of suitable auxiliary twistor. This procedure entails reducing the singularity set for each of the W_α- and Z^β-integrations involved in the expressions (3.17)–(3.20), but leaves both the relevant contour structure and the integration prescription unaffected. These modifications can be trivially made, and we will not consider them explicitly here. It is worth remarking, however, that such a reduction can actually be avoided by performing the integrations one at a time.

Let us consider the left-handed massive elementary fields. We introduce suitable auxiliary twistors $\underset{m}{A}{}^\beta$ and $\underset{m}{B}{}^\beta$, defined by

$$\oint_{\Gamma_m} \frac{\Delta \underset{m}{A}}{\left(\underset{m}{A}{}^\alpha \overset{2m-1}{W}_\alpha\right)\left(\underset{m}{A}{}^\beta \overset{2m}{W}_\beta\right)\left(\underset{m}{A}{}^\gamma \overset{2m+1}{W}_\gamma\right)\left(\underset{m}{A}{}^\delta \overset{2m+2}{W}_\delta\right)} = \frac{(-1)}{\varepsilon^{\alpha\beta\gamma\delta}\overset{2m-1}{W}_\alpha \overset{2m}{W}_\beta \overset{2m+1}{W}_\gamma \overset{2m+2}{W}_\delta}, \tag{4.1}$$

and

$$\oint_{\gamma_m} \frac{\Delta \underset{m}{B}}{\left(\underset{m}{B}{}^\alpha \overset{2m-2}{W}_\alpha\right)\left(\underset{m}{B}{}^\beta \overset{2m-1}{W}_\beta\right)\left(\underset{m}{B}{}^\gamma \overset{2m}{W}_\gamma\right)\left(\underset{m}{B}{}^\delta \overset{2m+1}{W}_\delta\right)} = \frac{(-1)}{\varepsilon^{\alpha\beta\gamma\delta}\overset{2m-2}{W}_\alpha \overset{2m-1}{W}_\beta \overset{2m}{W}_\gamma \overset{2m+1}{W}_\delta}, \tag{4.2}$$

where the Δ-differential forms are formally the same as the projective ones occurring in (3.14), and each integral is taken over a suitable $S^1 \times S^1 \times S^1$-contour (for further details, see Ref. [6]). The above integrals are represented as auxiliary twistor vertices in Fig. 4.

In (4.1) and (4.2), the label m takes the values $1, 2, 3, \ldots, K$ and $1, 2, 3, \ldots, K+1$, respectively. It can trivially label the ε-pieces entering into the field integrals for even-order and odd-order fields. In both cases,

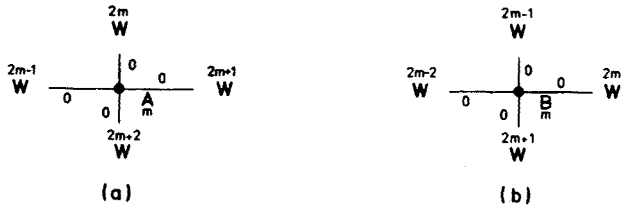


Fig. 4. Standard twistor vertices representing the integrations od suitable auxiliary twistors. These structures take up the alternating-twistor pieces that enter into the integrands of the twistorial scattering integrals for left-handed fields: (a) an auxiliary vertex for higher even-order contributions; (b) an auxiliary vertex for odd-order contributions.

the suitability of $\underset{m}{A}^{\beta}$ and $\underset{m}{B}^{\beta}$ is that the twistors associated with the involved W-vertices behave as fixed twistors whenever the evaluation of each of the auxiliary vertices is explicitly carried out. It is of some interest to observe that the numbers of black vertices involved in the respective MSG's are always equal to K and $K+1$. Notice also that for (4.1), $K \in \mathbb{N}$, while for (4.2), $K \in \mathbb{N} \cup \{0\}$. On the basis of the above prescription, we can set up an equality involving the MSG associated with any left-handed massive field and a twistor diagram. Some of these structures are shown in Fig. 5. It is clear that the massless free contribution carries no ε-piece. Hence, at least in the first instance, the relevant diagrammatic equality is that shown in Fig. 6. For simplicity the overall factors have been omitted here. In these twistor diagrams, the loops represent the holomorphic twistor-datum one-forms which are involved in the explicit twistorial scattering integrals. The dashed and dotted lines starting at points of the loops indicate, respectively, that these one-forms involve twistor derivatives of the type $\partial/\partial \overset{2}{W}_{\mu}$ along with twistor one-forms of the type $d\overset{1}{W}_{\mu}$. In each twistor diagram, the "outgoing" dashed line represents the (projective) one-form $\overset{N+2}{o}_{A} I^{\mu\nu} \overset{N+2}{W}_{\mu}\, d\overset{N+2}{W}_{\nu}$ which clearly involves the "outgoing" spinor that enters into the definition of some appropriate FSBS. In fact, the number carried by this line is related to the spin of the fields through $2(|-1/2|+1)$. Furthermore, the double line emanating from the $\overset{N+1}{W}$-twistor vertex of each massive diagram is associated with an integration over a contour [4] whose topology turns out to be $S^1 \times S^1 \times S^1$, and not $S^1 \times S^2$. Indeed, this double line appears to satisfy the "strong" four-lines rule, and seems to be related to the "simplicity" of all poles occurring in the denominators of the integrands involved in (4.1) and (4.2).

The corresponding twistor diagrams for the right-handed fields can be

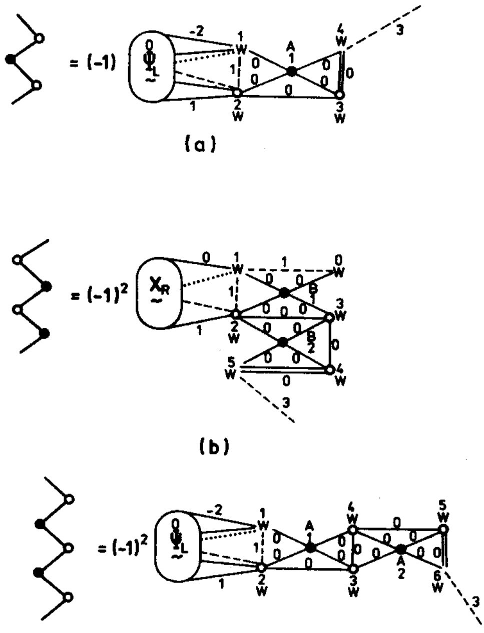


Fig. 5. Diagrammatic equalities involving MSG's and twistor diagrams for some left-handed contributions. For a field of order N the twistor diagram particularly carries either $N/2$ or $(N+1)/2$ auxiliary (filled) twistor vertices according to whether N is even or odd. These numbers of twistor vertices appear to be equal to the numbers of black vertices of the respective MSG's: (a) equality for $\overset{2}{\psi}_A(x)$; (b) equality for $\overset{3}{\psi}_A(x)$; (c) equality for $\overset{4}{\psi}_A(x)$.

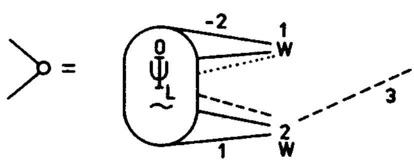


Fig. 6. Diagrammatic equality for the left-handed massless free contribution. The absence of internal edges in the relevant MSG entails absence of auxiliary vertices in the corresponding twistor diagram.

obtained from the previous ones as follows. First, we make the replacements

$$\overset{0}{\Psi}_{\mathrm{L}} \to \underset{0}{X}_{\mathrm{R}}, \quad X_{\mathrm{R}} \to \Psi_{\mathrm{L}}, \quad \overset{m}{W}_{\alpha} \to \underset{m}{Z}^{\beta},$$

m running over the same values as in (4.1) and (4.2). Second, we interchange white and black twistor vertices, keeping the same line structures.

It is evident that this twistor-vertex interchange rule includes replacing the elements of the pair $\{\underset{m}{A}^{\beta}, \underset{m}{B}^{\beta}\}$ by suitable dual twistors $\overset{m}{X}_{\alpha}, \overset{m}{Y}_{\alpha}$, and appears to be appropriate only for massive contributions. Two examples are depicted in Fig. 7.

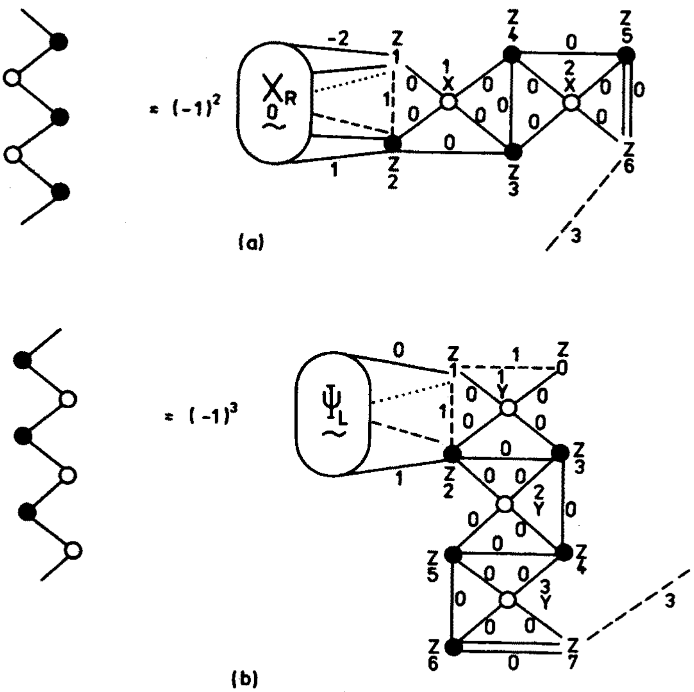


Fig. 7. Diagrammatic equalities involving MSG's and twistor diagrams for two right-handed contributions. For a field of order N the twistor diagram particularly carries either $N/2$ or $(N+1)/2$ auxiliary (hollow) twistor vertices according as N is even or odd. These numbers of twistor vertices appear to be equal to the numbers of white vertices of the corresponding MSG's: (a) equality for $\underset{4}{\chi}^{A'}(x)$; (b) equality for $\underset{5}{\chi}^{A'}(x)$.

5. Concluding remarks and outlook

We saw how the entire set of mass-scattering formulae can be derived from a set of simple rules for the colored graphs that describe the scattering processes. These rules enable us to write down the $\mathbb{RM}$-scattering integral for any elementary field and translate it immediately into twistorial term without performing any explicit calculation. Thus, the MSG's together with the relationship with (3.5) actually carry all the relevant information about the processes. As regards this fact, it may well be said that the general pattern of the structure of the twistor diagrams, associated with the graphical expansions for the entire fields, became clear. However, it is believed that alternative mass-scattering twistor diagrams can be actually drawn. In relation to this belief, we can expect that the eventual modifications perhaps involve introducing new definitions of the contours over which the twistorial field integrals must be taken. We believe also that investigation along these lines can provide significant twistorial results.

It is worth remarking explicitly that, whenever the integrals associated with the auxiliary vertices are actually brought into the twistorial expressions for the elementary fields, all the resulting integrals turn out to be independent of each other. Obviously, the integration of the auxiliary twistors effectively brings the initial structures back again. In case the involved W_α- and Z^β-integrals have to be performed first, the auxiliary twistors are held fixed. Therefore, it seems to be worthwhile to construct explicit projective pictures describing these integration procedures. A detailed discussion concerning this situation will not be entered into here, however.

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